[3] E. M. Stein, *Classes H^p, multiplicateurs et fonctions de Littlewood-Paley*, C. R. Acad. Sci. Paris 263 (1966), p. 716-719.

[4] Mary Weiss, *On symmetric derivatives in L^p*, Studia Math. 24 (1964), p. 89-100.

[5] A. Zygmund, *Über einige Sätze aus der Theorie der divergenten Reihen*, Bull. Intern. Acad. Pol. Sci. et Lettres, 1927, p. 309-331.

[6] — *Trigonometric series*, 2nd ed., Cambridge 1959, vol. I and II.

UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE
UNIVERSITY OF CHICAGO

Reçu par la Rédaction le 22. 8. 1967

STUDIA MATHEMATICA, T. XXX (1968)

Absolute continuity of vector-valued finitely additive set functions, I

by

W. ORLICZ (Poznań)

1. Throughout this paper we shall use the following notations and notions. X denotes a real Banach space and Ξ its conjugate space, ξ stands for a functional from Ξ. By Ξ_0 we denote a set of functionals from Ξ such that $\|\xi\| \leqslant 1$ and $\sup|\xi(x)| \geqslant c\|x\|$ for some $c > 0$, where the supremum is taken over all ξ in Ξ_0. The sets Ξ_0 are called *fundamental*. X is called *weakly sequentially compact* with respect to Ξ_0 if for every sequence ξ_n, where $\xi_n \epsilon \Xi_0$, there exists a subsequence $\xi_{n_i}(x)$ converging for any $x \epsilon X$. It is a simple matter to prove that a separable space X is weakly sequentially compact with respect to $\Xi_0 = \{\xi: \|\xi\| \leqslant 1\}$. There exist also non-separable spaces weakly sequentially compact with respect to some Ξ_0. As an example of such kind of spaces one can take the space M_0 of real-valued functions $f(\cdot)$, bounded on $\langle a, b\rangle$, having for any $t \epsilon (a, b)$ the right- and left-hand limit and the limits $f(a+), f(b-)$. Here the norm for $f \epsilon M_0$ is defined as $\sup\limits_{\langle a,b\rangle} |f(t)|$, and for Ξ_0 one can take the set of all functionals of the form

$$\xi(x) = \pm n \int\limits_{u}^{v+1/n} f(t)\,dt,$$

where $a \leqslant u, v+1/n \leqslant b, n = 1, 2, \ldots$ By E we will denote an abstract set of points (elements), the symbol $e_n\uparrow$ or $e_n\downarrow$ will stand for a sequence of sets e_n in E such that $e_1 \subset e_2 \subset \ldots$ or $e_1 \supset e_2 \supset \ldots$ respectively. $e_n\downarrow e$ or $e_n\uparrow e$ means $e_n\downarrow, e = \bigcap\limits_1^\infty e_n$, or $e_n\uparrow, e = \bigcup\limits_1^\infty e_n$ respectively.

Besides the notation given before we shall use throughout the paper the letter $\boldsymbol{E}$ to denote the class of zero-one sequences $\boldsymbol{\varepsilon} = \{\varepsilon_i\}$, that is to say with terms $\varepsilon_i = 1, 0$. If e is a set of points from E, then $\varepsilon_i a$ means a if $\varepsilon_i = 1$, and the empty set if $\varepsilon_i = 0$. For a sequence of sets a_i the symbol $a(\boldsymbol{\varepsilon})$ or $a^n(\boldsymbol{\varepsilon})$ denote the set $\bigcup\limits_1^\infty \varepsilon_i a_i$, or $\bigcup\limits_{i\geqslant n} \varepsilon_i a_i$, respectively. $\mathscr{E}, \mathscr{F}, \ldots$ always denotes a ring or an algebra of subsets from E. The class $\mathscr{E}$ is called *σ-ring* or *σ-algebra* respectively if the sum of countably many sets from $\mathscr{E}$ always belongs to $\mathscr{E}$ and $\mathscr{E}$ is a ring or an algebra respectively. $\mathscr{E}_a$ stands for the σ-algebra of all subsets from E. By $\mu(:)$, $\lambda(:)$, $\eta(:)$ we denote set functions defined on $\mathscr{E}$ and assuming real values, by $x(:)$, ... vector-valued set functions from $\mathscr{E}$ to some X. A set function $x(:)$ is called *additive* on $\mathscr{E}$ if $x(e_1 \cup e_2) = x(e_1)+x(e_2)$ for any disjoint sets e_1, e_2 in $\mathscr{E}$, it is called *σ-additive* on $\mathscr{E}$ if for any sequence e_i of mutually disjoint sets from $\mathscr{E}$ such that $e = \bigcup_1^\infty e_i \in \mathscr{E}$ we have $x(e) = x(e_1)+x(e_2)+\ldots$

A scalar set function $\eta(:)$ defined on $\mathscr{E}$ is said to be a *subadditive measure* on $\mathscr{E}$ if it satisfies the following conditions:

1) $\eta(\emptyset) = 0$, where $\emptyset$ denotes the empty set,

2) $\eta(e_1) \leqslant \eta(e_2)$ for $e_1 \subset e_2$ and $e_1, e_2 \in \mathscr{E}$,

3) $\eta(e_1 \cup e_2) \leqslant \eta(e_1)+\eta(e_2)$ for $e_1 \cap e_2 = \emptyset$ and $e_1, e_2 \in \mathscr{E}$.

Evidently $\eta(e) \geqslant 0$ for $e \in \mathscr{E}$.

Replacing condition 3) by the condition

3′) $\eta(\bigcup_{i=1}^\infty e_i) \leqslant \eta(e_1)+\eta(e_2)+\ldots$,

where e_i are disjoint sets in $\mathscr{E}$ and $\bigcup_1^\infty e_i \in \mathscr{E}$, we get the definition of a *σ-subadditive measure* on $\mathscr{E}$.

If we replace in 3) or in 3′) the sign $\leqslant$ by $=$ we obtain the definition of an additive measure or σ-additive measure on $\mathscr{E}$, respectively (in what follows we shall say briefly: an additive measure, a σ-additive measure). A subadditive measure $\eta(:)$ can be always extended to a subadditive measure $\bar{\eta}(:)$ defined on the algebra of all subsets of E setting–as usual– $\bar{\eta}(e) = \inf \eta(a)$, where the infimum is taken for all sets $a \in \mathscr{E}$ covering e if such a set exists, $\bar{\eta}(e) = \infty$ if no set in $\mathscr{E}$ covers e.

