and thus M^*_q is bounded. But in virtue of Lemma 1, $\log M^*_q$ is a convex function of $1/q$ $(1 \leqslant q \leqslant \infty)$, and it follows that M^*_q is bounded (independently of N and the choice of $n(t)$ and $\lambda(n)$) for all q in the range $2 < q < \infty$. Thus

$$\int_0^1 | s_{n(t)}(t) |^q \, dt \leqslant A_q \sum_{n=0}^{\infty} | c_n |^q (\lambda(n)+1)^{q-2}. \tag{9.1}$$

We can choose $\lambda(n)$ so that the right hand side of (9.1) is identically

$$\sum_{n=0}^{\infty} c_n^{*q} (n+1)^{q-2},$$

and we have

$$\int_0^1 S^q(t)\, dt = \sup \int_0^1 | s_{n(t)}(t) |^q \, dt \leqslant A_q \sum_{n=0}^{\infty} c_n^{*q} (n+1)^{q-2}.$$

References.

G. H. Hardy and J. E. Littlewood, [1] Some new properties of Fourier constants, Math. Ann. 97 (1926) p. 159—209.

[2], Notes on the theory of series (XIII): Some new properties of Fourier constants, Journal of the London Math. Soc. 6 (1931) p. 3—9.

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On a theorem of Privaloff

by

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1. Fejér has proved the following theorem. If a trigonometrical series

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{1}$$

is uniformly convergent $(0 \leqslant \theta \leqslant 2\pi)$, the conjugate series

$$\sum_{n=1}^{\infty} (a_n \sin n\theta - b_n \cos n\theta) \tag{2}$$

is convergent almost everywhere in $(0, 2\pi)$[1].

Fejér's result has been extended by Privaloff who has shown that, if the partial sums of the series (1) are uniformly bounded in $(0, 2\pi)$ and the series itself is convergent in a set E of positive measure, the series (2) is convergent almost everywhere in E[2]. We are going to prove a little more general theorem.

Theorem. *If the partial sums s_n of the series* (1) *1° satisfy an inequality*

$$s_n(\theta) > -\varphi(\theta) \qquad (0 \leqslant \theta \leqslant 2\pi), \tag{3}$$

where φ is integrable L[3]*, 2° the series* (1) *is convergent in a set E of positive measure, then* (2) *is convergent almost everywhere in E.*

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[1]) L. Fejér, Über konjugierte trigonometrische Reihen, Crelles Journal 144 (1913).

[2]) I. I. Privaloff, Sur la convergence des séries trigonométriques conjuguées (in russian, with french résumé), Recueil de la Societé Math. de Moscou, 32 (1925) p. 357—363.

[3]) In particular if $s_n \geqslant 0$.

2. PRIVALOFF, in proving his theorem, worked with POISSON's formula. We find it more convenient to use instead a formula, due to F. RIESZ, for the derivative of a trigonometrical polynomial. This formula, which has proved useful in similar problems[4]), is

$$\frac{s_n'(\theta)}{n+1}=\frac{2}{\pi}\int_{-\pi}^{\pi} s_n(t+\theta)\sin(n+1)t K_n(t)\,dt, \tag{4}$$

where K_n denotes FEJÉR's well known kernel. It may be obtained[5]), by considering the obvious equality

$$\frac{s_n'(\theta)}{n+1}=\frac{1}{\pi(n+1)}\int_{-\pi}^{\pi} s_n(t+\theta)\{\sin t+2\sin 2t+\ldots+n\sin nt\}\,dt$$

and adding to the sum in the brackets the expression

$$(n+1)\sin(n+1)t+n\sin(n+2)t+\ldots+\sin(2n+1)t.$$

In the same way we get the formula for the conjugate polynomial $\bar{s}_n$:

$$\frac{\bar{s}_n'(\theta)}{n+1}=\frac{2}{\pi}\int_{-\pi}^{\pi} s_n(t+\theta)\cos(n+1)tK_n(t)\,dt \tag{5}$$

due to G. SZEGÖ[6]).

From (3) it follows that the integrals of $|s_n(\theta)|$ in $(0, 2\pi)$ form a bounded sequence, and so the series (1), when integrated, is, without its linear term, the FOURIER series of a function of bounded variation. Consequently the series (1) and (2) are summable $(C, 1)$ almost everywhere in $(0, 2\pi)$. Denoting the partial sums and the arithmetical means of the series (2) respectively by $\bar{s}_n(\theta)$, $\bar{\sigma}_n(\theta)$, we have the equality

$$\bar{\sigma}_n(\theta)-\bar{s}_n(\theta)=\frac{s_n'(\theta)}{n+1}$$

[4]) Cf. for instance, R. E. A. C. Paley and A. Zygmund, On the partial sums of Fourier series, Studia Math. 2 (1930) p. 221—227.

