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On the semi-groups of isometries

by

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Introduction. Let Z be the additive group of integers and Z_+ the semi-group of non-negative integers. Let T be an isometric operator on a Hilbert space H and $\{T_n\}_{n\in Z_+}$ the semi-group of isometries on H defined by $T_n = T^n$ $(n\in Z_+)$. The well-known theorem of Wold says that the space H may be decomposed uniquely in the form

$$H = H_u \oplus H_t$$

in such a way that H_u reduces T_n to a unitary operator for all $n\in Z_+$ and each closed subspace of H_t with this property is equal to $\{0\}$. Moreover, there exists a Hilbert space N such that the restriction of the semigroup $\{T_n\}_{n\in Z_+}$ to the space H_t is unitarily equivalent to the semi-group of unilateral translations in $H^2(N)$.

Similar properties hold true also, in the case of a semigroup of isometries $\{T_s\}_{s\in S}$, where S is a subsemi-group of an abelian group G, such that $S \cap S^{-1} = \{1\}$, with suitable definition of the space $H^2(N; S)$ and of the semi-group of unilateral translations in $H^2(N; S)$.

Unlike the classical case there appear three terms in the decomposition formula of Wold; the unitary part of $\{T_s\}_{s\in S}$ the *totally non-unitary part of* $\{T_s\}_{s\in S}$ which is the *translation part* and once a *part which in the case of semi-group* Z_+ does not appear and which we propose to call the *strange part* of $\{T_s\}_{s\in S}$.

In the section 2 of this paper we shall prove a generalization of the Wold decomposition theorem (Theorem 3).

Using this theorem we shall give in section 3 an intrinsic characterization of the semi-group of unilateral translations in the space $H^2(N; S)$ (Theorem 4). We shall find also a generalization of Wermer's result about double invariant subspaces of translation operator [10] (Theorem 5).

All this facts can be formulated in a function language. In this language, in the case when G is totally ordered by S we find the results of Helson and Lowdeslager [3]. About these and other facts related to [1] and [7] we shall treat in a next paper.

1. The basic notation and definitions. Throughout the present paper G will be an abelian group and S will be a sub semi-group of G such that $S \cap S^{-1} = \{1\}$. Without loss of generality we can suppose that $G = SS^{-1}$.

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H.

A *semi-group of isometries* $\{T_s\}_{s\in S}$ *on* H is a mapping $s \to T_s$ of S in $B(H)$ such that T_s is an isometric operator on H for any $s \in S$, $T_1 = I$ and $T_{s_1}T_{s_2} = T_{s_1 s_2}$ for $s_1, s_2 \in S$.

If $s_1\sigma_1^{-1} = s_2\sigma_2^{-1}$, then we have $s_1\sigma_2 = s_2\sigma_1$ and thus $T_{s_1}T_{\sigma_2} = T_{s_2}T_{\sigma_1}$. Then we have

$$T^*_{\sigma_1}T_{s_1} = T^*_{\sigma_1}T^*_{\sigma_2}T_{\sigma_2}T_{s_1} = T^*_{\sigma_2}T^*_{\sigma_1}T_{\sigma_1}T_{s_2} = T^*_{\sigma_2}T_{s_2}.$$

Hence, if for $g = s\sigma^{-1}$ we put $T_g = T^*_\sigma T_s$, then we obtain a well defined mapping $g \to T_g$ of G in $B(H)$ which extends the semi-group $\{T_s\}_{s\in S}$ and verifies

$$T_{g^{-1}} = T^*_g \qquad (g \in G).$$

Using the Ito dilation theorem (see [5]) for a semi-group of isometries, we conclude that there exist a Hilbert space K which contains H as a subspace and a unitary reprezentation $g \to U_g$ of G on K such that

$$T_g h = PU_g h \qquad (h \in H, g \in G),$$

where P is the orthogonal projection of K on H. The group $\{U_g\}_{g\in G}$ is called the *unitary dilation* of the semi-group $\{T_s\}_{s\in S}$.

