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# Sobolev inequalities and extension theorems for functions with certain $L^p$-derivatives

by

ROBERT S. STRICHARTZ (Cambridge, Mass.)

**1. Introduction.** We wish to generalize some theorems about the Sobolev spaces

$$L_k^p(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leqslant k\},$$

$\Omega$ an open set in $R^n$ and the derivatives existing in the distribution sense, to spaces of functions having only certain specified derivatives in $L^p(\Omega)$. To state the theorems we need some conditions on $\Omega$ which we now define.

Definition 1. $\Omega$ is said to satisfy the *weak cone condition* if there exists a finite open covering $U_1, \ldots, U_N$ of $\Omega$ and finite cones $\gamma_1, \ldots, \gamma_N$ such that

$$\gamma_j + U_j \subseteq \Omega, \quad j = 1, \ldots, N. \tag{1.1}$$

$\Omega$ is said to satisfy the *strong cone condition* if there exists a finite open covering $U_1, \ldots, U_N$ of $\partial\Omega$ with positive Lebesgue number (i.e., there exists $\varepsilon > 0$ such that the $\varepsilon$-ball about each point in $\partial\Omega$ is entirely contained in some $U_j$) and finite cones $\gamma_1, \ldots, \gamma_N$ such that

$$\gamma_j + (U_j \cap \Omega) \subseteq \Omega. \tag{1.2}$$

We can now state the three theorems we will generalize. We assume throughout $1 < p < \infty$.

THEOREM 1 (Calderón [2]). *Let $\Omega$ satisfy the strong cone condition. Then for each $k$ there exists a bounded linear extension operator $\mathscr{E}_k$: $L_k^p(\Omega) \to L_k^p(R^n)$.*

By *extension operator* we mean $\mathscr{E}_k f = f$ on $\Omega$.

THEOREM 2 (Sobolev). *Let $\Omega$ satisfy the weak cone condition. Then we have the continuous inclusions*

(a) $$L_k^p(\Omega) \subseteq L_j^q(\Omega) \quad \textit{if} \quad \frac{1}{p} \geqslant \frac{1}{q} \geqslant \frac{1}{p} - \frac{k-j}{n} > 0,$$

(b) $$L_k^p(\Omega) \subseteq C_0^j(\Omega) \quad \textit{if} \quad j < \frac{k}{n} - \frac{1}{p},$$

*where* (b) *means that each function in* $L_k^p(\Omega)$ *can be modified on a set of measure zero so that it is j-times continuously differentiable and it, together with its derivatives of order* $\leqslant j$ *are bounded and tend to zero as* $x$ *tends to infinity in* $\Omega$.

THEOREM 3 (Smith). *Let* $\Omega$ *be bounded and satisfy the weak cone condition. If* $f\in\mathcal{D}'(\Omega)$ *and* $(\partial/\partial x_j)^k f\in L^p(\Omega)$ *for* $j=1,\ldots,n$, *then* $f\in L_k^p(\Omega)$ *and*

$$\|f\|_{p,k}\leqslant C\left(\sum_{j=1}^{n}\left\|\left(\frac{\partial}{\partial x_j}\right)^k f\right\|_p+\|f\|_p\right).$$

Proofs may be found in [1], [2] and [9]. Although there exist versions of the above theorems for $L^1$ and $L^\infty$, our methods are only valid in the range $1<p<\infty$.

In Section 2 we define our spaces and prove generalizations of Theorems 1, 2 and 3. The proofs are based on a generalization of the Sobolev representation formula (Lemma 1) and use estimates for singular integrals with mixed homogeneity given in [4]. In Section 3 we present some counterexamples to show that the conditions on $\Omega$ cannot be completely relaxed. In Section 4 we apply the results to $L_k^p(\Omega)$ to obtain an extension theorem with loss of smoothness if $\Omega$ has a rough boundary. In Section 5 we give applications to partial differential equations, including an extension theorem for solutions of certain systems of homogeneous constant coefficient linear equations.

**2.** Let us fix once and for all a basis $(x_1,\ldots,x_n)$ in $R^n$, so that if $(y_1,\ldots,y_n)$ is any other basis we have

$$y_i=\sum_{j=1}^{n}L_{ij}x_j$$

for some real, non-singular $(n\times n)$-matrix $L$. If $\beta=(\beta_1,\ldots,\beta_n)$ is an $n$-tuple of non-negative integers, we will denote by $D_L^\beta$ the differential operator $(\partial/\partial y_1)^{\beta_1}\ldots(\partial/\partial y_n)^{\beta_n}$.

Definition 2. Let $a$ denote a sequence $a(1),\ldots,a(m)$ of $n$-tuples of non-negative integers $a(k)=\{a_1(k),\ldots,a_n(k)\}$, and let $L$ denote a sequence $L(1),\ldots,L(m)$ of real, non-singular $(n\times n)$-matrices. We denote by $L_{a,L}^p(\Omega)$ the space of all functions $f\in L^p(\Omega)$ such that

$$D_L^\beta f=D_{L(1)}^{\beta(1)}\ldots D_{L(m)}^{\beta(m)}f\in L^p(\Omega)$$

for all $\beta$ such that

$$\text{(2.1)}\qquad \sum_{j=1}^{n}\frac{\beta_j(k)}{a_j(k)}\leqslant 1,\qquad k=1,\ldots,m$$



(by convention $0/0=0$), the derivatives existing in the distribution sense.

We equip $L_{a,L}^p(\Omega)$ with a norm, denoted by $\|\ \|_{p,a,L}$, defined by

$$\|f\|_{p,a,L}=\sum\|D_L^\beta f\|_p,$$

the sum extending over all $\beta$ satisfying (1). If each $L(k)$ is the identity matrix, we will just write $L_a^p(\Omega)$.

PROPOSITION 1. $L_{a,L}^p(\Omega)$ *is complete. Any bounded linear operator* $T:L^p(R^n)\to L^p(R^n)$ *which commutes with translations is also bounded:* $L_{a,L}^p(R^n)\to L_{a,L}^p(R^n)$.

Proof. The proof is an exercise in distribution theory, which we leave to the reader.