[5]) F. Riesz, Sur les polynomes trigonométriques, Comptes Rendus 158 (1914) p. 1657—1661.

[6]) G. Szegö, Über einen Satz des Herrn Serge Bernstein, Schriften der Königsberger Gelehrten Gesellschaft, Naturwissenschaftliche Klasse 5 (1928) p. 59—70. As a matter of fact, formulae (4) and (5) have been proved (loc. cit.) in a little stronger form, but the actual form is more convenient to us.

and so our problem reduces to showing that $s_n'(\theta)=o(n)$ almost everywhere in E.

3. We begin by proving a result, which although more general than PRIVALOFF's, is a special case of the theorem enounced. We suppose namely that (1) is the Fourier series of a function $f\in L^p$, and that $\varphi\in L^p$ $(p>1)$. In that case there exists a function $\psi\in L$ (and even $\psi\in L^p$), such that[7])

$$|s_n(\theta)|<\psi(\theta) \qquad (0\leqslant\theta\leqslant 2\pi). \tag{6}$$

Lemma. If the partial sums of the series (1) satisfy the inequality (6) with $\psi\in L$, and if in a set $\mathcal{E}$, $m(\mathcal{E})>0$, we have

$$|s_n(\theta)|\leqslant\epsilon \qquad (n\geqslant n_0), \tag{7}$$

then

$$\overline{\lim_{n\to\infty}}\left|\frac{s_n'(\theta)}{n+1}\right|\leqslant 2\varepsilon \tag{8}$$

almost everywhere in $\mathcal{E}$.

In fact, we may suppose without loss of generality that $\varphi(\theta)\geqslant 0$. Let $\chi(\theta)$ be the function equal to 1 in $\mathcal{E}$ and to 0 elsewhere (mod 2π) and $\psi^*(\theta)=\chi(\theta)\psi(\theta)\geqslant 0$. Let θ_0 be an arbitrary point of $\mathcal{E}$ where the integral of $\psi^*(\theta)$ has 0 for its derivative[8]). Then, decomposing in the inequality

$$\left|\frac{s_n'(\theta_0)}{n+1}\right|=\left|\frac{2}{\pi}\int_{-\pi}^{\pi} s_n(t)\sin(n+1)(t-\theta_0)K_n(t-\theta_0)\,dt\right|$$

$$\leqslant\frac{2}{\pi}\int_{-\pi}^{\pi}\psi(t)K_n(t-\theta_0)\,dt$$

the last integral into two, extended over the sets $\mathcal{E}$ and $C\mathcal{E}$, and denoting them by A_n and B_n, we have (for $n\geqslant n_0$)

$$|A_n|\leqslant\frac{2}{\pi}\,\varepsilon\int_{\mathcal{E}} K_n(t-\theta_0)\,dt\leqslant\frac{2}{\pi}\,\varepsilon\int_{-\pi}^{\pi}K_n(t)\,dt=2\varepsilon,$$

$$|B_n|\leqslant\frac{2}{\pi}\int_{C\mathcal{E}}\psi(t)K_n(t-\theta_0)\,dt=\frac{2}{\pi}\int_{-\pi}^{\pi}\psi^*(t)K_n(t-\theta_0)\,dt.$$

[7]) See R. E. A. C. Paley and A. Zygmund, loc. cit.

[8]) Considering such points is suggested by Privaloff's paper.

Using LEBESGUE's well known test for the summability $(C, 1)$ of FOURIER series we see that $B_n \to 0$ and (8) follows at once.

