The subspace $M$ is doubly invariant. Indeed, if

$$m = \sum_{a} T_{\alpha} n_{\alpha}$$

with $n_{\alpha} \in N_{\alpha}$,

we infer by (3.6) that

$$T_{\alpha} m = \sum_{a} T_{\alpha} n_{\alpha} \in M, \quad T_{\alpha}^{*} m = \sum_{a} T_{\alpha}^{*} n_{\alpha} \in M.$$ 

Since $M \cap N = N_{\alpha} \neq \{0\}$, we obtain $M = M$ and thus $N = N_{\alpha}$ is a 1-dimensional space. This completes the proof of the theorem.

**Corollary (Wermer).** An isometric non-unitary operator $T$ on a Hilbert space $H$ is unitarily equivalent to the translation operator on $H^2$ if and only if $T$ has no proper doubly invariant subspaces.

**References**


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**STUDIA MATHEMATICA, Y. XXX. (1968)**

**On multipliers preserving convergence of trigonometric series almost everywhere**

by

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1. Consider a trigonometric series $\sum_{n=1}^{\infty} c_n e^{i\alpha n}$, which in the case $c_n = x_n$ can also be written in the form

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n \theta + b_n \sin n \theta) = \sum_{n=0}^{\infty} A_n(\theta),$$

say. Its conjugate is

$$\sum_{n=1}^{\infty} (a_n \cos n \theta - b_n \sin n \theta) = \sum_{n=0}^{\infty} B_n(\theta),$$

say (with $B_0 = 0$).

One of the topics of the theory of trigonometric series that enjoyed popularity a few decades ago was the problem of the behavior (convergence or summability, at individual points or almost everywhere) of the series

$$\sum_{n=0}^{\infty} A_n(\theta) e^{i\alpha n}, \sum_{n=0}^{\infty} B_n(\theta) e^{i\alpha n},$$

where $\alpha$ is a constant. The problem has obvious connections with differentiability or integrability (in general, of fractional order) of functions, and $\alpha$ was almost exclusively real. In this note we consider complex values of $\alpha$, $\alpha = \beta + i\gamma$, but in view of the fact that the case of real $\alpha$ has been exhaustively dealt with we limit ourselves to a purely imaginary, $\alpha = i\gamma$, which shows some novel features. The problem we are discussing here arose out of some concrete applications but the latter are not considered here.

The main result of the paper is the following

**Theorem.** If the series $\sum_{n=0}^{\infty} A_n(\theta)$ is summable $(C, k)$, $k > -1$, at each point of a set $E$ of positive measure, then the series $\sum_{n=0}^{\infty} A_n(\theta) e^{i\alpha n}$ is summable $(C, k)$ almost everywhere in $E$. In particular, the convergence of $\sum_{n=0}^{\infty} A_n(\theta)$ in $E$ implies the convergence of $\sum_{n=0}^{\infty} A_n(\theta) e^{i\alpha n}$ almost everywhere in $E$. 

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Here γ is a real number distinct from 0. Without loss of generality we may assume, wherever needed, that \(a_0 = 0\), in which case successive termwise integrations of the series \(\sum A_n(\theta)\) lead again to trigonometric series. We also assume that the \(a_n^k\) and \(b_n^k\) are real numbers.

2. The proof which follows systematically uses the notion of differentiability in the metric \(L^p\), \(1 < p < \infty\), and we recall the definition (for more details see [1]). We say that the function \(f(\theta)\), defined almost everywhere in the neighborhood of the point \(\theta_0\), has an \(m\)-th differential at \(\theta_0\), in \(L^p\), if there is a polynomial

\[
P(t) = \sum_{j=0}^{m} a_j t^j
\]

of degree \(\leq m\) such that

\[
\lim_{h \to 0} \frac{1}{h^m} \int_{\theta_0}^{\theta_0+h} \left[ f(\theta_0 + t) - P(t) \right]^2 dt = 0
\]

(2.1)

The polynomial \(P(t)\) is called the \(m\)-th differential (in \(L^p\)) of \(f\) at \(\theta_0\), and the numbers \(a_0, a_1, \ldots, a_m\) are the \(m\)-th derivatives (in \(L^p\)) of \(f\) at \(\theta_0\). The differentials in \(L^p\) have a number of properties missing in the classical case \(p = \infty\) (when \(f(\theta_0 + t) = f(\theta_0) + o(t^m)\)) and for this reason are both interesting and useful to consider.