A closed subspace M of H is called *invariant under* $\{T_s\}_{s\in S}$ of, shortly, *invariant*, if $T_s M \subset M$ for $s \in S$. A closed subspace M of H is called *doubly invariant* if $T_s M \subset M$ and $T^*_s M \subset M$ for $s \in S$. If M is doubly invariant, then we denote by $T_s|M$ the restriction of T_s to M and by $\{T_s|M\}_{s\in S}$ the corresponding semi-group of isometries on M.

If A is a subset of a Hilbert space H, we denote by clm $[A]$ the closed linear manifold spanned by A in H.

The semi-group $\{T_s\}_{s\in S}$ is called *unitary* if T_s is a unitary operator on H for every $s \in S$.

The semi-group $\{T_s\}_{s\in S}$ is called *completely non-unitary* if for any doubly invariant subspace M for which $\{T_s|M\}_{s\in S}$ is unitary, we have $M = \{0\}$.

The semi-group $\{T_s\}_{s\in S}$ is called *quasi-unitary* if

$$\operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T^*_\sigma T_s H\Big] = H. \tag{1.1}$$

The semi-group $\{T_s\}_{s\in S}$ is called *totally non-unitary* if for any doubly invariant subspace M for which $\{T_s|M\}_{s\in S}$ is quasi-unitary we have $M = \{0\}$.

The semi-group $\{T_s\}_{s\in S}$ is called *strange* if it is completely non-unitary and quasi-unitary.

It is clear that if the semi-group $\{T_s\}_{s\in S}$ is unitary, then it is quasi-unitary, and that if $\{T_s\}_{s\in S}$ is totally non-unitary, then it is completely non-unitary.

If the semi-group $\{T_s\}_{s\in S}$ is quasi-unitary and M is a doubly invariant subspace, then the semi-group $\{T_s|M\}_{s\in S}$ is quasi-unitary.

Indeed, since M is doubly invariant, we have

$$\operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T^*_\sigma T_s M\Big] \subset M.$$

Let now $m \in M$ be orthogonal to $\operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T^*_\sigma T_s M\Big]$. If $h = h_1 + h_2$ with $h_1 \in M$ and $h_2 \in M^\perp$ is an element of H, then we have, for $s\sigma^{-1} \notin S^{-1}$,

$$(m, T^*_\sigma T_s h) = (m, T^*_\sigma T_s h_1) + (m, T^*_\sigma T_s h_2) = (T^*_s T_\sigma m, h_2) = 0,$$

Thus, m is orthogonal to $\operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T^*_\sigma T_s H\Big] = H$, i.e. $m = 0$.

Now, it is clear that if $\{T_s\}_{s\in S}$ *is unitary* (*completely non-unitary, quasi-unitary, totally non-unitary, strange*) *and M is a doubly invarianta then $\{T_s|M\}_{s\in S}$ has the same property.*

If G is the additive group of integers, S the semi-group of non-negative integers and $T_n = T^n$ for all $n \geqslant 0$ where T is an isometric operator on H, then

$$\operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T^*_\sigma T_s H\Big] = TH$$

and thus the quasi-unitary property is the same as the unitary one, the totally non-unitary property is the same as the completely non-unitary one and the strange property does not appear.

If G is the additive group of real numbers, S the semi-group of non-negative real numers and $\{T_s\}_{s\in S}$ is a strongly continuous semi-group of isometries on H, then it is clear that

$$\operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\in S^{-1}} T^*_\sigma T_s H\Big] = H$$

and thus every such semi-group is quasi-unitary.

2. The Wold decomposition. In this section we shall prove het Wold's decomposition theorem in the form announced in Introduction.