Let $\eta(:)$ be a subadditive measure on $\mathscr{E}$. A set function $x(:)$ defined on $\mathscr{E}$ is called *absolutely continuous* with respect to $\eta(:)$ (briefly: a.c. with respect to $\eta(:)$) if $\eta(e_n) \to 0$ as $n \to \infty$, $e_n \in \mathscr{E}$, implies $x(e_n) \to 0$ as $n \to \infty$. $x(:)$ is *weakly absolutely continuous* with respect to $\eta(:)$ (briefly: w.a.c. with respect to $\eta(:)$) if $e_n\downarrow$, $\eta(e_n) \to 0$ as $n \to \infty$ implies $x(e_n) \to 0$.

The definition of the absolute continuity (weak absolute continuity) of a subadditive measure $\mu(:)$ with respect to $\eta(:)$ is quite analogical.

For a vector-valued additive set function $x(:): \mathscr{E} \to X$, we define a set function $v(:, x)$, the variation of $x(:)$ on $\mathscr{E}$, as

$$v(e, x) = \sup \|\lambda_1 x(e_1)+\lambda_2 x(e_2)+\ldots+\lambda_n x(e_n)\| \quad \text{for } e \in \mathscr{E},$$

where the supremum is taken with respect to all disjoint e_i lying in $\mathscr{E}$, contained in e, and all real scalars λ_i, $|\lambda_i| \leqslant 1$. The value $v(e, x)$ is called the *variation* of $x(:)$ on e; if $v(e, x) < \infty$, then $x(:)$ is of *bounded variation* on e. The variation $v(:, x)$ is a subadditive measure on $\mathscr{E}$. If $x(:)$ is a.c. with respect to a subadditive measure $\eta(:)$, then $v(:, x)$ is also a.c. with respect to $\eta(:)$. If $x(:)$ is w.a.c. on $\mathscr{E}$, then $\eta(e) = 0$ implies $v(e, x) = 0$; the weak absolute continuity of $v(:, x)$ may be guaranteed by certain additional assumptions concerning either $\mathscr{E}$ or $x(:)$. (cf. the theorems in 3, 4.4).

2. Throughout section 2 $\tilde{\mathscr{E}}$ means a fixed σ-algebra of subsets from E (called *fundamental algebra* $\tilde{\mathscr{E}}$); $\eta(:)$ is a finite, subadditive measure defined on $\tilde{\mathscr{E}}$. Let a ring $\mathscr{E} \subset \tilde{\mathscr{E}}$ be given. Let us form a class $\mathscr{E}_J^\eta$ of sets from $\tilde{\mathscr{E}}$ as follows: $e \in \mathscr{E}_J^\eta$ if and only if there are sets $e_n \in \mathscr{E}$ such that $\eta(e_n \dot{-} e) \to 0$ as $n \to \infty$, the symbol $a \dot{-} b$ means the symmetric difference of the sets a and b. By routine arguments we verify that $\mathscr{E}_J^\eta$ is a ring. The class $\mathscr{E}_J^\eta$ will be called the *Jordan ring* of sets, generated by $\mathscr{E}$ and $\eta(:)$ in $\tilde{\mathscr{E}}$. It will by called a *Jordan algebra* if $\mathscr{E}_J^\eta$ is an algebra of subsets of E. In particular, $\mathscr{E}_J^\eta$ is a Jordan algebra if $\mathscr{E}$ is an algebra and $e \in \mathscr{E}_J^\eta$ if and only if for any $\varepsilon > 0$ there are sets $e_1, e_2 \in \mathscr{E}$, $e_1 \subset e \subset e_2$ such that $\eta(e_2 - e_1) < \varepsilon$. We denote by $\mathscr{S}$ the set of algebras $\mathscr{E}_J^\eta$ of this sort. Notice that any set e in $\tilde{\mathscr{E}}$ which η-measure equals 0 belongs to $\mathscr{E}_J^\eta$ for an arbitrary $\mathscr{E}$. If $\mathscr{E}_J^\eta \subset \mathscr{S}$, then for $e \in \mathscr{E}_J^\eta$ we obtain

$$\sup_{e_1} \eta(e_1) = \inf_{e_2} \eta(e_2) = \eta(e),$$

where the supremum is taken for all $e_1 \subset e$, $e_1 \in \mathscr{E}$, and the infimum is taken for all $e_2 \supset e$, $e_2 \in \mathscr{E}$. The formation of different Jordan-rings depends on the choice of $\tilde{\mathscr{E}}$ and $\eta(:)$. A well known useful procedure to obtain a finite subadditive measure $\eta(:)$ on $\mathscr{E} = \mathscr{E}_a$ consists in the following: we assume that $\mathscr{F}$ is an algebra of subsets of E and that a finite measure $\nu(:)$ is defined on $\mathscr{F}$. We set either $\eta(e) = \inf \nu(a)$, where the infimum is taken for all a in $\mathscr{F}$ which cover e, or $\eta(e) = \inf(\nu(a_1)+\nu(a_2)+\ldots)$, where the infimum is taken for all coverings $\bigcup_1^\infty a_i$, $a_i \in \mathscr{F}$. Another good example of $\eta(:)$ is obtained assuming that $\mathscr{F}$ is an algebra on which, a vector-valued, additive bounded set function $x(:)$ is defined. We set $\tilde{\mathscr{E}} = \mathscr{E}_a$, $\eta(e) = \inf v(a, x)$, for $e \in \mathscr{E}_a$, where the infimum is taken for $a \in \mathscr{F}$, $e \subset a$. Still another way to obtain $\eta(:)$ is to start with a σ-algebra of subsets $\mathscr{F}$ on which a vector-valued additive set function $x(:)$ is defined such that $\|x(e_1)\| \leqslant \|x(e_2)\|$ for $e_1, e_2 \in \mathscr{F}$, $e_1 \subset e_2$, and to set $\tilde{\mathscr{E}} = \mathscr{F}$, $\eta(e) = \|x(e)\|$ (cf. 6, [3]).

2.1. *If* $e_n \in \tilde{\mathscr{E}}$, $e_n\downarrow$, $\eta(e_n) \to 0$ *as* $n \to \infty$, $a_n \subset e_n - e_{n+1}$, $a_n \in \mathscr{E}_J^\eta$, *then for an arbitrary zero-one sequence* $\varepsilon = \{\varepsilon_i\}$ *the set* $a(\varepsilon) = \bigcup_1^\infty \varepsilon_i a_i$ *belongs* *to* $\mathcal{E}_J^\eta$ *and for* $a^n(\varepsilon) = \bigcup_{i\geq n} \varepsilon_i a_i$ *we have* $\eta(a^n(\varepsilon)) \to 0$ *as* $n\to\infty$, *uniformly with respect to* $\varepsilon \in \boldsymbol{E}$.