The proof of our theorem is based on a few lemmas which we now state.

**Lemma 1.** If a trigonometric series \(\sum A_n(\theta)\) is summable \((C, k)\), \(k = 0, 1, \ldots\), in \(E\) at \(\theta_0\), and the numbers \(a_0, a_1, \ldots, a_m\) are the \(m\)-th derivatives (in \(L^p\)) of \(f\) at \(\theta_0\). The differentials in \(L^p\) have a number of properties missing in the classical case \(p = \infty\) (when \(f(\theta_0 + t) = f(\theta_0) + o(t^m)\)) and for this reason are both interesting and useful to consider.

This lemma is known (see [4]) and we take it for granted here.

**Lemma 2.** Suppose that a trigonometric series \(\sum A_n(\theta)\) is the Fourier series of a function \(F(\theta)\), \(1 < p < \infty\), and that \(F\) has an \(m\)-th derivative in \(L^p\) at the point \(\theta_0\), equal to \(s\). Then the series obtained by differentiating \(\sum A_n(\theta)\) termwise \(m\) times is summable \((C, m+2)\) at \(\theta_0\), to \(s\).

This lemma holds even with \((C, m+2)\) replaced by \((C, m+c), c > 0\), but the index of summability is of no importance and, in the form stated, the lemma is a simple corollary of known results.

For, in the first place, since differentiability in \(L^p\) clearly implies differentiability in \(L^{p1}\) if \(p_1 < p\), we may assume that \(p = 1\). Thus we have (2.1) with \(f\) replaced by \(F\) and \(p = 1\). Omitting the sign of absolute value we see that the indefinite integral \(G\) of \(F\) has an \((m+1)\)-th derivative at \(\theta_0\) equal to \(s\), in the classical sense. But then, by a very well known result (see [6], p. 99) the series obtained by differentiating the Fourier series of \(G\) termwise \(m+1\) times — or what is the same thing differentiating the Fourier series of \(F\) termwise \(m\) times — is summable \((C, m+1)\) at \(\theta_0\), to \(s\), and this is our lemma 2.

**Lemma 3.** If \(F(\theta) = \sum A_n(\theta)\), \(1 < p < \infty\), then \(\sum A_n(\theta)\) is the Fourier series of a function \(F(\theta)\) which is also in \(L^p\). Moreover, if \(F\) has an \(m\)-th derivative in \(L^p\) at every point of a set \(E\) of positive measure, then \(F\) has an \(m\)-th derivative in \(L^p\) almost everywhere in \(E\).

The first part of this lemma is well known (see [6], p. 99) and another proof is contained implicitly in [3]). To the proof of the second part we return in the next section. Here we only observe that we require \(p\) to be strictly greater than 1.

**Lemma 4.** If a \((C, k)\) series \(\sum_{n=0}^{\infty} a_n\) is summable \((C, k)\), \(k > -1\), and the series \(\sum_{n=0}^{\infty} a_n n^m\) (\(\gamma\) real) is Abel summable, then it is also summable \((C, k)\).

The proof of lemma 3 is given in §§ 3, 4 below. That of lemma 4 is briefly discussed in § 5. We shall now deduce our theorem from the lemmas above.