Let $a=(a_1,\ldots,a_n)$ now be an $n$-tuple of positive integers. For simplicity of notation we set $a_k=1/a_k$ and $a=(a_1,\ldots,a_n)$. A non-empty, open set $\Gamma\subseteq R^n$ will be called an *a-cone* if $x\in\Gamma$ implies

$$t^a x=(t^{a_1}x_1,\ldots,t^{a_n}x_n)\in\Gamma\quad\text{ for all } t>0.$$

A *finite a-cone* is the intersection of an $a$-cone with a ball about the origin. A function $g$ defined on an $a$-cone $\Gamma$ is called *a-homogeneous of degree s* if

$$\text{(2.2)}\qquad g(t^a x)=t^s g(x).$$

Let $\Sigma$ denote the unit sphere. Following [4] we define $\varrho(x)$ to be the unique constant $\varrho$ for which $\varrho^{-a}x\in\Sigma$. It is easy to see that $\varrho(x)$ is $a$-homogeneous of degree $+1$. We introduce "polar coordinates" with respect to $\varrho$ and $\Sigma$ as follows: we identify $E_n-\{0\}$ with $(0,\infty)\times\Sigma$ by the map $x\to(\varrho,\nu)$, where $\varrho=\varrho(x)$ and $\nu=\varrho^{-a}x$. This map is a diffeomorphism and its jacobian $J$ is $a$-homogeneous of degree $|a|-1$, where $|a|=\sum_{j=1}^{n}a_j$. We have the integral formula

$$\int_{E_n}g(x)\,dx=\int_\Sigma\int_0^\infty g(\varrho,\nu)J(\varrho,\nu)\,d\varrho\,d\nu.$$

We can now derive our basic representation lemma. We fix a finite $a$-cone $\gamma=\Gamma\cap B$, where $\Gamma$ lies strictly in some half-space, and $B$ is a ball about the origin.

LEMMA 1. *There exist functions* $\psi,\varphi_1,\ldots,\varphi_n\in L^1(R^n)$ *such that, if we define* $\mathcal{E}_\gamma\colon L^p(R^n)^{n+1}\to L^p(R^n)$ *by*

$$\text{(2.3)}\qquad \mathcal{E}_\gamma(f,f_1,\ldots,f_n)=f*\psi+\sum_{j=1}^{n}f_j*\varphi_j,$$

*we have* $\mathcal{E}_\gamma(f,f_1,\ldots,f_n)=f$ *on* $\Omega'$ *provided* $f_j=(\partial/\partial x_j)^{a_j}f$ *on* $\Omega$, *where*

$$\Omega'=\{x\in\Omega: x+\gamma\subseteq\Omega\}.$$

*Furthermore, $\psi, \varphi_1, \ldots, \varphi_n$ are supported on $\gamma$, $\psi$ is $C^\infty$ and vanishes in a neighborhood of the origin, $\varphi_1, \ldots, \varphi_n$ are $C^\infty$ on $R^n - \{0\}$ and locally a-homogeneous of degree $1-|a|$ (i.e., agree with an a-homogeneous function in a neighborhood of the origin).*

Proof. Let $\gamma$ be an $a$-cone such that $\gamma'^{|a|} = \gamma' + \ldots + \gamma' \subseteq \gamma$.

Let $\varphi$ be locally $a$-homogeneous of degree $1-|a|$, $C^\infty$ on $R^n - \{0\}$, vanishing outside $\gamma'$, non-negative and not identically zero. Then $J\varphi$ is locally $a$-homogeneous of degree zero. We normalize $\varphi$ so that

$$\int_{\Sigma} \lim_{\varrho\to 0} J(\varrho, \nu)\varphi(\varrho, \nu)\,d\nu = 1.$$

Let

$$\psi_0 = \frac{1}{J}\frac{\partial}{\partial\varrho}(J\varphi) \quad \text{and} \quad \theta_j = \frac{\partial x_j}{\partial\varrho}\varphi.$$

Then we have

$$f(x) = \iint \varphi(\varrho, \nu) J(\varrho, \nu)\frac{\partial}{\partial\varrho} f\big(x - (\varrho, \nu)\big)\,d\varrho\,d\nu + \\ + \iint \frac{\partial}{\partial\varrho}(\varphi J)(\varrho, \nu) f\big(x - (\varrho, \nu)\big)\,d\varrho\,d\nu$$

by integration by parts. Substituting

$$\frac{\partial}{\partial\varrho} f\big(x-(\varrho, \nu)\big) = \sum_{j=1}^{n} \frac{\partial x_j}{\partial\varrho}(\varrho, \nu)\frac{\partial f}{\partial x_j}\big(x-(\varrho, \nu)\big),$$

and remembering that $dx = J\,d\varrho\,d\nu$, we have

$$f = \psi_0 * f + \sum_{j=1}^{n} \theta_j * \frac{\partial f}{\partial x_j}. \tag{2.4}$$

Thus also

$$D^\beta f = \psi_0 * D^\beta f + \sum_{j=1}^{n} \theta_j * D^\beta \frac{\partial f}{\partial x_j}. \tag{2.5}$$

Now we substitute (2.5) in the right-hand side of (2.4) according to the following scheme: if $D^\beta f$ appears with $\beta_j < a_j$ for all $j = 1, \ldots, n$, and it is not in a convolution containing $\psi_0$, then substitute the right-hand side of (2.5). This process eventually terminates to give

$$f = \psi_0 * \Big(\sum_{\gamma_j < a_j} C_\gamma \psi_\gamma * D^\gamma f\Big) + \sum_{\substack{\beta_j \le a_j \\ \beta_k = a_k}} C_\beta \psi_\beta * D^\beta f, \tag{2.6}$$



the $c$'s being binomial constants, and

$$\psi_\beta = \theta_1^{(\beta_1)} * \theta_2^{(\beta_2)} * \ldots * \theta_n^{(\beta_n)}, \quad \theta_1^{(\beta_1)} = \theta_1 * \ldots * \theta_1 \quad (\beta_1 \text{ times}).$$

Let

$$\psi = \Big(\sum_{\gamma_j < a_j} (-1)^\gamma C_\gamma (D^\gamma \psi_0) * \psi_\gamma\Big)$$

and let

$$\varphi_k = \sum_{\substack{\beta_j \le a_j \\ \beta_k = a_k}} (-1)^{\beta - \beta_k} C_\beta D^{\beta - \beta_k} \psi_\beta.$$

Then by integration by parts we have, at least formally

$$f = \psi * f + \sum_{k=1}^{n} \varphi_k * \left(\frac{\partial}{\partial x_k}\right)^{a_k} f. \tag{2.7}$$

Now $\gamma'$ was contained in some half-space, so if $g$ and $h$ are supported in $\gamma'$, $C^\infty$ in $R^n - \{0\}$, and locally $a$-homogeneous of degrees $s - |a|$ and $t - |a|$ respectively, $s, t > 0$, then $g * h$ is supported in $\gamma' + \gamma'$, is $C^\infty$ in $R^n - \{0\}$, and is locally $a$-homogeneous of degree $s + t - |a|$.