Proof. Let $\bar{e} = \varepsilon_1 a_1 \cup \varepsilon_2 a_2 \cup \ldots \cup \varepsilon_{n-1}a_{n-1}$; then $\eta(a(\varepsilon)-\bar{e}) \leqslant \eta(e_n) < \varepsilon/2$ for sufficiently large n. Choosing a set $\bar{a}$ in $\mathcal{E}$ such that $\eta(\bar{e} \dot{-} \bar{a}) < \varepsilon/2$ we obtain $\eta(a(\varepsilon)\dot{-}\bar{a}) < \varepsilon$, hence $a(\varepsilon) \in \mathcal{E}_J^\eta$. Therefore $a^n(\varepsilon) \in \mathcal{E}_J^\eta$ and because of $a^n(\varepsilon) \subset e_n$ we get $\eta(a^n(\varepsilon)) \leqslant \eta(e_n) \to 0$ as $n\to\infty$, independently of the zero-one sequence ε.

2.2. Let $\mathcal{E}_0$ be a σ-ring of subsets of E. A set function $x(:)$ on $\mathcal{E}_0 \cap \mathcal{E}_J^\eta$, is called *$\sigma$-additive* with respect to $\eta(:)$ if for $e_i \in \mathcal{E}_0 \cap \mathcal{E}_J^\eta$, $\bigcup_1^\infty e_i \in \mathcal{E}_0 \cap \mathcal{E}_J^\eta$, $e_i \cap e_j = \emptyset$ for $i \neq j$, such that $\eta(e_n \cup e_{n+1} \cup \ldots) \to 0$ as $n\to\infty$, $x(\bigcup_1^\infty e_i) = x(e_1)+x(e_2)+\ldots$.

If $x(:)$ is σ-additive on $\mathcal{E}_0 \cap \mathcal{E}_J^\eta$ with respect to $\eta(:)$, then from $e_n \downarrow \emptyset$, $\eta(e_n)\to 0$ as $n\to\infty$ it follows that $x(e_n)\to 0$ as $n\to\infty$.

Indeed, 2.1 and the σ-additivity of $x(:)$ with respect to $\eta(:)$ imply $x(e_n) = (x(e_n)-x(e_{n+1}))+(x(e_{n+1})-x(e_{n+2}))+\ldots$; thus $x(e_n)\to 0$ as $n\to\infty$.

If $x(:)$ is σ-additive on $\mathcal{E}_0 \cap \mathcal{E}_J^\eta$ with respect to $\eta(:)$ and $e_n \uparrow e$, $e \in \mathcal{E}_J^\eta$ $\eta(e-e_n)\to 0$, as $n\to\infty$, or $e_n\downarrow e$, $e\in\mathcal{E}_J^\eta$, $\eta(e_n - e)\to 0$ as $n\to\infty$ respectively, then

$$\liminf_{n\to\infty} v(e_n, x) \geqslant v(e, x).$$

This follows from the preceding remark by the application of the definition of the variation $v(:, x)$.

(a) *An additive set function $x(:)$ which is w.a.c. with respect to $\eta(:)$ is σ-additive with respect to $\eta(:)$.*

(b) *If $x(:)$ is σ-additive on $\mathcal{E}_0 \cap \mathcal{E}_J^\eta$ and $\eta(e) = 0$, where $e \in \mathcal{E}_0 \cap \mathcal{E}_J^\eta$, implies $x(e) = 0$, then $x(:)$ is w.a.c. on $\mathcal{E}_0 \cap \mathcal{E}_J^\eta$ with respect to $\eta(:)$.*

Ad (b). If $e_n\downarrow$, $\eta(e_n)\to 0$ as $n\to\infty$, then for the set $a = \bigcap_1^\infty e_n$, $\tilde{\mathcal{E}}$ being a σ-algebra, we have $a \in \mathcal{E}_0 \cap \mathcal{E}_J^\eta$, $\eta(a) = 0$; consequently $x(a) = 0$. Since $e_n - a = (e_n - e_{n+1}) \cup (e_{n+1} - e_{n+2}) \cup \ldots$ we obtain by the σ-additivity of $x(:)$, $x(e_n - a) = x(e_n - e_{n+1}) + x(e_{n+1}-e_{n+2})+\ldots$ Therefore the series $x(e_1-e_2)+x(e_2-e_3)+\ldots$ converges and consequently $x(e_n - a) = x(e_n)\to 0$ as $n\to\infty$.

3. *Let $\eta(:)$ be a finite subadditive measure on a ring $\mathcal{E}$, $\mu(:)$ a real valued, additive, bounded set function on $\mathcal{E}$. If $\mu(:)$ is weakly absolutely continuous with respect to $\eta(:)$ on $\mathcal{E}$, then it is absolutely continuous with respect to $\eta(:)$.*

Proof. Let us write $v^+(e) = \sup \mu(a)$, $v^-(e) = -\inf \mu(a)$, $v(e) = v^+(e)+v^-(e)$, where $e\in\mathcal{E}$ and the supremum or the infimum respectively is taken with respect to $a \in \mathcal{E}$, $a \subset e$. It is well known that $v^+(:)$, $v^-(:)$, $v(:)$ are bounded, additive set function on $\mathcal{E}$. Let us write

$$\mu = \lim_{\delta\to 0}\big(\sup_{\eta(e)\leqslant\delta} v^+(e)\big);$$

we have $0 \leqslant \mu < \infty$. Let us assume that $\mu > 0$. First, we shall prove the following lemma:

Let positive $\varepsilon_1, \varepsilon_2$ be given and let $\mu - (\varepsilon_1+\varepsilon_2) > 0$, $\mu + \varepsilon_1 \geqslant v^+(e)$ for $\eta(e) < \delta$. Let for a set $a \in \mathcal{E}$, such that $\eta(a) < \delta$, the inequality $\mu(a) > \mu - \varepsilon_2$ hold. Moreover, let us assume that $\bar{\varepsilon}_1, \bar{\varepsilon}_2 > 0$ are given and $\mu - (\varepsilon_1 + \varepsilon_2) - (\bar{\varepsilon}_1+\bar{\varepsilon}_2) > 0$. Then there exists $\bar{\delta} < \delta/2$ such that

$$(+) \qquad \mu + \bar{\varepsilon}_1 \geqslant v^+(e) \quad \textit{for} \quad \eta(e) < \bar{\delta},$$

and a set $\bar{a} \in \mathcal{E}$, $\bar{a} \subset a$, $\eta(\bar{a}) < \bar{\delta}$ such that

$$\mu(\bar{a}) > \mu - (\varepsilon_1+\varepsilon_2) - (\bar{\varepsilon}_1+\bar{\varepsilon}_2).$$