Suppose that \(\sum A_n(\theta)\), with \(a_0 = 0\), is summable \((C, k)\), \(k > -1\), at each point of a set \(E\), \(|E| > 0\); in particular, it is summable \((C, k')\), where \(k'\) is the least integer \(\geq k\). By lemma 1, the sum \(\sum \gamma(\theta)\) of the series obtained by integrating \(\sum A_n(\theta)\) termwise \(k+1\) times has almost everywhere in \(E\) a \((k+1)\)-st derivative, in the metric \(L^p\), \(p < \infty\). Suppose, e.g., that \(k' + 1 > 1\), so that

\[
G(\theta) = (-1)^{k+2} \sum A_n(\theta) n^{k+1}
\]

By lemma 3, the function

\[
\tilde{G}(\theta) = (-1)^{k+2} \sum A_n(\theta) n^{k+1} \gamma
\]

has a \((k+1)\)-st derivative in \(L^p\), \(1 < p < \infty\), at almost all points of \(E\). By lemma 2, the last series differentiated termwise \(k' + 1\) times, that is the series \(\sum A_n(\theta) n^{r}\), is summable \((C, k' + 3)\) almost everywhere in \(E\); in particular, it is Abel summable almost everywhere in \(E\). Finally, by lemma 4, at each point where \(\sum A_n(\theta) n^{r}\) is Abel summable, and so almost everywhere in \(E\), it is summable \((C, k)\). This completes the proof of the theorem provided we supply the proofs of lemmas 3 and 4.

3. In this and next sections we prove lemmas 3 and 4, which is of independent interest, for general \(1 < p < \infty\). It should however be observed that for the proof of theorem 1 we need only some fixed \(p\), e.g. \(p = 2\), in which case the fact that \(\sum A_n(\theta) n^{r}\) is, like \(\sum A_n(\theta)\), in \(L^p\) is obvious.
The proof of lemma 3 resembles that of the fact that if a function has an m-th derivative in $L^p$, $1 < p < \infty$, at each point of a set $F$, then its Hilbert transform (conjugate function) has the same property almost everywhere in $E$ (see [1], theorems 6, 7). The main idea is in both cases the same but unfortunately there is enough formal difference, of a not completely trivial nature, not to leave the reader to take care of the required changes. Thus we must go through certain details of computation.

Let $C_n^m$ be the Cesàro numbers defined by the generating function

$$\sum_{\alpha=0}^{\infty} C_n^\alpha z^{\alpha} = (1 - z)^{-1 - \alpha}.$$  

We have $C_n^\alpha \sim n^{\alpha / (1 + 4)}$ $(n \to \infty)$, and in particular

$$C_n^{m} \sim n^{m / (1 + 4)}.$$

We first prove lemma 3 with factors $n^k$ replaced by $C_n^m$, and we shall see later that this easily leads to lemma 3 as stated.

Suppose therefore that $F \sim \sum A_n(\theta) P_n$, $1 < p < \infty$, that $F$ has an m-th derivative in $L^p$ at each point of a set $E$; $|E| > 0$. It is known (see [1], theorems 9, 10) that given any $\varepsilon > 0$ we can find a closed set $P \in E$, $|E - P| < \varepsilon$, and a decomposition

$$P = G + H,$$

where $G$ and $H$ are again periodic, $G$ is in $C(m)$ and $G(\theta), \theta = j$, $j = 0, 1, \ldots, m$, coincides with the j-th derivative of $F$ in $L^p$; thus the derivatives of order $j = 0, 1, \ldots, m$ of $H$ are all 0 on $P$. Moreover,

$$\int_{\theta + P} |H(\theta + t)|^m dt < \varepsilon$$

with $\varepsilon$ independent of $\theta$ and

$$\int_{\theta + P} |H(\theta + t)|^m dt = o(n^m)$$

uniformly in $\theta \in P$. In particular, also

$$h^{-1} \int_{\theta} H(\theta + t) dt = o(h^m) \quad \theta \to 0 \theta \in P.$$

Basic for the proof of theorem 1 is also the fact (see [1], theorem 16) that

$$\frac{1}{h} \int_{\theta} H(\theta + t) dt < \infty$$

for almost all $\theta \in P$.