Now let us compute the $a$-homegeneity of the above functions. For $\theta_j$ it is $a_j - |a|$, hence for $\psi_\beta$ it is $\big(\sum_{j=1}^{n} \beta_j a_j\big) - |a|$, and for $D^{\beta-\beta_k}\psi_\beta$ it is $\beta_k a_k - |a|$, since $\partial/\partial x_j$ reduces the $a$-homogeneity by $a_j$. In particular, if $\beta_k = a_k$, it is $1 - |a|$. Thus $\varphi_k$ satisfies the conditions of Lemma 1. Note that the integrability of a good locally $a$-homogeneous function is equivalent to the degree being $> -|a|$. Now $\psi_0$ is $C^\infty$ and vanishes in a neighborhood of the origin, hence $\psi$ has the same properties. Thus it remains to establish $\mathscr{E}_\gamma(f, f_1, \ldots, f_n) = f$ on $\Omega'$ if $f_j = (\partial/\partial x_j)^{a_j} f$ on $\Omega$.

Note that all the functions $\psi, \varphi_1, \ldots, \varphi_n$ appearing in (2.7), and all the functions appearing in the derivation of (2.7) (excluding, of course, $f$ and its derivatives) are integrable and supported in $\gamma^N$. Thus the integration by parts is justified, and for values of $x \in \Omega'$ we need only have $f$ defined on $\Omega$ for (2.7) to hold. But there (2.7) is just $\mathscr{E}_\gamma(f, f_1, \ldots, f_n) = f$.

Remark. We can also prove, by the same method, a variant of the lemma in which the term $f * \psi$ does not appear, but at the cost of having $\varphi_j$ supported in an entire $a$-cone $\Gamma$. In this case $\varphi_j$ is no longer integrable, and we must put further conditions on $f$ to have $f_j * \varphi_j$ convergent.

Now suppose $a_j = 0$ for some $j$. By relabeling coordinates we can suppose $a = (a_1, \ldots, a_{n'}, 0, \ldots, 0)$, where $a_j \ne 0$ for $j = 1, \ldots, n'$. Let $a' = (a_1, \ldots, a_{n'})$. We say that $\gamma \subseteq R^n$ is an $a$-cone if there exists an $a'$-cone $\gamma' \subseteq R^{n'}$ such that

$$\gamma = \gamma' \times \{0\} = \{x : (x_1, \ldots, x_{n'}) \in \gamma' \text{ and } x_{n'+1} = \ldots = x_n = 0\}.$$

Corollary. *In this case Lemma 1 is again valid, except that* $\psi, \varphi_1, \ldots, \varphi_{n'}$ *are now measures supported in* $\gamma'$ *of the form* $\psi = \psi' \times \delta$, $\varphi_j = \varphi_j' \times \delta$ [1] *and* $\psi', \varphi_1', \ldots, \varphi_n'$ *satisfy the conditions of Lemma* 1 *with respect to* $a', \gamma'$.

Proof. In fact, this is precisely what we get if we apply Lemma 1 for $a', \gamma'$ to each section of $f$ parallel to $R^{n'}$ and put it together via Fubini's theorem.

Definition 3. $\Omega$ is said to satisfy the *weak a-cone condition* if there exists a finite open cover $U_1, \ldots, U_N$ of $\Omega$ and finite $a$-cones $\gamma_1, \ldots, \gamma_N$ such that

$$\gamma_j + U_j \subseteq \Omega, \quad j = 1, \ldots, N. \tag{2.8}$$

$\Omega$ is said to satisfy the *strong a-cone condition* if there exists a finite open cover $U_1, \ldots, U_N$ of $\partial\Omega$ with positive Lebesgue number and finite $a$-cones $\gamma_1, \ldots, \gamma_N$ such that

$$\gamma_j + (U_j \cap \Omega) \subseteq \Omega, \quad j = 1, \ldots, N. \tag{2.9}$$

We define similarly the $a$-$L$-cone conditions by requiring the $\gamma_j$ to be $a$-cones with respect to the coordinates $(y_1, \ldots, y_n) = L(x_1, \ldots, x_n)$.

Theorem 1'. *Let* $a(1), \ldots, a(m)$, $L(1), \ldots, L(m)$ *be as in Definition* 2. *Suppose* $\Omega$ *satisfies the strong* $a(k)$-$L(k)$-*cone condition for* $k = 1, \ldots, m$. *Then there exists a bounded linear extension operator* $\mathscr{E}_{a,L}: L^p_{a,L}(\Omega) \to L^p_{a,L}(R^n)$.

Proof. The proof is by induction on $m$. The case $m = 0$ is trivial, for then, without assumptions on $\Omega$,

$$\mathscr{E}_0 f(x) = \begin{cases} f(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \notin \Omega \end{cases}$$

is a bounded linear extension operator $L^p(\Omega) \to L^p(R^n)$.