THEOREM 1. *Let $\{T_s\}_{s\in S}$ be a semi-group of isometries on a Hilbert space H. The the space H may be decomposed in the form*

$$H = H_u \oplus H_c \tag{2.1}$$

in such a way that H_u *and* H_c *are doubly invariant subspaces,* $\{T_s|H_u\}_{s\epsilon S}$ *is unitary and* $\{T_s|H_c\}_{s\epsilon S}$ *is completely non-unitary. The decomposition is unique and we have*

(2.2) $$H_u = \{h;\, h\epsilon H : \|T_s^* h\| = \|h\| \text{ for all } s\epsilon S\}.$$

Proof. Let $\{U_g\}_{g\epsilon G}$ be the unitary dilation of $\{T_s\}_{s\epsilon S}$ and

(2.3) $$H_u = \{h\colon h\epsilon H,\ U_g h\epsilon H,\ g\epsilon G\}.$$

Then $H = H_u \oplus H_c$ is the desired decomposition and it is unique. The proof of this fact is essentially the same as the proof of Theorem 5.1 from [6] and we omit it.

Let us prove formula (2.2). Let $h\epsilon H_u$. We have

$$\|T_s^* h\| = \|PU_s^* h\| = \|U_s^* h\| = \|h\|$$

for every $s\epsilon S$. Conversely, if $\|T_s^* h\| = \|h\|$ for all $s\epsilon S$, we have

$$\|PU_s h\| = \|T_s h\| = \|h\| = \|U_s h\| \quad\text{and}\quad \|PU_{s^{-1}} h\| = \|T_s^* h\| = \|h\| = \|U_{s^{-1}} h\|$$

for $s\epsilon S$. Thus $U_s h\epsilon H$, $U_{s^{-1}} h\epsilon H$ for $s\epsilon S$. Then, if $g\epsilon G$, $g = s\sigma^{-1}$ we have

$$\begin{aligned}\|PU_g h\| &= \|PU_{s\sigma^{-1}} h\| = \|PU_s U_{\sigma^{-1}} h\| \\ &= \|PU_s PU_{\sigma^{-1}} h\| = \|T_s T_\sigma^* h\| = \|h\| = \|U_g h\|.\end{aligned}$$

Thus $U_g h\epsilon H$ for $g\epsilon G$. Hence $h\epsilon H_u$.

This completes the proof of formula (2.2) and of the theorem.

THEOREM 2. *Let* $\{T_s\}_{s\epsilon S}$ *be a semi group of isometries on* H. *The space* H *may be decomposed uniquely in the form*

(2.4) $$H = H_q \oplus H_t$$

in such a way that H_q *and* H_t *are doubly invariant subspaces,* $\{T_s|H_q\}_{s\epsilon S}$ *is quasi-unitary and* $\{T_s|H_t\}_{s\epsilon S}$ *is totally non-unitary.*

Proof. Let us put

(2.5) $$N = [\bigcup_{\sigma^{-1}s\notin S^{-1}}{}' T_\sigma^* T_s H].$$

For $n\epsilon N$, $h\epsilon H$ and $s\sigma^{-1}\notin S$ we have

$$(T_\sigma^* T_s n, h) = (n, T_s^* T_\sigma h) = 0.$$

Hence

(2.6) $$T_\sigma^* T_s n = 0 \quad (n\epsilon N;\ s\sigma^{-1}\notin S).$$

The closed subspaces $T_s N$ are mutually orthogonal. Indeed, since $S \cap S^{-1} = \{1\}$, we have

$$(T_s n, T_\sigma m) = (n, T_s^* T_\sigma m) = (T_\sigma^* T_s n, m) = 0$$

for $n, m\epsilon N$, $s, \sigma\epsilon S$, $s \neq \sigma$.

Let us put

(2.7) $$H_t = \bigoplus_{s\epsilon S} T_s N$$

and write $H = H_q \oplus H_t$.