If we write

$$\tilde{F}(\theta) \sim \sum_{n=1}^{\infty} A_n(\theta) C_n^{m},$$

then $\tilde{F} = \tilde{G} + \tilde{H}$. Since the multipliers $C_n^{m}$ preserve the class $L^p$, $1 < p < \infty$ (see [21], p. 232, theorem 4.14), it is clear that $\tilde{G}(\theta)$ has almost everywhere an m-th derivative in $L^p$ and it is enough to prove that $\tilde{H}(\theta)$ has an m-th derivative in $L^p$ almost everywhere in $P$.

Let

$$K(r, t) = \left( 1 + \sum_{n=1}^{\infty} C_n^{m} r^n \cos nt \right) = \frac{1}{2} \left( \frac{1}{1 - re^{i\theta}} + \frac{1}{1 - re^{-i\theta}} \right).$$

Clearly, for almost all $\theta$, $\tilde{H}(\theta)$ is the limit of the Abel means of its Fourier series, i.e.,

$$\tilde{H}(\theta) = \lim_{n \to \infty} \frac{1}{\pi} \int_{-\pi}^{\pi} H(\theta + t) K(r, t) dt$$

$$= \lim_{n \to \infty} \frac{1}{\pi} \int_{-\pi}^{\pi} [H(\theta + t) - \tilde{H}(\theta)] K(r, t) dt + \tilde{H}(\theta).$$

On the other hand, it is well known (see [2]) that if $f(x) \in L^p(-\infty, +\infty)$, $1 < p < \infty$, then the expression

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x + t) - f(x)}{t^{(p - 1)/2}} dt + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x + t)}{t^{(p - 1)/2}} dt = \lim_{n \to \infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \tilde{H}(\theta)$$

exists almost everywhere (clearly, it is only the existence of $\int$ that requires proof; the integral $\int$ converges absolutely and uniformly); moreover, if $1 < p < \infty$, we have

$$\|g\|_p \leq A_p \|f\|_p,$$

where $A_p$ depends on $p$ only. All these facts remain essentially unchanged if in the definition of $g(x)$ instead of the decomposition $\int_{-\pi}^{\pi}$ we use $\int_{-\infty}^{\infty}$, $0 < a < \infty$, the constant $A_p$ in (3.5) remaining the same (this follows by a change of variable). Also, each integral $\int_{-\pi}^{\pi}$ and $\int_{-\infty}^{\infty}$ satisfies an inequality analogous to (3.5).
Using these facts it is easy to deduce from (3.3) that for periodic functions \( H(\theta) \) that are merely integrable we have

\[
\hat{H}(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ H(\theta + t) - H(\theta) \right] K(1, t) \, dt + H(\theta),
\]

where

\[
K(1, t) = \lim_{\epsilon \to 0} K(r, t) = \frac{1}{2} \left( \frac{1}{1 - e^{it}} \right)^{1-\| \theta \|} + \frac{1}{1 - e^{-it}} \left( \frac{1}{1 - e^{it}} \right)^{1-\| \theta \|}
\]

and if \( H \in L^p \), \( 1 < p < \infty \), then

\[
\|\hat{H}\|_p \leq A_p \|H\|_p,
\]

where the norms are over the interval \((0, 2\pi)\).

For example, in order to deduce (3.6) from (3.3) it is enough to verify that \( |K(r', t)| \leq A_0 \), where \( R = 1 - \rho^2 \) and that \( |K(r, t) - K(1, t)| \leq \rho \| \theta \| \rho^2 \) for \( \rho < \| \theta \| < \pi \), so that the expression equal to \( K(r, t) \) for \( |r| = \rho \) and equal to \( K(r, t) - K(1, t) \) for \( \rho < |r| < \pi \) is majorized by a fixed multiple of the Poisson kernel.