In fact, let $\bar{\delta}$ be less than $\min(\delta/2, \delta - \eta(a))$ and so small that inequality $(+)$ holds. Next, let us choose a set $e_0 \in \mathcal{E}$ in such a manner that $\eta(e_0) < \bar{\delta}$, $\mu(e_0) > \mu - \bar{\varepsilon}_2$. Suppose that $\mu(e_0 - a) \geqslant \varepsilon_1 + \varepsilon_2$. Because of $\eta(a \cup (e_0 - a)) < \eta(a) + (\delta - \eta(a)) = \delta$ we get

$$\mu + \varepsilon_1 \geqslant v^+(a \cup (e_0 - a)) \geqslant \mu(a) + \mu(e_0 - a) > \mu - \varepsilon_2 + (\varepsilon_1 + \varepsilon_2) = \mu + \varepsilon_1,$$

a contradiction. Hence we have $\mu(e_0 - e_0 \cap a) < \varepsilon_1 + \varepsilon_2$ and defining $\bar{a} = e_0 \cap a$ we get

$$\mu(\bar{a}) = \mu(e_0) - \mu(e_0 - \bar{a}) > \mu - \bar{\varepsilon}_2 - (\varepsilon_1+\varepsilon_2) > \mu - (\varepsilon_1+\varepsilon_2) - (\bar{\varepsilon}_1+\bar{\varepsilon}_2) > 0,$$

$\bar{a} \subset a$, $\eta(\bar{a}) < \bar{\delta}$.

By means of the above lemma we can define by induction a sequence of positive numbers δ_n and a sequence of sets e_n from $\mathcal{E}$ having the following properties for $n = 1, 2, \ldots$:

(a) $\delta_{n+1} < \delta_1/2^n$;
(b) $\mu + \mu/2^{n+3} \geqslant v^+(e)$ for $\eta(e) < \delta_n$;
(c) $e_n\downarrow$, $\mu(e_n) \geqslant 0$;
(d) $\eta(e_n) < \delta_n$;
(e) $\mu(e_n) > \mu - (1/2^2 + 1/2^3 + \ldots + 1/2^{n+1})\mu > \mu/2$.

In virtue of (a), (c) and (d) we obtain $\mu(e_n)\to 0$ as $n\to\infty$ and by (e) it follows that $\mu = 0$–a contradiction. In a similar way we can prove

$$\lim_{\delta\to 0}\big(\sup_{\eta(e)\leqslant\delta} v^-(e)\big) = 0,$$

and since $v(:) = v^+(:) + v^-(:)$ we infer that $v(:)$ is a.c. with respect to $\eta(:)$. Consequently, $\mu(:)$ is a.c. with respect to $\eta(:)$ as well.

3.1. *Suppose on a ring $\mathcal{E}$ a finite subadditive measure $\eta(:)$ is defined and $\eta(:)$ fulfills the following axiom of completeness* ([3]):

(c) *If* $e_i \in \mathscr{E}$ *for* $i = 1, 2, \ldots, e_i \cap e_j = 0$ *for* $i \neq j$ *and* $\sum_1^\infty \eta(e_i) < \infty$, *then* $\bigcup_1^\infty e_i \in \mathscr{E}$ *and*

$$\eta\Big(\bigcup_{i=1}^\infty e_i\Big) \leqslant \sum_{i=1}^\infty \eta(e_i).$$

Suppose that for $n = 1, 2, \ldots, x_n(:): \mathscr{E} \to X$ *is a vector-valued, additive set function, a.c. with respect to* $\eta(:)$. *If for any* e *in* $\mathscr{E}$ $x_n(e) \to x(e)$ *as* $n \to \infty$, *then* $x_n(e_n) \to 0$ *as* $n \to \infty$ *if* $\eta(e_n) \to 0$ *as* $n \to \infty$ *and* $x(:)$ *is additive, a.c. with respect to* $\eta(:)$ *on* $\mathscr{E}$.

The theorem is well known if $\mathscr{E}$ is a σ-ring and $\eta(:)$ is a finite σ-additive measure on $\mathscr{E}$. Since the proof of the theorem under more general hypotheses given above is quite analogous to the proof in the classical case, we omit it.

4. Throughout the present section and the following sections we assume that a fundamental measure space $(\tilde{\mathscr{E}}, \eta)$ is given. $\mathscr{E}_0$ means a σ-ring of subsets of E, $\mathscr{E}_J^\eta$ the Jordan-ring generated by $\eta(:)$ and $\mathscr{E}$ in $\tilde{\mathscr{E}}$, where $\mathscr{E}$ denotes a subring of $\tilde{\mathscr{E}}$. $x_n(:), x(:)$ means vector-valued, additive set functions on $\mathscr{E}_0 \cap \mathscr{E}_J^\eta$ to X. Moreover, the notation $v_n(e)$ for $v(e, x_n)$, $v(e)$ for $v(e, x)$, variations taken on $\mathscr{E}_0 \cap \mathscr{E}_J^\eta$, will be used systematically. We shall use also the notation

$$\gamma(e) = \limsup_{n\to\infty} v_n(e)$$

for $e \in \mathscr{E}_0 \cap \mathscr{E}_J^\eta$.

4.1. *Assume that* $x_n(:)$ *are* σ*-additive with respect to* $\eta(:)$ *on* $\mathscr{E}_0 \cap \mathscr{E}_J^\eta$. *Assume that*

(a) $\quad (*) \quad x_n(e) \to x(e)$ *as* $n \to \infty$,

for $e \in \mathscr{E}_0 \cap \mathscr{E}_J^\eta$,

(b) $\quad (**) \quad v_n(e) \to 0$ *as* $n \to \infty$,

for any e *in* $\mathscr{E}_0 \cap \mathscr{E}_J^\eta$ *for which* $\eta(e) = 0$.

Under the assumptions given above the following statements hold:

(α) *For any sequence of sets* e_n *in* $\mathscr{E}_0 \cap \mathscr{E}_J^\eta$ *such that* $e_n \downarrow$, $\eta(e_n) \to 0$ *as* $n \to \infty$, *we have* $x_n(e_n) \to 0$ *as* $n \to \infty$;

(β) *The set function* $\gamma(:)$ *is subadditive and w.a.c. with respect to* $\eta(:)$ *on* $\mathscr{E}_0 \cap \mathscr{E}_J^\eta$;

(γ) *The set function* $x(:)$ *is additive and w.a.c. with respect to* $\eta(:)$ *on* $\mathscr{E}_0 \cap \mathscr{E}_J^\eta$.