Suppose the theorem is true for $m-1$. Then $\Omega$ satisfies the strong $a(k)$-$L(k)$-cone conditions for $k = 1, \ldots, m-1$, hence there exists a bounded linear extension operator

$$\mathscr{F}: L^p_{a(1),\ldots,a(m-1),L(1),\ldots,L(m-1)}(\Omega) \to L^p_{a(1),\ldots,a(m-1),L(1),\ldots,L(m-1)}(R^n).$$

Let $(y_1, \ldots, y_n) = L(m)(x_1, \ldots, x_n)$. Then

$$\left(\frac{\partial}{\partial y_k}\right)^{a_k(m)} f \in L^p_{a(1),\ldots,a(m-1),L(1),\ldots,L(m-1)}(\Omega) \quad \text{if} \quad f \in L^p_{a,L}(\Omega).$$

Hence

$$\mathscr{F}\left(\frac{\partial}{\partial y_k}\right)^{a_k(m)} f \in L^p_{a(1),\ldots,a(m-1),L(1),\ldots,L(m-1)}(R^n).$$

[1] $\delta$ is the Dirac measure at the origin of $R^{n-n'}$.



Now $\Omega$ satisfies the strong $a(m)$-$L(m)$-cone condition. Let $U_1, \ldots, U_N$ be the open cover of $\partial\Omega$ guaranteed by Definition 3, with Lebesgue number $\varepsilon$. Let

$$U_0 = \left\{x \in R^n: d(x, \partial\Omega) > \frac{\varepsilon}{3}\right\}.$$

Then $U_0, \ldots, U_N$ forms an open cover of $R^n$ with Lebesgue number $\varepsilon/3$. It is a routine matter now to construct a $C^\infty$-partition of unity $h_0, \ldots, h_N$ with $h_k$ supported in $U_k$, such that all derivatives of $h_k$ are bounded.

Now define $\mathscr{E}_{a,L}$ as follows:

$$\mathscr{E}_{a,L} f = h_0 \mathscr{E}_0 f + \sum_{k=1}^{N} h_k \mathscr{E}_{\gamma k}\left(\mathscr{F}f, \mathscr{F}\left(\frac{\partial}{\partial y_1}\right)^{a_1(m)} f, \ldots, \mathscr{F}\left(\frac{\partial}{\partial y_n}\right)^{a_n(m)} f\right), \tag{2.10}$$

where $\gamma_k$ is the cone associated with $U_k$ and $\mathscr{E}_{\gamma k}$ is given by Lemma 1 or the corollary. Now

$$\left(\frac{\partial}{\partial y_k}\right)^{a_k(m)} \mathscr{F}f = \mathscr{F}\left(\frac{\partial}{\partial y_k}\right)^{a_k(m)} f \quad \text{on } \Omega,$$

hence by Lemma 1

$$\mathscr{E}_{\gamma k}\left(\mathscr{F}f, \mathscr{F}\left(\frac{\partial}{\partial y_1}\right)^{a_1(m)} f, \ldots, \mathscr{F}\left(\frac{\partial}{\partial y_n}\right)^{a_n(m)} f\right) = \mathscr{F}f = f \quad \text{on } \Omega'.$$

But, by (2.9), $\Omega' \supseteq U_k$. Thus $\mathscr{E}_{a,L} f = f$ on $\Omega$. It remains to show that $\mathscr{E}_{a,L}: L^p_{a,L}(\Omega) \to L^p_{a,L}(R^n)$ and is bounded. This will follow once we have

$$\mathscr{E}_{\gamma k}: \quad \left(L^p_{a(1),\ldots,a(m-1),L(1),\ldots,L(m-1)}(R^n)\right)^{n+1} \to L^p_{a,L}(R^n) \text{ bounded}.$$

But $\mathscr{E}_{\gamma k}$ is a convolution operator hence by Proposition 1 it is enough to show that $\mathscr{E}_{\gamma k}$ is bounded from $L^p(R^n)^{n+1}$ to $L^p_{a(m),L(m)}(R^n)$, or $D^\beta_{L(m)} \mathscr{E}_{\gamma k}$ is bounded from $L^p(R^n)^{n+1}$ to $L^p(R^n)$ if $\beta_1/a_1 + \ldots + \beta_n/a_n \leqslant 1$. Now

$$\mathscr{E}_{\gamma k}(f, f_1, \ldots, f_n) = \psi * f + \sum_{j=1}^{n} \varphi_j * f_j.$$

Let us assume $a_j \neq 0$ for $j = 1, \ldots, n$. Then $\psi \in C^\infty_{\text{com}}(R^n)$ so $f \to D^\beta_{L(m)}(\psi * f)$ is bounded from $L^p(R^n)$ to $L^p(R^n)$. If $\beta_1/a_1 + \ldots + \beta_n/a_n < 1$, then $D^\beta_{L(m)} \varphi_j \in L^1(R^n)$; hence

$$f_j \to D^\beta_{L(m)}(\varphi_j * f_j) = f_j * D^\beta_{L(m)} \varphi_j$$

is bounded from $L^p(R^n)$ to $L^p(R^n)$. However, in case $\beta_1/a_1 + \ldots + \beta_n/a_n = 1$, $D^\beta_{L(m)} \varphi_j$ is locally $a(m)$-homogeneous of degree $-|a(m)|$, hence not integrable. Here we have to use the theory of singular integrals with mixed homogeneity developed in [4]. There it is shown that if $g$ is $C^\infty(R^n-\{0\})$ and $a(m)$-homogeneous of degree $-|a(m)|$ and also

$$\int_\Sigma g(1,\nu)J(1,\nu)\,d\nu = 0, \tag{2.11}$$

then the principal value convolution with $g$ defines a bounded linear operator on $L^p(R^n)$. It is also shown in the Appendix of [4] that (2.11) is equivalent to

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(y_1,\dots,y_n)\,dy_1\dots dy_{n-1} = 0 \tag{2.12}$$

for some fixed $y_n \neq 0$. Thus if $g$ is a derivative $D^\beta_{L(m)}h$, then it satisfies condition (2.11). Furthermore, it is easy to modify the proof in the homogeneous case (see e.g. [1], theorem 11.4) to show that

$$D^\beta_{L(m)}(h*f) = cf + PV(g*f).$$

Taking $h = \varphi_j$ in a neighborhood of the origin, $h$ $a(m)$-homogeneous we get that $f_j \to D^\beta_{L(m)}(\varphi_j * f)$ is bounded from $L^p(R^n)$ to $L^p(R^n)$.