We also add that since \( K(t) \) is infinitely differentiable for \( t \neq 0 \), the second integral (3.6) shows that we do not affect the differentiability properties of \( H \) at the point \( \theta \) if we modify \( H \) away from \( \theta \).

4. If \( \varphi(z) \) denotes the function \( (1 - e^{iz})/(iz)^{1-\| \theta \|} \), regular for \( |z| < 2\pi \), we have

\[
K(1, t) = \frac{1}{2} \left[ \frac{\varphi(t)}{(it)^{1-\| \theta \|}} + \frac{\varphi(-t)}{(-it)^{1-\| \theta \|}} \right],
\]

and taking a sufficiently large number of terms of the Taylor series of \( \varphi \) we easily see that the problem reduces to showing that the functions

\[
\int_{-\pi}^{\pi} \left[ H(t) - H(\theta) \right] \frac{(1 - e^{it})^k}{|t|^{1+\| \theta \|}} \, dt,
\]

have \( m \)-th derivatives in \( L^p \) at almost all points of \( P_1 \); the exponent \( k \) here takes a finite number of values \( 0, 1, 2, \ldots \). We may restrict ourselves to the first integral (4.1). We shall discuss only the case \( k = 0 \) which is, in some sense, the most subtle (it corresponds to the constant term of the Taylor development of \( \varphi \)); for other values of \( k \) the proof is parallel.

Thus we will prove that the function

\[
S(\theta) = \int_{-\pi}^{\pi} \left[ H(t) - H(\theta) \right] \frac{dt}{(t-\theta)^{1+\| \theta \|}}
\]

has an \( m \)-th derivative in \( L^p \) at almost all points of \( P \).

Let us consider any point \( \theta \in P \) at which the integral (3.2) is finite. We shall prove that \( S \) has an \( m \)-th derivative in \( L^p \) at \( \theta \). Assume, as we may, that \( \theta_0 = 0 \). We fix a small \( h > 0 \) and we consider the function \( S(\theta) \) in the interval \((-h, h)\).

We write

\[
S(\theta) = \int_{-h}^{h} \frac{H(t) - H(\theta)}{(t-\theta)^{1+\| \theta \|}} \, dt + \int_{-h}^{h} \frac{H(t)}{(t-\theta)^{1+\| \theta \|}} \, dt + \int_{-h}^{h} \frac{H(\theta)}{(t-\theta)^{1+\| \theta \|}} \, dt
\]

say. Since, for \( |\theta| \leq h \),

\[
S_1(\theta) = \int_{-h}^{h} \frac{H(t + \theta) - H(\theta)}{t^{1+\| \theta \|}} \, dt = \int_{-h}^{h} \frac{H(t)}{(t+\theta)^{1+\| \theta \|}} \, dt + \int_{-h}^{h} \frac{H(\theta)}{(t+\theta)^{1+\| \theta \|}} \, dt,
\]

where \( H(t) = H(t) \) for \( |t| \leq \theta \), \( H_1(t) = 0 \) elsewhere, an application of (3.5) (and the remarks that follow it) shows that

\[
\left( h^{-1} \int_{-h}^{h} |H(\theta)| \right)^{1/p} \leq \left( h^{-1} \int_{-h}^{h} |S_1(\theta)| \right)^{1/p} \leq A_p \left( h^{-1} \int_{-h}^{h} |H(\theta)| \right)^{1/p}
\]

\[
\leq A_p \left( h^{-1} \int_{-h}^{h} |H(\theta)| \right)^{1/p} = o(h^m),
\]

by (3.1). Also, since \( |S_1(\theta)| \leq A_p |H(\theta)| \),

\[
\left( h^{-1} \int_{-h}^{h} |S_1(\theta)| \right)^{1/p} = o(h^m).
\]