Proof. Let $e_n \in \mathscr{E}_0 \cap \mathscr{E}_J^\eta$ for $n = 1, 2, \ldots, e_n \downarrow$, $\eta(e_n) \to 0$ as $n \to \infty$. Set $a = \bigcap_1^\infty e_i$; evidently $a \in \mathscr{E}_0 \cap \mathscr{E}_J^\eta$, $\eta(a) = 0$. For $q \geqslant p$, $p = 1, 2, \ldots$

we have $e_p - e_q \uparrow e_p - a$, $\eta(e_q - a) \to 0$ as $q \to \infty$; thus $\eta\big((e_p - a) - (e_p - e_q)\big) \to 0$ as $q \to \infty$ and by the σ-additivity of $x_k(:)$ with respect to $\eta(:)$ the inequality

(i) $$\liminf_{q\to\infty} v_k(e_p - e_q) \geqslant v_k(e_p - a)$$

holds for $p = 1, 2, \ldots$.

We claim that

(ii) $$\lim_{p,q\to\infty} \big(\sup_k v_k(e_p - e_q)\big) = 0.$$

In fact, in the contrary case there exist $\varepsilon_0 > 0, k_1, k_2, \ldots$ and $p_1 < p_2 < \ldots$ such that $v_{k_i}(e_{p_i} - e_{p_{i+1}}) \geqslant \varepsilon_0$ for $i = 1, 2, \ldots$ Choose the sets $a_i \in \mathscr{E}_0 \cap \mathscr{E}_J^\eta$, $a_i \in e_{p_i} - e_{p_{i+1}}$ in such a manner that $2\|x_{k_i}(a_i)\| \geqslant \varepsilon_0$ for $i = 1, 2, \ldots$ (this is always possible as follows from the definition of the variation of a set function). Set $a(\boldsymbol{\varepsilon}) = \bigcup_1^\infty \varepsilon_i a_i$; by 2.1 and by the σ-additivity of $x_n(:)$ with respect to $\eta(:)$ we get $a(\boldsymbol{\varepsilon}) \in \mathscr{E}_0 \cap \mathscr{E}_J^\eta$,

$$x_k\big(a(\boldsymbol{\varepsilon})\big) = \varepsilon_1 x_k(a_1) + \varepsilon_2 x_k(a_2) + \ldots$$

for any $\boldsymbol{\varepsilon} \in \boldsymbol{E}$. But, by (a), $x_k\big(a(\boldsymbol{\varepsilon})\big) \to x\big(a(\boldsymbol{\varepsilon})\big)$ as $k \to \infty$ for any $\boldsymbol{\varepsilon} \in \boldsymbol{E}$. Therefore, by a known lemma (cf. [1])

$$\sup_k \|x_k(a_i)\| \to 0 \quad \text{as } i \to \infty,$$

a contradiction. Now, let us observe that by the subadditivity of $v_k(:)$ the inequality

(iii) $$\sup_{k\geqslant k_0} v_k(e_p) \leqslant \sup_k v_k(e_p - a) + \sup_{k\geqslant k_0} v_k(a)$$

holds. But $\eta(a) = 0$; thus, in virtue of (b),

$$\gamma(a) \leqslant \sup_{k\geqslant k_0} v_k(a) \to 0$$

as $k_0 \to \infty$. Hence $\gamma(a) = 0$. By (i), (ii)

$$\sup_k v_k(e_p - a) \to 0 \quad \text{as } p \to \infty.$$

By (iii) for sufficiently large p the inequality $\|x_p(e_p)\| < \varepsilon$ holds, thus (α) is proved. From (iii) it follows also that

$$\gamma(e_p) \leqslant \sup_k v_k(e_p - a) + \gamma(a)$$

and since $\gamma(a) = 0$, we obtain $\gamma(e_p) \to 0$ as $p \to \infty$. The subadditivity of $\gamma(:)$ being evident, this completes the proof of (β). To prove (γ) let us observe that $\|x(e)\| \leqslant v(e, x) \leqslant \gamma(e)$ and that by additivity of $x_n(:)$ the limit set function $x(:)$ is additive as well.

4.2. *Suppose that $\mathcal{E}^\eta_J$ is a Jordan algebra belonging to the class $\mathcal{S}$, $\mathcal{E}\subset\mathcal{E}_0$. Let $x_n(:)$ be additive on $\mathcal{E}_0\cap\mathcal{E}^\eta_J$ for $n=1,2,\ldots$ and let 4.1(b) be satisfied. Suppose that the relation 4.1(*) holds for any e in $\mathcal{E}$. Suppose that $\gamma(:)$ is w.a.c. on $\mathcal{E}$. Then 4.1(a) is satisfied.*

Proof. Suppose $e\in\mathcal{E}_0\cap\mathcal{E}^\eta_J$. Then, we can find sets e_n^2, e_n^1 in $\mathcal{E}$, such that $e_n^2\downarrow, e_n^1\uparrow, e_n^1\subset e\subset e_n^2$ for $n=1,2,\ldots$ $\eta(e_n^2-e)\to 0$ as $n\to\infty$, $\eta(e-e_n^1)\to 0$ as $n\to\infty$. Since $\eta(e_n^2-e_n^1)\to 0$, $e_n^2-e_n^1\downarrow$, as $n\to\infty$ we get $\gamma(e_k^2-e_k^1)<\varepsilon$ for a certain index k and consequently for sufficiently large n the inequalities

$$\|x_n(e)-x_n(e_k^2)\|\leqslant v_n(e_k^2-e_k^1)<\gamma(e_k^2-e_k^1)+\varepsilon<2\varepsilon$$

hold. Hence

$$\|x_p(e)-x_q(e)\|\leqslant\|x_p(e)-x_p(e_k^2)\|+\|x_p(e_k^2)-x_q(e_k^2)\|+\|x_q(e)-x_q(e_k^2)\|<5\varepsilon$$

for sufficiently large p and q and the theorem is proved.

4.3. *Assume that $x_n(:)$ are σ-additive with respect to $\eta(:)$ on $\mathcal{E}_0\cap\mathcal{E}^\eta_J$, and that 4.1(b) is satisfied. Then, each of conditions, 4.1(α) or 4.1 (β), is equivalent to the condition:*

(δ) *For any sequence of sets e_n in $\mathcal{E}_0\cap\mathcal{E}^\eta_J$ such that $e_n\downarrow$, $\eta(e_n)\to 0$ as $n\to\infty$ the relation*

$$\sup_{k\geqslant n} v_k(e_n)\to 0$$

as $n\to\infty$ holds.