The subspaces H_q and H_t are doubly invariant. Indeed, obviously H_t is invariant. If $m\epsilon H_q$ and $s\epsilon S$, then

$$(T_\sigma m, T_s n) = (m, T_\sigma^* T_s n) = 0$$

for all $\sigma\epsilon S$ for which $\sigma^{-1}s\notin S$ and for every $n\epsilon N$ because of (2.6). If $\sigma^{-1}s = s_1\epsilon S$, then $s = \sigma s_1$ and we have

$$(T_\sigma m, T_s n) = (m, T_\sigma^* T_\sigma T_{s_1} n) = (m, T_{s_1} n) = 0$$

for all $n\epsilon N$, because of definition of H_q. Thus $T_\sigma m$ is orthogonal to H_t, i.e. $T_\sigma m\epsilon H_q$. It results that both H_q and H_t are doubly invariant.

The semi-group $\{T_s|H_q\}_{s\epsilon S}$ is quasi-unitary. Indeed, since H_q is doubly invariant, we have

$$\operatorname{clm}[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H_q] \subset H_q.$$

Let $m\epsilon H_q$ be orthogonal to $\operatorname{clm}[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H_q]$ and let $h\epsilon H$ be of the form $h = h_1 + h_2$ with $h_1\epsilon H_q$, $h_2\epsilon H_t$. We have

$$(m, T_\sigma^* T_s h) = (m, T_\sigma^* T_s h_1) + (m, T_\sigma^* T_s h_2) = (T_\sigma m, T_s h_2) = 0$$

for $s, \sigma\epsilon S$, $s\sigma^{-1}\notin S^{-1}$. Thus m is othogonal to

$$\operatorname{clm}[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H] \supset H_q,$$

i.e. $m = 0$. Hence

$$H_q = \operatorname{clm}[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H_q]$$

and thus $\{T_s|H_q\}_{s\epsilon S}$ is quasi-unitary.

Le M be a doubly invariant subspace included in H_t and such that $\{T_s|M\}_{s\epsilon S}$ is quasi-unitary. Since

$$M = \operatorname{clm}[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s M],$$

M is orthogonal to N. Then, for $m\epsilon M$, $n\epsilon N$ and $s\epsilon S$ we have

$$(m, T_s n) = (T_s^* m, n) = 0.$$

Hence M is orthogonal to $H_t \supset M$, i.e. $M = \{0\}$. Thus $\{T_s|H_t\}_{s\epsilon S}$ is totally non-unitary. Let now

$$H = H_1 \oplus H_2$$

be another decomposition of H such that H_1 and H_2 are doubly invariant subspaces, $\{T_s|H_1\}_{s\in S}$ is quasi-unitary and $\{T_s|H_2\}_{s\in S}$ is totally non-unitary. Since

$$H_1 = \operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H_1\Big],$$

we have $H_1 \subset H_q$. Let us put

$$H_q = H_1 \oplus M.$$

Then it is clear that M is a doubly invariant subspace and thus $\{T_s|M\}_{s\in S}$ is quasi-unitary. But $M \subset H_2$, thus $M = \{0\}$ and we conclude that $H_1 = H_q$, $H_2 = H_t$.

This completes the proof of the theorem.

THEOREM 3 (Wold, Helson-Lowdenslager). *Let $\{T_s\}_{s\in S}$ be a semi-group of isometries on a Hilbert space H. The space H may be decomposed uniquely in the form*

$$H = H_u \oplus H_s \oplus H_t \tag{2.8}$$

in such a way that H_u, H_s and H_t are doubly invariant subspaces, $\{T_s|H_u\}_{s\in S}$ is unitary, $\{T_s|H_s\}_{s\in S}$ is strange and $\{T_s|H_t\}_{s\in S}$ is totally non-unitary.

Proof. The decomposition may be obtained by applying successively the preceding theorems. When considering the uniqueness, we remark, for example, that $M = H_u \oplus H_s$ is doubly invariant subspace and $\{T_s|M\}_{s\in S}$ is quasi-unitary.