Now observe that \( 2h \leq \theta + 3h \leq 4h \), so that

\[
S_2 = \int_{-2h}^{2h} \frac{H(t)}{(t-\theta)^{1+\| \theta \|}} \, dt - \int_{-2h}^{2h} \frac{H(t)}{(t+\theta)^{1+\| \theta \|}} \, dt = S_{21} + S_{22},
\]

Clearly,

\[
|S_{21}| \leq h^{-1} \int_{-h}^{h} |H(t)| \, dt = o(h^m),
\]

Thus we will prove that the function

\[
S(\theta) = \int_{-\pi}^{\pi} \left[ H(t) - H(\theta) \right] \frac{dt}{(t-\theta)^{1+\| \theta \|}}
\]

has an \( m \)-th derivative in \( L^p \) at almost all points of \( P \).

Let us consider any point \( \theta_0 \in P \) at which the integral (3.2) is finite. We shall prove that \( S \) has an \( m \)-th derivative in \( L^p \) at \( \theta \). Assume, as we may, that \( \theta_0 = 0 \). We fix a small \( h > 0 \) and we consider the function \( S(\theta) \) in the interval \((-h, h)\).

We write

\[
S(\theta) = \int_{-h}^{h} \frac{H(t) - H(\theta)}{(t-\theta)^{1+\| \theta \|}} \, dt + \int_{-h}^{h} \frac{H(t)}{(t-\theta)^{1+\| \theta \|}} \, dt + \int_{-h}^{h} \frac{H(\theta)}{(t-\theta)^{1+\| \theta \|}} \, dt
\]

say. Since, for \( |\theta| \leq h \),

\[
S_1(\theta) = \int_{-h}^{h} \frac{H(t + \theta) - H(\theta)}{t^{1+\| \theta \|}} \, dt = \int_{-h}^{h} \frac{H(t)}{(t+\theta)^{1+\| \theta \|}} \, dt + \int_{-h}^{h} \frac{H(\theta)}{(t+\theta)^{1+\| \theta \|}} \, dt,
\]

where \( H(t) = H(t) \) for \( |t| \leq \theta \), \( H_1(t) = 0 \) elsewhere, an application of (3.5) (and the remarks that follow it) shows that

\[
\left( h^{-1} \int_{-h}^{h} |H(\theta)| \right)^{1/p} \leq \left( h^{-1} \int_{-h}^{h} |S_1(\theta)| \right)^{1/p} \leq A_p \left( h^{-1} \int_{-h}^{h} |H(\theta)| \right)^{1/p}
\]

\[
\leq A_p \left( h^{-1} \int_{-h}^{h} |H(\theta)| \right)^{1/p} = o(h^m),
\]

by (3.1). Also, since \( |S_1(\theta)| \leq A_p |H(\theta)| \),

\[
\left( h^{-1} \int_{-h}^{h} |S_1(\theta)| \right)^{1/p} = o(h^m).
\]

Now observe that \( 2h \leq \theta + 3h \leq 4h \), so that

\[
S_2 = \int_{-2h}^{2h} \frac{H(t)}{(t-\theta)^{1+\| \theta \|}} \, dt - \int_{-2h}^{2h} \frac{H(t)}{(t+\theta)^{1+\| \theta \|}} \, dt = S_{21} + S_{22},
\]

Clearly,

\[
|S_{21}| \leq h^{-1} \int_{-h}^{h} |H(t)| \, dt = o(h^m),
\]
on the other hand,

\[ S_{+,1}(\theta) = \int_{1}^{n} \frac{H(t)}{t^{1+\nu}} dt \]

\[ = \int_{1}^{n} \frac{H(t)}{t^{1+\nu}} \left[ \sum_{j=0}^{\infty} \frac{\theta^j}{t^j} + O\left(\frac{\theta^{m+1}}{t^{m+1}}\right) \right] dt \]

\[ = \sum_{j=0}^{\infty} \frac{\theta^j}{t^{1+\nu}} \int_{1}^{n} H(t) \, dt + O\left(\frac{\theta^{m+1}}{t^{m+1}} \int_{1}^{n} H(t) \, dt \right). \]