4. Now, if (1) converges in a set E, it may be found, by the theorem of EGOROFF, a subset $\mathcal{E}$ of E, its measure differing as little as we like from $m(E)$, in which (1) would converge uniformly. It follows, that, for a given $\varepsilon > 0$, there exists an integer n_0 such that

$$|s_n(\theta) - s_{n_0}(\theta)| < \varepsilon \qquad (\theta \in \mathcal{E},\ n \geqslant n_0).$$

We may represent the series (1) as the sum of two, the first being the polynomial s_{n_0}. Let $t_n = s_n - s_{n_0}$ $(n > n_0)$ denote the partial sums of the second. We have then

$$|t_n(\theta)| < \varepsilon \quad (n \geqslant n_0,\ \theta \in \mathcal{E});\ |t_n(\theta)| \leqslant \psi(\theta) + M = \psi_1(\theta) \quad (0 \leqslant \theta \leqslant 2\pi),$$

where $M = \operatorname{Max} |s_{n_0}(\theta)|$. Hence, by our lemma,

$$\overline{\lim_{n\to\infty}} \left| \frac{t_n'(\theta)}{n+1} \right| \leqslant 2\varepsilon,$$

$$\overline{\lim_{n\to\infty}} \frac{|s_n'(\theta)|}{n+1} \leqslant 2\varepsilon,$$

$$\lim_{n\to\infty} \frac{s_n'(\theta)}{n+1} = 0,$$

almost everywhere in $\mathcal{E}$. Evidently the last equality holds also almost everywhere in E.

5. Now we pass to the general case (3), where ψ is integrable L. In this case it may be shown[9]) that we have still (6), but with $\psi \in L^{1-\varepsilon}$ and so our previous argument fails. In the subsequent proof we shall not use any result of the paper just referred to.

As we have already mentionned, the series (1), under the condition (3), is the differentiated FOURIER series of a function of bounded variation. In other words, (1) is, what may be called, the FOURIER-STIELTJES series, i. e. there exists a function $F(\theta)$ of bounded variation, such that

$$(9) \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n\theta \, dF(\theta), \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta \, dF(\theta) \qquad (n = 0, 1, 2, \ldots).$$

[9]) See Studia Mathematica, loc. cit.

Generally $F(\pi) \neq F(-\pi)$, but if $a_0 = 0$, then $F(\pi) = F(-\pi)$ and we may define $F(\theta)$ for all θ, setting $F(\theta + 2\pi) = F(\theta)$. The first arithmetical means $\sigma_n(\theta)$ of the series (1) may then be written in the form

$$(10) \qquad \sigma_n(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t-\theta)\, dF(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t)\, d_t F(t+\theta)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t)\, d_t [F(\theta+t) - F(\theta-t)],$$

where the symbol d_t means that the variation is taken with respect to t. As in the general case $F(\theta)$ is the sum of $\frac{1}{2} a_0 t$ and of a periodic function, it will be readily seen that (10) remains true even if $a_0 \neq 0$.

6. Let β be any positive integer such that $\beta/n < \pi$. Taking into account that

$$(11) \qquad K_n(t) < \frac{\pi^2}{8} n \quad (0 \leqslant t \leqslant \beta/n), \qquad K_n(t) < \frac{\pi^2}{4nt^2} \quad (\beta/n \leqslant t \leqslant \pi),$$

we get from (10) that

$$(12) \qquad |\sigma_n(\theta)| \leqslant n \int_0^{\beta/n} |d_t(F(\theta+t) - F(\theta-t))|$$

$$+ \frac{1}{n} \int_{\beta/n}^{\pi} \frac{|d_t(F(\theta+t) - F(\theta-t))|}{t^2}.$$

Replacing in (5) $s_n(t+\theta)$ by $(s_n(t+\theta) + \varphi(t+\theta)) - \varphi(t+\theta)$, we have

$$(13) \qquad \frac{|\overline{s_n'}(\theta)|}{n+1} \leqslant \frac{2}{\pi} \int_{-\pi}^{\pi} (s_n(\theta+t) + \varphi(\theta+t)) K_n(t)\, dt$$

$$+ \frac{2}{\pi} \int_{-\pi}^{\pi} \varphi(t+\theta) K_n(t)\, dt = 2\sigma_n(\theta) + \frac{4}{\pi} \int_{-\pi}^{\pi} \varphi(\theta+t) K_n(t)\, dt.$$

From the equality

$$s_n(\theta) = \sigma_n(\theta) + \frac{\overline{s}_n'(\theta)}{n+1}$$

we get

$$s_n(\theta) + \varphi(\theta) = |s_n(\theta) + \varphi(\theta)| \leqslant |\sigma_n(\theta) + \varphi(\theta)| + \frac{|\overline{s}_n'(\theta)|}{n+1}$$

$$= \sigma_n(\theta) + \varphi(\theta) + \left|\frac{s_n'(\theta)}{n+1}\right|.$$