Implications (δ) $\Rightarrow$ (α) and (δ) $\Rightarrow$ (β) are trivial. To show the opposite implication it is enough to prove that $e_n\downarrow$, $\eta(e_n)\to 0$ as $n\to\infty$, under condition (α) or (β), implies relation 4.1(ii), for the relation in question implies

$$\sup_k v_k(e_n-a)\to 0 \quad \text{as } n\to\infty,$$

where

$$a=\bigcap_1^\infty e_n\in\mathcal{E}_0\cap\mathcal{E}^\eta_J,\qquad \eta(a)=0,$$

and inequality 4.1(iii) holds. To this end assume that for a sequence $e_n\downarrow$, $\eta(e_n)\to 0$ as $n\to\infty$, relation 4.1(ii) is not satisfied. Then, as in course of the proof in 4.1, we can choose indices $k_1,k_2,\ldots,p_1<p_2<\ldots$ and sets $a_i\in\mathcal{E}_0\cap\mathcal{E}^\eta_J$ such that for $\boldsymbol{\varepsilon}\in\boldsymbol{E}$ the set $a(\boldsymbol{\varepsilon})=\bigcup\varepsilon_i a_i$ belongs to $\mathcal{E}_0\cap\mathcal{E}^\eta_J$, if $a^n(\boldsymbol{\varepsilon})=\bigcup_{i\geqslant n}\varepsilon_i a_i$ then $\eta(a^n(\boldsymbol{\varepsilon}))\to 0$ as $n\to\infty$, and

(i) $$2\|x_{k_n}(a_n)\|\geqslant\varepsilon_0>0 \quad \text{for } n=1,2,\ldots$$

In virtue of the σ-additivity of $x_n(:)$ with respect to $\eta(:)$ we can assume $k_1<k_2<\ldots$ Suppose now that 4.1(β) is fulfilled. Then it is possible to assume that

(ii) $$\|x_{k_n}(a_i)\|<\frac{1}{2^i}\quad \text{for } i<n,\ n=2,\ldots$$

Indeed, $\gamma(a_n)\leqslant\gamma(a^n(\boldsymbol{\varepsilon}))\to 0$ as $n\to\infty$ for $\boldsymbol{\varepsilon}=1,1,\ldots$; thus for a subsequence a_{n_i} we have $\gamma(a_{n_i})<1/2^{i+1}$, and, step by step, we can define a subsequence k_{l_i} with the desired property. Choose $\xi_n\in\Xi_0$ and a subsequence k_{p_n} in such a manner that:

1° $\xi_n(x_{k_{p_n}}(a_{p_n}))\geqslant\varepsilon_0/4$, $\|\xi_n\|=1$;

2° the limit

$$\lim_{p_n\to\infty}\xi_n(x_{k_{p_n}}(a_i))=\lambda_i$$

exists for $i=1,2,\ldots$

In virtue of (ii) we have

$$\sum_{i=1}^\infty|\lambda_i|<\infty.$$

Set $k_{p_n}=l_n$; for any $\boldsymbol{\varepsilon}\in\boldsymbol{E}$ the equations

$$\xi_n(x_{l_n}(a(\boldsymbol{\varepsilon})))=\varepsilon_1\xi_n(x_{l_n}(a_1))+\varepsilon_2\xi_n(x_{l_n}(a_2))+\ldots$$

and

$$\xi_n(x_{l_n}(a^i(\boldsymbol{\varepsilon})))=\varepsilon_i\xi_n(x_{l_n}(a_i))+\varepsilon_{i+1}\xi_n(x_{l_n}(a_{i+1}))+\ldots$$

hold. By $\eta(a^j(\boldsymbol{\varepsilon}))\to 0$ as $j\to\infty$ and by 4.1(β)

$$|\xi_n(x_{l_n}(a^j(\boldsymbol{\varepsilon})))|\leqslant\varepsilon\quad \text{for } j\geqslant j_0 \text{ and } n\geqslant n_0(j).$$

Since for $j\geqslant j_0$

$$\limsup_{n\to\infty}\Big|\xi_n(x_{l_n}(a(\boldsymbol{\varepsilon})))-\sum_{i=1}^\infty\varepsilon_i\lambda_i\Big|$$

$$\leqslant\limsup_{n\to\infty}\sum_{i=1}^{j-1}|\varepsilon_i\xi_n(x_{l_n}(a_i))-\varepsilon_i\lambda_i|+\sum_{i=j}^\infty|\lambda_i|+\limsup_{n\to\infty}|\xi_n(x_{l_n}(a^j(\boldsymbol{\varepsilon})))|$$

$$\leqslant\sum_{i=j}^\infty|\lambda_i|+\varepsilon,$$

we obtain

$$\lim_{n\to\infty}\xi_n(x_{l_n}(a(\boldsymbol{\varepsilon})))=\sum_{i=1}^\infty\varepsilon_i\lambda_i$$

for any $\boldsymbol{\varepsilon}\in\boldsymbol{E}$. Hence

$$\sup_n|\xi_n(x_{l_n}(a_i))|\to 0\quad \text{as } i\to\infty,$$

but this contradicts 1°. Thus 4.1(ii) is proved. Suppose that 4.1 (α) is fulfilled. Because of $\bigcup_1^\infty a_i\in\mathcal{E}_0\cap\mathcal{E}^\eta_J$, $\eta(\bigcup_{i\geqslant k}^\infty a_i)\to 0$ as $k\to\infty$, the series

$x_n(a_1)+x_n(a_2)+\ldots$ is convergent. Hence we can select an increasing sequence of indices p_n such that

$$\|x_{k_{p_n}}(a_{p_m})\| \leqslant \varepsilon_0 2^{-m}$$

for $m = n+1, n+2, \ldots$ In virtue of (i) we obtain

$$2\|x_{k_{p_n}}(a_{p_n} \cup a_{p_{n+1}} \cup \ldots)\| \geqslant \varepsilon_0(1-2^{-n}),$$

and, on the other hand,

$$x_{k_{p_n}}(a_{p_n} \cup a_{p_{n+1}} \cup \ldots) \to 0 \quad \text{as } n \to \infty,$$

a contradiction. Thus 4.1(ii) is proved.

4.4. *Let on a ring of sets $\mathscr{F}$ a subadditive measure $\eta(:)$ be defined. The measure $\eta(:)$ satisfies condition (D) on $\mathscr{F}$ if for any e in $\mathscr{F}$ for which $0 < \eta(e) < \infty$ and for arbitrary positive $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1+\varepsilon_2 > 1$, there exist disjoint sets e_1, e_2 belonging to $\mathscr{F}$, $e = e_1 \cup e_2$, with positive η-measure and satisfying the inequalities $\eta(e_1) \leqslant \varepsilon_1\eta(e)$, $\eta(e_2) \leqslant \varepsilon_2\eta(e)$.*

Let the assumption of theorem 4.1 be satisfied, moreover, let $\eta(:)$ satisfy condition (D) on $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$. Under these assumptions $\gamma(e_0) < \infty$ for any $e_0 \in \mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$.