Observe now that, by hypothesis, the integral \( \int_{1}^{n} \frac{|H(t)|}{t^{m+1}} \, dt \) is finite, so that

\[ \int_{1}^{n} |H(t)| t^{-m-2} \, dt = o(h^{m-1}) \]

and the last term on the right of \( o(h^{m}) \), also, if \( j = 0, 1, \ldots, m \),

\[ \int_{1}^{n} \frac{H(t)}{t^{j+\nu}} \, dt = \int_{1}^{n} \frac{F(t)}{t^{j+\nu}} \, dt + O(h^{m-j}) \]

as easily seen by integration by parts and applying (3.1). Thus \( S_{+,1}(\theta) \) is a fixed polynomial of degree \( \leq m \) in \( \theta \) plus an error term \( o(h^{m}) \). Collecting results we obtain that \( S(\theta) \) does indeed have an \( m \)-th derivative in \( L^p \) at \( \theta = 0 \).

We have thus proved lemma 3 with factors \( n^{\nu} \) replaced by \( C_{v}^{m} \). But the result holds also for the factors \( C_{v}^{m-j} \), where \( s \) is any positive integer. For the new generating function is

\[ \sum_{s=0}^{\infty} C_{v}^{m-s} z^{s} = \frac{(1-z)^{j}}{(1-z)^{j+1}} \]

and it is clear that the argument above holds for kernels \( \delta^{j}(1-z)^{i+j} \), \( j = 0, 1, 2, \ldots \). Since the asymptotic formula

\[ C_{v}^{m} = n^{\nu} (\lambda_{0} + \lambda_{1} n^{-1} + \ldots + \lambda_{j} n^{-j} + O(n^{-j-1})); \]

leads to

\[ n^{\nu} = \mu_{0} C_{v}^{m} + \mu_{1} C_{v}^{m-1} + \ldots + \mu_{j} C_{v}^{m-j} + O(n^{-j-1}); \]

taking \( s \) sufficiently large we are easily led to lemma 3 as stated.

5. It remains to consider lemma 4. We do not give a proof here since it is implicitly contained in paper [5] (Sate 2, p. 317). It is shown there that if the Cesàro means \( \sigma_{n}^{m} \) of a series \( \sum \frac{X_n}{t^{j}} \) satisfy the condition \( \sigma_{n}^{m} = o(m) \), where \( m \) is a sequence monotonically tending to \( +\infty \) and such that

\[ \frac{f'}{\mu_{j}} = O\left(\frac{1}{n^{j} \mu_{n}} \right) (j = 0, 1, \ldots, k), \]

\( k' \) being the integer \( \geq k \), then \( \sum \mu_{n} / \mu_{n} \) is either summable \((C, k)\) or else is not Abel summable. Actually the proof without any change gives the following result: if \( \{1/\mu_{n}\} \) is a bounded sequence satisfying \( \sigma^{j}(1/\mu_{n}) = O(n^{-\nu}) \) for \( j = 0, 1, \ldots, k' \) and \( \sum \mu_{n} \) is summable \((C, k)\), then \( \sum \mu_{n}/\mu_{n} \) is also summable \((C, k)\), provided it is Abel summable.

The result clearly applies in the case when \( \mu_{n} = n^{-\nu} (n > 0) \).

Since the case \( k = 0 \) of theorem 1 is of special interest, it may be worth pointing out that in this case lemma 4 is immediate. For it is very well known that if a series \( \sum \frac{X_n}{n} \) is Abel summable and

\[ \sum_{n}^{\infty} X_{n} = o(n) \]

then \( \sum X_{n} \) converges. But (5.1) is immediate, by summation by parts, if \( \sum \mu_{n} = o(n^{\nu}) \) and \( \sum \mu_{n} \) converges.