Consequently, using (13),

$$(14) \qquad s_n(\theta) \leqslant \sigma_n(\theta) + \left|\frac{\overline{s}_n'(\theta)}{n+1}\right| \leqslant \tau_n(\theta),$$

where τ_n denotes the arithmetical means of the series we get by multiplying (1) by 3 and adding the fourfold FOURIER series of $\varphi(\theta)$. This new series is also a FOURIER-STIELTJES series and so for $\tau_n(\theta)$ we have the inequality analogous to (12)

$$(15) \qquad |\tau_n(\theta)| \leqslant n \int_0^{\beta/n} |d_t(G(\theta+t) - G(\theta-t))|$$

$$+ \frac{1}{n} \int_{\beta/n}^{\pi} \frac{|d_t(G(\theta+t) - G(\theta-t))|}{t^2},$$

where G is a function of bounded variation. From (3) and (14) it follows that

$$(16) \qquad |s_n(\theta)| \leqslant |\tau_n(\theta)| + \varphi(\theta).$$

Now we are going to prove the following

Lemma. If the partial sums of the series (1) satisfy (3) with φ integrable L, and if in a set $\mathcal{E}$ of positive measure we have (7), then (8) is true almost everywhere in $\mathcal{E}$.

Let θ_0 be a point of density of $\mathcal{E}$, such that

$$(17) \qquad \int_0^h |d_t G(\theta_0 \pm t)| = O(h).$$

It is well known that (17) is true for allmost all θ_0. Integrating by part one deduces easily from (17) the inequality

$$(18) \qquad \int_h^{\pi} \frac{|d_t G(\theta_0 \pm t)|}{t^2} = O(h^{-1}).$$

Without loss of generality we may suppose that $\theta_0 = 0$. Let $\mathcal{E}(\alpha, \beta)$ denote the portion of $\mathcal{E}$ belonging to an interval (α, β).

From (4) we get (putting $C\mathcal{E} = H$)

$$\frac{|s_n'(\theta)|}{n+1} \leqslant \frac{2}{\pi} \varepsilon \int_{\mathcal{E}} K_n(t)\,dt + \frac{2}{\pi} \int_H |s_n(t)| K_n(t)\,dt$$

and it remains to be shown that the last integral tends to 0.

Inequality (16) shows that the integral considered is less than

$$(19) \qquad \frac{2}{\pi} \int_H \varphi(t) K_n(t)\,dt + \frac{2}{\pi} \int_H |\tau_n(t)| K_n(t)\,dt.$$

Putting $\varphi^*(\theta) = \varphi(\theta)\chi(\theta)$, where $\chi(\theta)$ is the characteristic function of the set $\mathcal{E}$, and supposing that the integral of $\varphi^*(\theta)$ has for $\theta = 0$ the derivative vanishing, we prove, as in the case of the preceeding lemma, that the first of the integrals (19) tends to zero. The second is equal to

$$\frac{2}{\pi} \int_{H(0,\pi)} |\tau_n(t)| K_n(t)\,dt + \frac{2}{\pi} \int_{H(-\pi,0)} |\tau_n(t)| K_n(t)\,dt.$$

We shall estimate only the first of these two terms, for the second may be dealt with in the same way.

8. Let us put $\alpha = \beta/2$,

$$(20) \qquad \int_{H(0,\pi)} |\tau_n(u)| K_n(u)\,du = \int_{H(0,\alpha/n)} + \int_{H(\alpha/n,\pi)} = A_n^1 + A_n^2.$$

Using (15) and (11) we have

$$A_n^1 \leqslant 2n^2 \int_{H(0,\alpha/n)} du \int_0^{\beta/n} (|d_t G(u+t)| + |d_t G(u-t)|)$$

$$+ 2 \int_{H(0,\alpha/n)} du \int_{\beta/n}^{\pi} \frac{|d_t G(u+t)| + |d_t G(u-t)|}{t^2} = A_n^{1,1} + A_n^{1,2}.$$

From (17) we deduce that

$$A_n^{1,1} = 2n^2 \int_{H(0,\alpha/n)} O\left(\frac{1}{n}\right) du = o(1).$$

From (18) it follows that

$$A_n^{1,2} = 2 \int_{H(0,\alpha/n)} O(n)\,du = o(1).$$

Further

$$A_n^2 \leqslant 4 \int\limits_{H(\alpha/n,\pi)} \frac{du}{u^2} \int\limits_0^{\beta/n} (|d_t G(u+t)| + d_t |G(u-t)|)$$

$$+\frac{4}{n^2} \int\limits_{H(\alpha/n,\pi)} \frac{du}{u^2} \int\limits_{\beta/n}^{\pi} \frac{|d_t G(u+t)| + |d_t G(u-t)|}{t^2} = A_n^{2,1} + A_n^{2,2}.$$