3. Unilateral translations. Let N be a Hilbert space. Denote by $L^2(N; G)$ the Hilbert space of all families $\tilde{h} = (h_g)$, $g \in G$, with $h_g \in N$, for which

$$\|\tilde{h}\|^2 = \sum_{g\in G} \|h_g\|^2 < \infty,$$

with the scalar product

$$(\tilde{h}, \tilde{k}) = \sum_{g\in G} (h_g, k_g),$$

where $\tilde{h} = (h_g)$, $\tilde{k} = (k_g)$.

By $H^2(N; S)$ we denote the closed subspace of $L^2(N; G)$ of elements of the form $\tilde{h} = (h_g)$ with $h_g = 0$ for $g \notin S$.

If we denote by P the orthogonal projection of $L^2(N; G)$ on $H^2(N; S)$, we have $P(h_g) = (h'_g)$, where $h'_g = h_g$ if $g \in S$ and $h'_g = 0$ if $g \notin S$.

The mapping of N into $H^2(N; S)$ defined by $n \to (n_g)$, where $n = n_1$ and $n_g = 0$ for $g \neq 1$ is one-to-one and isometric so that N may be embeded isometrically in $H^2(N; S)$.

For any $s \in S$ let us define on $H^2(N; S)$ the operator

$$T_s(h_g) = (h_{s^{-1}g}). \tag{3.1}$$

It is easy to see that $\{T_s\}_{s\in S}$ is a semi-group of isometries on $H^2(N; S)$. This semi-group is called the *semi-group of unilateral translations in* $H^2(N; S)$.

Note that if $\{T_s\}_{s\in S}$ is the semi-group of unilateral translations in $H^2(N; S)$, then

$$T_s^*(h_g) = P(h_{sg}) \tag{3.2}$$

for $s \in S$ and

$$H^2(N; S) = \bigoplus_{s\in S} T_s N. \tag{3.3}$$

If G is the group of integers and S the semi-group of non-negative integers, then we write $H^2(N)$ instead of $H^2(N; S)$. If N is one-dimensional space, then we write $H^2(S)$ instead of $H^2(N; S)$ and H^2 instead of $H^2(N)$.

THEOREM 4. *Let $\{T_s\}_{s\in S}$ be a semi-group of isometries on the Hilbert space H. The following conditions are equivalent;*

(a) *There exists a Hilbert space N such that the semi-group $\{T_s\}_{s\in S}$ is unitarily equivalent to the semi-group of unilateral translations in $H^2(N; S)$.*

(b) *The semi-group $\{T_s\}_{s\in S}$ is totally non-unitary.*

(c) *If M is doubly invariant subspace of H such that*

$$M \subset \operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H\Big],$$

hen $M = \{0\}$.

(d) *If M is doubly invariant subspace of H such that*

$$M \cap \Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H\Big]^\perp = \{0\},$$

then $M = \{0\}$.

Proof. The implications (d) → (c), (c) → (b) are obvious. If $\{T_s\}_{s\in S}$ is totally non-unitary, then by Theorem 2 we have

$$H = H_t = \bigoplus_{s\in S} T_s N, \tag{3.4}$$

where

$$N = \Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H\Big]^\perp.$$

Now, it is clear that H is unitarily isomorphic to $H^2(N; S)$ and the semi-group $\{T_s\}_{s\in S}$ is unitarily equivalent to the semi-group of unilateral translations in $H^2(N; S)$. Thus the implication (b) → (a) is proved.

Let us prove now the implication (a) → (c). Let N be a Hilbert space and $\{T_s\}_{s\in S}$ the semi-group of unilateral translations in $H^2(N; S)$.