6. The theorem of this paper has connection with some recent results of E. M. Stein. As he pointed out to us, the methods of his paper [3] give the following theorem:

If a function

\[ F(x) = \sum_{n}^{\infty} c_{n} x^{n} \]

analytic in \(|x| < 1\) has a non-tangential limit at each point of a set \( E \) situated on \(|x| = 1\), then the function

\[ G(x) = \sum_{n}^{\infty} c_{n} x^{n} \]

has a non-tangential limit almost everywhere in \( E \).

References

Absolute continuity of vector-valued finitely additive set functions, I

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1. Throughout this paper we shall use the following notations and notions. \( X \) denotes a real Banach space and \( \mathbb{E} \) its conjugate space, \( \varepsilon \) stands for a functional from \( \mathbb{E} \). By \( \mathbb{E}_b \) we denote a set of functionals from \( \mathbb{E} \) such that \( \|\varepsilon\| \leq 1 \) and \( \sup \{\varepsilon(x) : x \in \mathbb{E} \} \geq \varepsilon(x) \) for some \( \varepsilon > 0 \), where the supremum is taken over all \( \varepsilon \) in \( \mathbb{E}_b \). The sets \( \mathbb{E}_b \) are called fundamental.

\( X \) is called weakly sequentially compact with respect to \( \mathbb{E}_b \) if for every sequence \( \varepsilon_n \), where \( \varepsilon_n \in \mathbb{E}_b \), there exists a subsequence \( \varepsilon_{n_k} \) converging for any \( x \in X \). It is a simple matter to prove that a separable space \( X \) is weakly sequentially compact with respect to \( \mathbb{E}_b = \{\varepsilon : \|\varepsilon\| \leq 1\} \). There exist also non-separable spaces weakly sequentially compact with respect to some \( \mathbb{E}_b \). As an example of such kind of spaces one can take the space \( M_0 \) of real-valued functions \( f(\cdot) \), bounded on \( (a, b) \), having for any \( t \in (a, b) \) the right- and left-hand limit and \( f(a^-, b^-) \).

Here the norm for \( f \in M_0 \) is defined as \( \sup_{t \in (a, b)} |f(t)| \), and for \( \mathbb{E}_b \) one can take the set of all functionals of the form

\[ \xi(\varepsilon) = \pm \int_a^b f(t) \, dt, \]

where \( a \leq u < v \leq b \), \( u = 1, 2, \ldots \). By \( \mathbb{E} \) we will denote an abstract set of points (elements), the symbol \( e_n \) or \( e_n \downarrow \) will stand for a sequence of sets \( e \) in \( \mathbb{E} \) such that \( e_n \subseteq e_{n+1} \subseteq \cdots \), or \( e_n \supseteq e_{n+1} \supseteq \cdots \), respectively.

\( e \downarrow e \) means \( e_n \downarrow e \), \( e = \bigcap_{n=1}^{\infty} e_n \), or \( e_n \downarrow e = \bigcup_{n=1}^{\infty} e_n \) respectively.

Besides the notation given before we shall use throughout the paper the letter \( E \) to denote the class of zero-one sequences \( s = (s_n) \), that is to say with terms \( s_n = 0, 1 \). If \( e \) is a set of points from \( E \), then \( e_n \) means \( e \) if \( e_n = 1 \), and the empty set if \( e_n = 0 \). For a sequence of sets \( e_n \) the symbol \( a(e) \) or \( a^*(e) \) denote the set \( \bigcup_{n=1}^{\infty} e_n \), or \( \bigcup_{n=1}^{\infty} e_n \), respectively. \( \mathcal{E}, \mathcal{F}, \cdots \) always denotes a ring or an algebra of subsets from \( E \). The class \( \mathcal{E} \) is