Thus the theorem is proved if $a_j \neq 0$, $j = 1,\dots,n$. In general, we must apply the corollary to Lemma 1 and reason as above with each section of $f$ restricted to each subspace parallel to $R^{n'}$.

THEOREM 2′. *Let $a(1),\dots,a(m)$, $L(1),\dots,L(m)$ be as above, and assume $a_j(k)\neq 0$ $(j=1,\dots,n;\ k=1,\dots,m)$. Suppose that $\Omega$ satisfies the weak $a(k)$-$L(k)$-cone condition for $k=1,\dots,m$. Then we have the following continuous inclusions:*

(a) $L^p_{a,L}(\Omega) \subseteq L^q(\Omega)$, *provided* $p \leqslant q < \infty$ *and*

$$\frac{1}{q} \geqslant \frac{1}{p} - \sum_{k=1}^m \frac{1}{|a(k)|};$$

(b) $L^p_{a,L}(\Omega) \subseteq C_0(\Omega)$ *after modification on a set of measure zero, provided*

$$\sum_{k=1}^m \frac{1}{|a(k)|} < \frac{1}{p}.$$

Proof. Again the proof is by induction on $m$. The induction argument goes as before, so we will only give the proof in the case $m=1$, $L = I$. Since $\Omega$ satisfies the weak $a$-cone condition, we have an open covering $U_1,\dots,U_N$ of $\Omega$ given in Definition 2. It clearly suffices to show that $L^p_a(\Omega)|_U \subseteq L^q(U)$ for $U$ one of the $U_j$'s. Let $\gamma$ be the associated $a$-cone. Then by Lemma 1 we have

$$f = \psi*f + \sum_{j=1}^n \varphi_j * f_j \ \text{ on } U, \quad \text{where} \quad f_j = \left(\frac{\partial}{\partial x_j}\right)^{a_j} f \in L^p(\Omega).$$



Now $\psi \in C^\infty_{\text{com}}$, hence $\psi * f \in C^\infty_0$. Also $\varphi_j$ is locally $a$-homogeneous of degree $1-|a|$. Thus $\varphi_j \in L^{1/(1-1/|a|),\infty}$, the Lorentz space of functions satisfying

$$m\{x\in R^n\colon |f(x)| \geqslant S\} \leqslant CS^{-1/(1-1/|a|)}.$$

Now the modern version of the Hardy-Littlewood and Sobolev Fractional Integration Theorem asserts that convolution is a bounded bilinear map from $L^p\times L^{r,\infty}$ to $L^q$, where $1/q = 1/p+1/r-1$, provided $1<p,\ q,\ r<\infty$. Applying this to $f*\varphi_j$ gives (a). To prove (b) we note that the conditions on $a$ and $p$ imply $\varphi_j \in L^{p'}$ where $1/p'+1/p = 1$. Thus (b) follows from the classical Hölder theorem.

COROLLARY. *We also have $D^\beta_L f = D^{\beta(1)}_{L(1)}\dots D^{\beta(m)}_{L(m)} f \in L^q(\Omega)$ if*

$$\sum \frac{\beta_j(k)}{a_j(k)} = \beta(k)\cdot a(k) \leqslant 1 \quad (p \leqslant q < \infty)$$

$$\text{and} \quad \frac{1}{q} \geqslant \frac{1}{p} - \sum_{k=1}^m \frac{1-\beta(k)\cdot a(k)}{|a(k)|} \geqslant 0$$

*or*

$$D^\beta_L f \in C_0(\Omega) \ \text{ if } \ \beta(k)a(k)\leqslant 1 \quad \text{and} \quad \frac{1}{p} < \sum_{k=1}^m \frac{1-\beta(k)\cdot a(k)}{|a(k)|}.$$

The corollary is proved by the same reasoning. Note, however, that the decomposition $D^\beta_L = D^{\beta(1)}_{L(1)}\dots D^{\beta(m)}_{L(m)}$ is not canonical, and that different decompositions may give different restrictions on $p$ and $q$.

We can also obtain inclusion theorems for non-negative multi-indices, although the formulae become more complicated. For simplicity we restrict to the case $L(1) = \dots = L(m) = I$.

THEOREM 2″. *Let $a(1),\dots,a(m)$ be multi-indices of non-negative integers such that*

$$\sum_{k=1}^m a_j(k) > 0 \quad \text{for} \quad j = 1,\dots,n.$$

*Suppose that $\Omega$ satisfies the weak $a(k)$-cone condition for $k=1,\dots,m$. Let $\beta(1),\dots,\beta(m)$ be such that*

$$0 \leqslant \beta_j(k) \leqslant a_j(k), \qquad \sum_{j=1}^n \frac{\beta_j(k)}{a_j(k)} \leqslant 1$$

*(recall $0/0 = 0$), and $\sum' a_j(k)-\beta_j(k) > 0$, the summation extending over all $k$ such that*

$$\sum_{j=1}^n \frac{\beta_j(k)}{a_j(k)} < 1.$$

*Then* $f \in L^p_a(\Omega)$ *implies* $D^\beta f = D^{\beta(1)} \ldots D^{\beta(m)} f \in L^q(\Omega)$ *if* $p \leqslant q < \infty$ *and*

$$\frac{1}{q} \geqslant \frac{1}{p} - \inf_j \sum_{a_j(k) > \beta_j(k)} \frac{1-\beta(k)\cdot a(k)}{\sum_{a_i(k)>0} a_i(k)} \geqslant 0$$

*or* $C_0(\Omega)$ *if the last term is* $< 0$.

Proof. First we note that the values of $k$ for which $\beta(k)a(k) = 1$ make no contribution to the theorem, so we may assume without loss of generality that they do not exist. Next we note that by intersecting the elements of the different open covers associated with the $a(k)$-cone conditions we obtain a common open cover $U_1, \ldots, U_N$ satisfying

$$U_j + \gamma_j(1)^{|a(1)|} + \ldots + \gamma_j(m)^{|a(m)|} \subseteq \Omega,$$

where $\gamma_j(k)$ is an $a(k)$-cone. Let $U$ be an element of this cover, and $\gamma(1), \ldots, \gamma(m)$ the associated cones. It is clearly sufficient to prove that $D^\beta f|_U \in L^q(U)$ or $C_0(U)$.