Firstly we shall prove relation

$$N = \Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H^2(N; S)\Big]^\perp, \tag{3.5}$$

where N is embeded in $H^2(N;S)$ as above. For this purpose let $n\in N$, $n=(n_g)$ with $n_g=0$ for $g\neq 1$. Using (3.2) we have

$$T_\sigma^* T_s n = T_\sigma^* T_s(n_g) = T_\sigma^*(n_{s^{-1}g}) = P(n_{s^{-1}\sigma g}) = (n_g'),$$

where

$$n_g' = \begin{cases} n_{s^{-1}\sigma g} & \text{if } g\in S, \\ 0 & \text{if } g\notin S. \end{cases}$$

But $n=0$ for $g\neq 1$, whence $n_g'=0$ for $g\neq\sigma^{-1}s$. Hence we have for $n\in N$ and $s\sigma^{-1}\notin S$

$$T_\sigma^* T_s n = 0. \tag{3.6}$$

Thus $(n, T_s^* T_\sigma h) = (T_\sigma^* T_s n, h) = 0$ for $s\sigma^{-1}\notin S$. Hence

$$N\subset \Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H^2(N;S)\Big]^\perp.$$

Conversely, if

$$\tilde n = (n_g)\in \Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H^2(N;S)\Big]^\perp,$$

then

$$(T_\sigma^* T_s\tilde n, \tilde h) = (\tilde n, T_s^* T_\sigma \tilde h) = 0$$

for all $\tilde h\in H^2(N;S)$ and $s\sigma^{-1}\notin S$. Hence $T_\sigma^* T_s\tilde n=0$ for any $s,\sigma\in S$, $s\sigma^{-1}\notin S$. In particular, $T_\sigma^*\tilde n=0$ for all $\sigma\in S$, $\sigma\neq 1$. If $T_\sigma^*\tilde n=(n_g')$, then $n_g'=0$ for all $g\in G$. But $n_\sigma = n_1' = 0$. Thus $n_\sigma=0$ for all $\sigma\neq 1$ and thus $n\in N$. This completes the proof of (3.5).

Let M be a doubly invariant subspace included in $\operatorname{clm}\big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma\times \times T_s H^2(N;S)\big]$. On account of (3.5) we have $M\subset N^\perp$ and thus

$$(T_s n, m) = (n, T_s^* m) = 0$$

because M is doubly invariant. Thus M is orthogonal to $T_s N$ for any $s\in S$ and using (3.2) we conclude that $N=\{0\}$.

This completes the proof of the implication (a) $\to$ (c).

To finish the proof of the theorem we shall prove the implication (b) $\to$ (d).

For this purpose let M be a doubly invariant subspace of H such that

$$M\cap\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H\Big]^\perp = \{0\}.$$

We have

$$M = \operatorname{clm}\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s M\Big].$$

Indeed, if $m\in M$ is orthogonal to $\operatorname{clm}\big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s M\big]$ and $h\in H$, then putting $h=h_1+h_2$, with $h_1\in M$, $h_2\in M^\perp$ we have

$$(m, T_\sigma^* T_s h) = (m, T_\sigma^* T_s h_1) + (m, T_\sigma^* T_s h_2) = 0$$

for $s,\sigma\in S$, $\sigma^{-1}s\notin S^{-1}$, because M is doubly invariant. Hence $m\in\big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H\big]$, i.e. $m=0$.

Using (b) we conclude that $M=\{0\}$ and the implication (b) $\to$ (d) is proved.

This completes the proof of the theorem.

The following theorem is a completion of Theorem 4 in the case when N is a 1-dimensional space.

THEOREM 5. *Let $\{T_s\}_{s\in S}$ be a semi-group of isometries on a Hilbert space H. Then $\{T_s\}_{s\in S}$ is unitary equivalent to the semi-group of unilateral translations in $H^2(S)$ if and only if it is not quasi-unitary and has no proper doubly invariant subspaces.*

Proof. Let $\{T_s\}_{s\in S}$ be the semigroup of unilateral translation in $H^2(S)=H^2(N;S)$ with N 1-dimensional. Let $M\subset H^2(S)$ be a doubly invariant subspace. If

$$M\cap\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H^2(N;S)\Big]^\perp = \{0\},$$

then, from Theorem 4, we obtain $M=\{0\}$. Suppose now that

$$M\cap\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H^2(S)\Big]\neq 0.$$