Proof. If $\eta(e_0) = 0$, then $\gamma(e_0) = 0$; so we can assume $\eta(e_0) > 0$. Moreover, $\eta(e_0) < \infty$, for $\eta(:)$ is finite on $\mathscr{E}$. If $\gamma(e_0) = \infty$, then applying condition (D) and the subadditivity of $\gamma(:)$ it is possible to construct a sequence of sets e_n belonging to $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$ such that $e_n\downarrow$, $\eta(e_n) \leqslant (2/3)^n\eta(e_0)$, $\gamma(e_n) = \infty$ for $n = 1, 2, \ldots$ But, by the weak absolute continuity of $\gamma(:)$, $\gamma(e_n) \to 0$ for $n \to \infty$ and we have got a contradiction.

Corollary. 1) If $\eta(:)$ satisfies condition (D) on $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$, $x_n(:)$ are w.a.c. with respect to $\eta(:)$ for $n = 1, 2, \ldots$, then by 4.4 (notice that $v_n(e) = 0$, if $\eta(e) = 0$) $v_n(e_0) < \infty$ for $n = 1, 2, \ldots$

Consequently, in this case we can replace $\gamma(e_0) < \infty$ in 4.4 by $\sup v_n(e_0) < \infty$.

2) A finite σ-additive atomless measure $\eta(:)$ defined on a σ-algebra $\mathscr{F}$ satisfies condition (D) on $\mathscr{F}$.

Examples of additive measures on $\mathscr{F}$ which satisfy condition (D) on $\mathscr{F}$ can be found in [5].

4.5. *Suppose $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$ is an algebra of subsets. If $\mu_n(:)$ is a real-valued set function, additive on $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$, w.a.c. with respect to $\eta(:)$ on $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$, for $n = 1, 2, \ldots$, $\mu_n(e) \to \mu(e)$ as $n \to \infty$ for any $e \in \mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$, then $\mu(:)$ is w.a.c with respect to $\eta(:)$. If in addition $\limsup\limits_{n\to\infty} v(E, \mu_n)$ is finite, then $\mu(:)$ is a.c. with respect to $\eta(:)$.*

This immediately follows from 4.1 and 3.

5. *Let X be a real Banach space which is sequentially weakly compact with respect to a set of functionals Ξ_0. Let $x(:)\colon \mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta} \to X$ be a vector-valued additive set function and let, for any $\xi \in \Xi$, $\xi(x(:))$ be w.a.c. with respect to $\eta(:)$ on $\mathscr{E}_0 \cap \mathscr{E}_{\eta}^{J}$. Then $x(:)$ is w.a.c. with respect to $\eta(:)$ on $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$.*

Proof. Let $e_n\downarrow$, $\eta(e_n) \to 0$ as $n \to \infty$. We have to show that $x(e_n) \to 0$ as $n \to \infty$. If this is not so, there is an increasing sequence of indices k_n such that $\|x(e_{k_n})\| \geqslant \varepsilon_0 > 0$. Hence for a certain positive $\bar{\varepsilon}$ and a $\xi_n \in \Xi_0$

$$|\xi_n(x(e_{k_n}))| \geqslant \bar{\varepsilon} \quad \text{for } n = 1, 2, \ldots$$

X being sequentially weakly compact with respect to Ξ_0 we can assume that the limit

$$\lim_{n\to\infty} \xi_n(x(e)) = \mu(e)$$

exists for any $e \in \mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$. But $\xi_n(x(:))$ is w.a.c. with respect to $\eta(:)$; hence the variation of $\xi_n(x(:))$ equals zero for any set in $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$ of η-measure 0. Thus we can apply 4.1 to $\xi_n(x(:))$ and consequently $\xi_n(x(e_{k_n})) \to 0$ as $n \to \infty$, which contradicts $|\xi_n(x(e_{k_n}))| \geqslant \bar{\varepsilon}$ for $n = 1, 2, \ldots$

5.1. *Theorem in section 5 remains valid if X is arbitrary Banach space and $\Xi_0 = \{\xi\colon \|\xi\| \leqslant 1, \xi \in \Xi\}$.*

Let $e_n\downarrow$, $\eta(e_n) \to 0$ as $n \to \infty$. We proceed analogously as in 4.1 with the aim to prove

(i) $$\lim_{p,q\to\infty} v(e_p - e_q) = 0.$$

If relation (i) is not satisfied, then there is an $\varepsilon_0 > 0$, an increasing sequence of indices p_i, disjoint sets a_i in $\mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$, such that $a_i \subset e_{p_i} - e_{p_{i+1}}$ $a(\boldsymbol{\varepsilon}) \in \mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$ for $\boldsymbol{\varepsilon} \in \boldsymbol{E}$, $\eta(a^n(\boldsymbol{\varepsilon})) \to 0$ as $n \to \infty$, $2\|x(a_i)\| \geqslant \varepsilon_0$ for $i = 1, 2, \ldots$ Since $\xi(x(:))$ is w.a.c. with respect to $\eta(:)$, we have by 2.2

$$\xi\big(x(a(\boldsymbol{\varepsilon}))\big) = \varepsilon_1\xi(x(a_1)) + \varepsilon_2\xi(x(a_2)) + \ldots$$

for any $\xi \in \Xi$, $\boldsymbol{\varepsilon} \in \boldsymbol{E}$. This implies that the values $x(a(\boldsymbol{\varepsilon}))$ belong to the linear subspace X_0 spanned by the elements $x(a_i)$. X_0 being separable we can apply the theorem in section 5 to $x(:)$ on $\mathscr{E}_1 = \mathscr{E}_1 \cap \mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$, where $\mathscr{E}_1$ is the σ-ring of sets $a(\boldsymbol{\varepsilon})$, $\boldsymbol{\varepsilon} \in \boldsymbol{E}$. In view of $\eta(a^n(\boldsymbol{\varepsilon})) \to 0$ as $n \to \infty$ we obtain $x(a_n) \to 0$ as $n \to \infty$, a contradiction with $2\|x(a_i)\| \geqslant \varepsilon_0$. Thus (i) is proved.