For simplicity we assume $m = 2$; the general case is analogous, but notationally more cumbersome. We relabel the coordinates so that $a_j(1) = 0$ if and only if $j > w$ and $a_j(2) = 0$ if and only if $j \leqslant v$. By hypotheses $v \leqslant w$. By Lemma 1 we have, in $U$,

$$f = \psi(1)*f + \sum_{j=1}^{w} \varphi_j(1) * \left(\frac{\partial}{\partial x_j}\right)^{a_j(1)} f$$

$$= \psi(1)*\psi(2)*f + \sum_{j=1}^{w} \psi(2)*\varphi_j(1)*\left(\frac{\partial}{\partial x_j}\right)^{a_j(1)} f + \sum_{j=v+1}^{n} \psi(1)*\varphi_j(2)*\left(\frac{\partial}{\partial x_j}\right)^{a_j(2)} f +$$

$$+ \sum_{j=1}^{w} \sum_{k=v+1}^{n} \varphi_j(1)*\varphi_k(2)*\left(\frac{\partial}{\partial x_j}\right)^{a_j(1)} \left(\frac{\partial}{\partial x_k}\right)^{a_k(2)} f.$$

As before, the difficult step is to show that convolution by $D^\beta(\varphi_j(1)*\varphi_k(2))$ is a bounded linear operator from $L^p$ to $L^q$. If we let $g = D^{\beta(1)}\varphi_j(1)$ and $h = D^{\beta(2)}\varphi_j(2)$, we have $g = g'(x_1, \ldots, x_w) \times \delta_1$ and $h = \delta_2 \times h'(x_{v+1}, \ldots, x_n)$, where $g$ is locally $(a_1(1), \ldots, a_w(1))$-homogeneous of degree

$$1 - \sum_{j=1}^{w} \big(a_j(1) + a_j(1)\beta_j(1)\big)$$

and $h$ is locally $(a_{v+1}(2), \ldots, a_n(2))$ homogeneous of degree

$$1 - \sum_{j=v+1}^{n} \big(a_j(2) + a_j(2)\beta_j(2)\big),$$

and both have compact support.

The proof will be complete once we have established



LEMMA 2. *Let* $g \in L^{p,\infty}(R^w)$, $h \in L^{q,\infty}(R^{n-v})$ *have compact support, and suppose* $v \leqslant w$. *Then the function* $\varphi$ *given by*

$$\varphi(x_1, \ldots, x_n) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(x_1, \ldots, x_v, x_{v+1} - y_{v+1}, \ldots, x_w - y_w) \times$$
$$\times h(y_{v+1}, \ldots, y_w, x_{w+1}, \ldots, x_n)\, dy_{v+1} \ldots dy_w$$

*if* $v < w$, $\varphi(x_1, \ldots, x_n) = g(x_1, \ldots, x_v) h(x_{v+1}, \ldots, x_n)$ *if* $v = w$, *defines, by convolution, a bounded operator from* $L^r(R^n)$ *to* $L^s(R^n)$, *where*

$$\frac{1}{s} = \frac{1}{r} + \max\left\{\frac{1}{p}; \frac{1}{q}\right\} - 1,$$

*provided* $1 < p, q, r, s < \infty$.

Proof. By the Marcinkiewicz Interpolation Theorem for Lorentz spaces [7] it suffices to prove the result under the assumption $g \in L^p(R^w)$, $h \in L^q(R^{n-v})$. Say $p \leqslant q$. Then, since $h$ has compact support, we have also $h \in L^p(R^{n-v})$. This implies $\varphi \in L^p(R^n)$ by Fubini's Theorem (note $L^p_{\text{com}} * L^p_{\text{com}} \subseteq L^p_{\text{com}}$). Now the result is just the standard convolution inequality.

Remark. In case $\Omega = R^n$ it is possible to identify the spaces $L^p_{a,L}(R^n)$ with spaces defined by Fourier transforms. For these spaces it is possible to obtain inclusion relations (see [11]). However, to obtain best possible results it is necessary to enlarge the class of spaces considered. This explains why the corollary to Theorem 2′ is so awkward to formulate.

Any such result about $L^p_{a,L}(R^n)$ can be transferred to domains satisfying weak cone conditions by means of the following relative extension theorem which is a corollary of the proof of Theorem 1′:

THEOREM 1″. *Let* $U \subseteq \Omega \subseteq R^n$ *be open sets such that* $U + \gamma(1) + \ldots + \gamma(m) \subseteq \Omega$, *where* $\gamma(k)$ *is some* $a(k)$-$L(k)$-*cone. Then there exists a bounded linear operator* $\mathscr{E}: L^p_{a,L}(\Omega) \to L^p_{a,L}(R^n)$ *such that* $\mathscr{E}f = f$ *on* $U$.

THEOREM 3′. *Suppose* $\Omega$ *is bounded and satisfies the weak* $a(k)$-$L(k)$-*cone condition for* $k = 1, \ldots, m$. *If* $f \in \mathscr{D}'(\Omega)$ *and* $D^\beta_L f = D^{\beta(1)}_{L(1)} \ldots D^{\beta(m)}_{L(m)} f \in L^p(\Omega)$ *for all* $D^\beta_L$ *such that* $\beta(k) = (0, \ldots, 0, a_j(k), 0, \ldots, 0)$ *for some* $j$ *depending on* $k$, *then* $f \in L^p_{a,L}(\Omega)$ *and*

$$\|f\|_{p,a,L} \leqslant C\Big(\|f\|_p + \sum \|D^\beta_L f\|_p\Big),$$

*the summation extending over all such* $D^\beta_L$.

Proof. Again the proof is by induction on $m$. We give the details only for the case $m = 1$, $L = I$.