From (3.5) it results that $M\cap N\neq 0$. Since N is a 1-dimensional space, we have $M\cap N=N$ and consequently

$$H^2(S) = \bigoplus_{s\in S} T_s(N)\subset M.$$

Let now $\{T_s\}_{s\in S}$ be a semi-group of isometries on H for which the condition of Theorem 5 is fulfilled. We have

$$H\cap\Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H\Big]^\perp = \Big[\bigcup_{\sigma^{-1}s\notin S^{-1}} T_\sigma^* T_s H\Big]^\perp \neq 0$$

because $\{T_s\}_{s\in S}$ is not quasi-unitary. Hence it is clear that $\{T_s\}_{s\in S}$ verifies assertion (d) of Theorem 4. It results from Theorem 4 that there exists an N such that $\{T_s\}_{s\in S}$ is unitarily equivalent to the semigroup of unilateral translations in $H^2(N;S)$. Let $n_0\in N$, $n_0\neq 0$ and let us denote by N_0 the 1-dimensional space spanned by n_0. Let us put

$$M = \bigoplus_{s\in S} T_s N_0.$$

The subspace M is doubly invariant. Indeed, if

$$m = \sum_{s\in S} T_s n_s \quad \text{with} \quad n_s \in N_0,$$

we infer by (3.6) that

$$T_\sigma m = \sum_{s\in S} T_{s\sigma} n_s \in M, \qquad T_\sigma^* m = \sum_{s_1=\sigma^{-1}s} T_{s_1} n_s \in M.$$

Since $M \cap N = N_0 \neq \{0\}$, we obtain $H = M$ and thus $N = N_0$ is a 1-dimensional space. This completes the proof of the theorem.

COROLLARY (Wermer). *An isometric non-unitary operator T on a Hilbert space H is unitarily equivalent to the translation operator on H^2 if and only if T has no proper doubly invariant subspaces.*

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On multipliers preserving convergence of trigonometric series almost everywhere

by

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1. Consider a trigonometric series $\sum_{-\infty}^{+\infty} c_n e^{in\theta}$, which in the case $c_{-n} = \bar{c}_n$ can also be written in the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n\cos n\theta + b_n \sin n\theta) = \sum_0^\infty A_n(\theta),$$

say. Its conjugate is

$$\sum_{-\infty}^{+\infty} (-i\operatorname{sign} n) c_n e^{in\theta} = \sum_1^\infty (a_n \sin n\theta - b_n\cos n\theta) = \sum_0^\infty B_n(\theta),$$

say (with $B_0 = 0$).

One of the topics of the theory of trigonometric series that enjoyed popularity a few decades ago was the problem of the behavior (convergence or summability, at individual points or almost everywhere) of the series $\sum A_n(\theta) n^a$, $\sum B_n(\theta) n^a$, where a is a constant. The problem has obvious connections with differentiability or integrability (in general, of fractional order) of functions, and a was almost exclusively real. In this note we consider complex values of a, $a = \beta + i\gamma$, but in view of the fact that the case of real a has been exhaustively dealt with we limit ourselves to a purely imaginary, $a = i\gamma$, which shows some novel features. The problem we are discussing here arose out of some concrete applications but the latter are not considered here.

The main result of the paper is the following

THEOREM. *If the series $\sum_{n=0}^{\infty} A_n(\theta)$ is summable (C, k), $k > -1$, at each point of a set E of positive measure, then the series $\sum_0^\infty A_n(\theta) n^{i\gamma}$ is summable (C, k) almost everywhere in E. In particular, the convergence of $\sum A_n(\theta)$ in E implies the convergence of $\sum_0^\infty A_n(\theta) n^{i\gamma}$ almost everywhere in E.*