Let $a = \bigcap\limits_{1}^{\infty} e_n$; since $a \in \mathscr{E}_0 \cap \mathscr{E}_{\mathfrak{J}}^{\eta}$, $\eta(a) = 0$, and the variation of $\xi(x(:))$ is equal to 0 for any set of η-measure zero, we have $v(a, x) = 0$. But

$$\limsup_{n\to\infty}\|x(e_n)\| \leqslant \lim_{n\to\infty} v(e_n - a, x) + v(a, x) \leqslant \lim_{p,q\to\infty} v(e_p - e_q, x) = 0.$$

Corollary. It may be expected that replacing in theorems 5,5.1 the assumption of the weak absolute continuity of $\xi(x(:))$ by the stronger assumption of the absolute continuity of $\xi(x(:))$ we can obtain a stronger result that $x(:)$ is a.c. As shows an example given by Fichtenholz [2] and mention in section 6, this is not true in general if the range of $x(:)$ belongs to an infinitely dimensional Banach space.

6. In this section we shall given some illustrative examples of Jordan-rings together with some comments.

I. Let E be the interval $\langle a, b\rangle$, $\mathscr{I}$ the algebra of subsets of E, elements of which are finite unions of intervals of the form $\langle c, d)$, where $a \leqslant c < d < b$, or $\langle c, b\rangle$, where $a \leqslant c < b$ and the empty set. We choose as $\tilde{\mathscr{E}}$ the algebra of all subsets of E; we define $\nu(e)$ for $e \in \mathscr{I}$ as the sum of lengths of disjoint intervals $e_1, \ldots, e_n \in \mathscr{I}$ whose union is e and we set $\eta(e) = \inf \nu(a)$ for an arbitrary set in $\tilde{\mathscr{E}}$, where $e \subset a$, $a \in \mathscr{I}$. $\mathscr{E}^\eta_\mathscr{J}$ generated by $\mathscr{I}$, $\eta(:)$ in $\tilde{\mathscr{E}}$ belongs to the class $\mathscr{S}$ and is the algebra of subsets of $\langle a, b\rangle$ measurable in the classical Jordan-sense and $\eta(e)$ is the Jordan measure of e for $e \in \mathscr{E}^\eta_\mathscr{J}$. Theorems 4.1 and 4.2 may be applied in some problems of convergence of sequences of quadrature formulae, integrals (cf. e.g. [2], [6], [7]) involving set functions on the algebra of Jordan-measurable sets. Fichtenholz [2] has given an example of real-valued set functions $\mu_n(:)$ defined on $\mathscr{E}^\eta_\mathscr{J}$ with the following properties: $\mu_n(:)$ are additive, non-negative, and a.c. with respect to $\eta(:)$, $\mu_n(e) \to \mu(e)$ as $n \to \infty$ if $e \in \mathscr{E}^\eta_\mathscr{J}$, for a certain sequence $e_n \in \mathscr{E}^\eta_\mathscr{J}$, $e_n \downarrow$, $\eta(e_n) \to 0$ as $n \to \infty$ $\limsup\limits_{n\to\infty} \mu_n(e_n) > 0$. Since $\sup\limits_n \mu_n(E) < \infty$, $\mu(:)$ is a.c. with respect to $\eta(:)$ (consequently extensable to an absolutely continuous additive set function on the algebra of Lebesgue-measurable sets). Assume $X = c_0$ = the space of sequences converging to 0, and define $x(:): \mathscr{E}^\eta_\mathscr{J} \to X$ setting $x(e) = \{\mu_n(e)\}$. For any $\xi \in \Xi$, $\xi(x(:))$ is a.c. with respect to $\eta(:)$ on $\mathscr{E}^\eta_\mathscr{J}$; $x(:)$ is w.a.c. with respect to $\eta(:)$, nevertheless $x(:)$ is not a.c. with respect to $\eta(:)$.

II. Let E be the set of natural numbers, $\tilde{\mathscr{E}}$ the algebra of all subsets of E and $\mathscr{F}$ the ring of finite subsets of E. Let $\varepsilon = \{\varepsilon_i\} \in E$ denote the characteristic function of a set $e \in \tilde{\mathscr{E}}$. Let

$$\eta(e) = \sup_n (\lambda_{n1}\varepsilon_1 + \lambda_{n2}\varepsilon_2 + \ldots),$$

where

$$\lambda_{ni} > 0, \quad \sum_{i=1}^{\infty} \lambda_{ni} \leqslant c \quad \text{for } n = 1, 2, \ldots$$

$\lambda_{ni} \to 0$ as $n \to \infty$ for $i = 1, 2, \ldots$ A set e belongs to the ring $\mathscr{E}^\eta_\mathscr{J}$ generated by $\mathscr{F}$ and $\eta(:)$ in $\tilde{\mathscr{E}}$ if and only if

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \lambda_{ni}\varepsilon_i = 0.$$

The measure $\eta(:)$ fulfills on $\mathscr{E}^\eta_\mathscr{J}$ the axiom of completeness (c) given in 3.1. Set, for $n = 1, 2, \ldots$, $\lambda_{ni} = 1/n$ if $1 \leqslant i \leqslant n$, $\lambda_{ni} = 0$ if $i > n$. Then $\eta(:)$ satisfies condition (D) on $\mathscr{E}^\eta_\mathscr{J}$ (this is also true under some others assumptions on λ_{ni}). Any real-valued set function $\mu(:)$, additive and w.a.c. on $\mathscr{E}^\eta_\mathscr{J}$ is of the form $\mu(e) = \sum c_i\varepsilon_i$, where $\sum\limits_1^\infty |c_i| < \infty$. As easily seen

$$v(e, x) = \sum_{i=1}^{\infty} |c_i|\,\varepsilon_i$$

(cf. [4], [5]).

References

[1] A. Alexiewicz, *On sequences of operations I*, Studia Math. 11 (1950), p. 1-30.

[2] G. Fichtenholz, *Sur les suites convergentes des intégrales*, Bull. Ac. Pol. Sci. (1924), p. 91-117.

[3] W. Orlicz, *Linear operations in Saks spaces I*, Studia Math. 11 (1950), p. 237-272.

[4] — *O szeregach doskonale zbieżnych w pewnych przestrzeniach funkcyjnych* (in Polish), Prace Matematyczne 1 (1955), p. 393-414.

[5] — *On some classes of modular spaces*, Studia Math. 26 (1966), p. 165-192.

[6] — *On some classes of finitely additive set functions*, Prace Matematyczne 11 (1968), p. 317-327.

[7] G. Pólya, *Über die Konvergenz von Quadraturverfahren*, Math. Zeitschrift 37 (1933), p. 264-286.

Reçu par la Rédaction le 9. 12. 1967