Assume first $a_j \neq 0$ for $j = 1, \ldots, n$. Again we apply Lemma 1 to $f$ in one of the sets $U$ of the open cover given in Definition 2. The terms of the form $\varphi_j*(\partial/\partial x_j)^{a_j}f$ are shown to be in $L^p_a(U)$ just as in the proof of Theorem 1′, since we have assumed $(\partial/\partial x_j)^{a_j}f\in L^p(\Omega)$. Now, however, $\psi*f$ is the difficult term since we only know $f\in\mathscr{D}'(\Omega)$. But $\psi$ is $C^\infty_{\text{com}}$ with support in $\gamma^{|a|}$ and vanishing in a neighborhood of the origin. This implies $\psi*f\in C^\infty(\overline{U})$, and, since $\Omega$ is bounded, $C^\infty(\overline{U})\subseteq L^p_a(U)$.

If some $a_j=0$, we have assumed $f\in L^p(\Omega)$ because $(0,\dots,0,a_j,0,\dots,0)=(0,0)$. The proof now proceeds just as in Theorem 1′.

**3.** We begin with some simple remarks about the cone conditions.

1. $\Omega$ satisfies the strong $a$-cone condition if and only if its complement does. This is not true for the weak $a$-cone condition.
2. If $\beta_j=Ca_j$, then the $a$ and $\beta$-cone conditions are the same.
3. Let $\Omega$ have the form $\{x: x_n\geqslant g(x_1,\dots,x_{n-1})\}$. Suppose $a_n\geqslant a_j$ for $j=1,\dots,n-1$. If $g$ satisfies a uniform Lipschitz condition of order $a_j/a_n$ in the $x_j$-direction, i.e.

$$|g(x)-g(y)|\leqslant C\sum_{j=1}^{n-1}|x_j-y_j|^{a_j/a_n},$$

then $\Omega$ satisfies the strong $a$-cone condition.

Now let us consider an example. In $R_2$ we define

$$\Omega_t=\{(x,y): y>|x|^t\}.$$

Let $\varphi$ be a $C^\infty_{\text{com}}$-function $=1$ in the unit ball. Let $f_s(x,y)=y^{-s}\varphi(x,y)$. We compute that $f_s\in L^p_{(l,m)}(\Omega)$ if and only if $s<(1+t)/tp-m$, and $f_s\in L^q(\Omega)$ if and only if $s<(1+t)/tq$. It follows that Theorem 2′ can only hold if $t\geqslant l/m$. Now Theorem 1′ holding for $\Omega$ would imply Theorem 2′ for $\Omega$ because Theorem 2′ holds for $R^n$. Thus Theorem 1′ also fails unless $t\geqslant l/m$. But this is exactly the weak $(l,m)$-cone condition.

Unfortunately, if $l>m$ none of the $\Omega_t$, $t>0$, satisfy the strong $(l,m)$-cone condition since this requires that the boundary be horizontal for an interval every time the height achieves a relative maximum or minimum.

This example can be modified to show that the conditions on $\Omega$ in Theorem 2′ are very close to being necessary. However, it is not clear whether there is some condition between the weak and strong $a$-cone conditions which is sufficient for Theorem 1′. For example, one may ask, in the classical case, whether Theorem 1 holds for the complement of $\Omega_t$ which always satisfies the weak cone condition.

Finally, we present an example to show that Theorem 3 does not hold for arbitrary bounded open sets. It seems unlikely, however, that the weak cone condition is really necessary.





First consider the unbounded $\Omega$ in $R^2$ given by $\Omega=\{(x,y): x\geqslant 0$ and $0\leqslant y\leqslant a(x)\}$, where $a(x)$ is a finite, positive, continuous, decreasing function satisfying

$$\int_0^\infty a(x)\,dx<\infty\quad\text{but}\quad\int_0^\infty x^p a(x)\,dx=+\infty.$$

Define $f(x,y)=x$ on $\Omega$. Then every derivative of $f$ is bounded, hence in $L^p(\Omega)$, but $f$ is not in $L^p(\Omega)$. Now $\Omega$ can be mapped onto a bounded domain $\Omega'$ by a $C^\infty$-diffeomorphism $\varphi$ such that all the derivatives of $\varphi^{-1}$ are bounded. In fact, we just wind $\Omega$ in a spiral, which remains inside a disk since

$$\int_0^\infty a(x)\,dx<\infty.$$

Then $f\circ\varphi^{-1}$ is not in $L^p(\Omega')$, but every derivative is.

**4. Definition 4.** Let $0<t\leqslant 1$. An open set $\Omega\subseteq R^n$ is said to have a *Lip t boundary* if there exists a finite open cover $U_1,\dots,U_N$ of $\partial\Omega$ with positive Lebesgue number, and matrices $L_1,\dots,L_N$ such that each pair $U$, $L$ satisfies:

If $(y_1,\dots,y_n)=L(x_1,\dots,x_n)$, there exists $g(y_1,\dots,y_{n-1})$ satisfying a uniform Lipschitz condition of order $t$ such that

$$\Omega\cap U=\{y: y_n>g(y_1,\dots,y_{n-1})\}\cap U.$$

THEOREM 4. *Let $t=k/m\leqslant 1$, and suppose $\Omega$ has Lip $t$ boundary. Then there exists a bounded linear extension operator $\mathscr{E}: L^p_m(\Omega)\to L^p_k(R^n)$.*

Proof. In case $t=1$ this is just Theorem 1. Let $k<m$. Let $U_1,\dots,U_N$, $L_1,\dots,L_N$ be as in Definition 4. Now trivially, $L^p_m(\Omega)\subseteq L^p_{(k,\dots,k,m),L(j)}(\Omega)$ for each $j=1,\dots,N$. But by Theorem 1″ we have a relative extension operator

$$\mathscr{F}_j: L^p_{(k,\dots,k,m),L(j)}(\Omega)\to L^p_{(k,\dots,k,m),L(j)}(R^n),$$

$\mathscr{F}_jf=f$ on $U_j$ in view of Remark 3 of Section 3 (it may be necessary to take $U$ a little smaller than originally given). But $L^p_{(k,\dots,k,m),L(j)}(R^n)\subseteq L^p_k(R^n)$. Thus $\mathscr{F}_j: L^p_m(\Omega)\to L^p_k(R^n)$. We obtain an extension operator from the relative extension operators via a partition of unity as in Theorem 1′.

Remark. We get as a corollary weaker versions of the Sobolev inequalities for domains with *Lip t* boundary. The details are straightforward and are left to the reader. Similar results have been obtained by Hurd [8], although his boundary conditions are of a different nature.

The following example of such a result may be of interest:

COROLLARY. *Suppose $\Omega$ has a Lip $t$ boundary for some $t > 0$. Then*

$$L^p_\infty(\Omega) = \bigcap_{k=1}^{\infty} L^p_k(\Omega) \subseteq C^\infty_0(\Omega).$$

Of course, the inclusion $L^p_\infty(\Omega) \subseteq C^\infty(\Omega)$ holds for all $\Omega$ since it is essentially a local result.

**5.** Here we give some applications to partial differential equations.

THEOREM 5. *Let $\Omega$ be bounded and satisfy the strong cone condition. If $f \in L^2_k(\Omega)$ and $P(D)$ is any constant coefficient linear partial differential operator, then the equation $P(D)u = f$ can be solved with $u \in L^2_k(\Omega)$; in fact, the solution can be given by a bounded linear transformation on $L^2_k(\Omega)$.*

Proof. Let $\mathscr{E}_k$ be the extension operator of Theorem 1, and let $E$ be a fundamental solution of $P(D)$ given in [5]. $E$ has the property that

$$E: L^2(R^n) \to L^2_{\mathrm{loc}}(R^n),$$

$E$ commutes with translations and $P(D)Ef = f$. It follows that

$$E: L^2_k(R^n) \to L^2_{k,\mathrm{loc}}(R^n).$$

The solution to $P(D)u = f$ is given by $u = E\mathscr{E}_k f|_U$. It is easily seen to have the desired properties since $\Omega$ is bounded.

THEOREM 6. *Let $\Omega$ be bounded and satisfy the strong cone condition and the strong $a(k)$-$L(k)$-cone condition for $k = 1, \ldots, m$. Let $D_1, \ldots, D_M$ be differential monomials of the form $D^{(1)}_{L(1)} \ldots D^{(m)}_{L(m)}$ with $\beta(k)a(k) \leqslant 1$ and including all such monomials with, for each $k$, $\beta(k) = (0, \ldots, 0, a_j(k), 0, \ldots, 0)$ for some $j$ depending on $k$. Let $\mathscr{P}$ be the (finitely-generated) module over the polynomials of $M$-tuples $P_1, \ldots, P_M$ of differential polynomials satisfying*

$$P_1 D_1 + \ldots + P_M D_M = 0.$$

*Let*

$$L^p_k(\Omega, \mathscr{P}) = \{u \in L^p_k(\Omega)^M : \mathscr{P}u = 0\}.$$

*Then there exists a bounded linear extension operator $\mathscr{E}_{\mathscr{P}}: L^p_k(\Omega, \mathscr{P}) \to L^p_k(R^n, \mathscr{P})$, provided $H^1(\Omega, R) = 0$.*

Proof. The fundamental theorem for over determined systems ([3], [6], [10]) states that for convex $\Omega$ the equations $u_j = D_j v$ $(j = 1, \ldots, M)$ will have a solution $v \in \mathscr{D}'(\Omega)$ if and only if $\mathscr{P}u = 0$. By Theorem 3'

$$v \in L^p_{a(1),\ldots,a(m+1),L(1),\ldots,L(m),I}, \quad \text{where} \quad a(m+1) = (k, \ldots, k).$$

Now $v$ is not unique, but the space of solutions to the homogeneous equations $D_j w = 0$ $(j = 1, \ldots, M)$ is finite-dimensional. It is this fact which both allows us to replace the hypothesis $\Omega$ convex by the cohomology condition $H^1(\Omega, R) = 0$, and also enables us to construct a bounded linear map $u \to v$ by normalizing with conditions $(v, \theta_k) = 0$ for certain $\theta_k \in C^\infty_{\mathrm{com}}(\Omega)$. The desired extension map is $u \to v \to D_j\mathscr{E}v$, where $\mathscr{E}$ is the extension operator of Theorem 1' for $L^p_{a(1),\ldots a,(m+1),L(1),\ldots,L(m),I}(\Omega)$.

If we take $m = 1$, $a = (1, \ldots, 1)$, $D_1, \ldots, D_n = \partial/\partial x_1, \ldots, \partial/\partial x_n$, the system $\mathscr{P}u = 0$ is equivalent to $\operatorname{curl} u = 0$ (i.e., $\partial u_k/\partial x_j = \partial u_j/\partial x_k$).

In general, pick a set of generators of $\mathscr{P}$, so that $\mathscr{P}u = 0$ is equivalent to

$$\sum_{j=1}^{M} P_j(i) u_j = 0 \quad \text{for} \quad i = 1, \ldots, N.$$

Then if we take $k - n/p$ greater than the highest degree of the $P_j(i)$, we have the above equations holding in the ordinary sense for $u \in L^p_k(\mathscr{P})$.



**References**


[1] S. Agmon, *Lectures on elliptic boundary value problems*, 1965.

[2] A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Symp. on Pure Math. 5 (1961), p. 33-49.

[3] L. Ehrenpreis, *A fundamental principle*, Intern. Symp. on Linear Spaces, Jerusalem 1961, p. 161-174.

[4] E. B. Fabes and N. M. Rivière, *Singular integrals with mixed homogeneity*, Studia Math. 27 (1966), p. 19-38.

[5] L. Hörmander, *Linear partial differential operators*, Berlin 1963.

[6] — *An introduction to complex analysis in several variables*, 1966.

[7] R. Hunt, *An extension of the Marcinkiewicz interpolation theorem*, Bull. Amer. Math. Soc. 70 (1964), p. 803-807; *Addendum*, ibidem 71 (1965), p. 396.

[8] A. Hurd, *Boundary regularity in the Sobolev embedding theorems*, Canadian J. of Math. 18 (1966), p. 350-356.

[9] J. Lions, *Problèmes aux limites dans les équations aux dérivées partielles*, Montréal 1962.

[10] B. Malgrange, *Sur les systèmes différentielles à coefficients constants*, Séminaire Leray, Collège de France, 1961-62, exposé 8 et 8a.

[11] L. P. Volevich and B. P. Paneyakh, *Certain spaces of generalized functions and embedding theorems*, Russian Mathematical Surveys 20 (1965), p. 1-74.


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