Let $\gamma = \beta^2 > \beta$. Then

$$A_n^{2,1} = 4 \int\limits_{H(\alpha/n,\gamma/n)} \frac{du}{u^2} \int\limits_0^{\beta/n} (|d_t G(u+t)| + |d_t G(u-t)|)$$

$$+4 \int\limits_{H(\gamma/n,\pi)} \frac{du}{u^2} \int\limits_0^{\beta/n} (|d_t G(u+t)| + |d_t G(u-t)|) = I_n + K_n.$$

$$I_n = 4 \int\limits_{H(\alpha/n,\gamma/n)} O\left(\frac{1}{n}\right) \frac{du}{u^2} = o(1).$$

If we set (for $h > 0$)

$$\int\limits_{-0}^{+h} |d\,G(t)| = L(h),$$

we may write

$$K_n = 4 \int\limits_{H(\gamma/n,\pi)} \frac{du}{u^2} \int\limits_{-\frac{\beta}{n}}^{\frac{\beta}{n}} |d_t G(u+t)|$$

$$= 4 \int\limits_{H(\gamma/n,\pi)} \frac{du}{u^2} \{L(u+\frac{\beta}{n}) - L(u-\frac{\beta}{n})\} \leqslant 4 \int\limits_{\gamma/n}^{\pi} \frac{du}{u^2} \{L(u+\frac{\beta}{n}) - L(u-\frac{\beta}{n})\}$$

$$= o(1) + 4 \int\limits_{\frac{\gamma+\beta}{n}}^{\pi} \frac{L(u)\,du}{(u-\frac{\beta}{n})^2} - 4 \int\limits_{\frac{\gamma-\beta}{n}}^{\pi} \frac{L(u)\,du}{(u+\frac{\beta}{n})^2}$$

$$\leqslant o(1) + 4 \int\limits_{\frac{\gamma-\beta}{n}}^{\pi} L(u) \left\{ \frac{1}{(u-\frac{\beta}{n})^2} - \frac{1}{(u+\frac{\beta}{n})^2} \right\} du$$

$$= o(1) + \frac{6\beta}{n} \int\limits_{\frac{\gamma-\beta}{n}}^{\pi} L(u) \frac{u\,du}{(u-\frac{\beta}{n})^2 (u+\frac{\beta}{n})^2}.$$

As $L(u) <$ const. u, it will be readily seen that the last integral is less than a given number $\delta > 0$, provided β *is* sufficiently large. Hence

$$A_n^{2,1} \leqslant o(1) + \delta.$$

Now

$$A_n^{2,2} \leqslant \frac{4}{n^2} \int\limits_{\alpha/n}^{\pi} \frac{du}{u^2} \int\limits_{\alpha/n}^{\pi} \frac{|d_t G(u+t)| + |d_t G(u-t)|}{t^2}$$

$$= \frac{8}{n^2} \int\limits_{\alpha/n}^{\pi} \frac{du}{u^2} \int\limits_{u}^{\pi} \frac{|d_t G(u+t)| + |d_t G(u-t)|}{t^2} \quad [10]$$

$$= o(1) + \frac{8}{n^2} \int\limits_{\alpha/n}^{\pi/2} \frac{du}{u^2} \int\limits_{u}^{2u} \frac{|d_t G(u+t)| + |d_t G(u-t)|}{t^2}$$

$$+ \frac{8}{n^2} \int\limits_{\alpha/n}^{\pi/2} \frac{du}{u^2} \int\limits_{2u}^{\pi} \frac{|d_t G(u+t)| + |d_t G(u-t)|}{t^2} = o(1) + P_n + Q_n.$$

$$P_n \leqslant \frac{8}{n^2} \int\limits_{\alpha/n}^{\pi/2} \frac{du}{u^2} \cdot \frac{1}{u^2} \int\limits_{u}^{cu} (|d_t G(u+t)| + |d_t G(u-t)|) \leqslant \frac{c}{n^2} \int\limits_{\alpha/n}^{\infty} \frac{du}{u^3}.$$

If $\alpha (=\beta/2)$ is large enough, the last expression is less than δ. From (18) (with $\theta_0 = 0$) we deduce that

$$Q_n = \frac{8}{n^2} \int\limits_{\alpha/n}^{\pi/2} \frac{1}{u^2} \cdot o\left(\frac{1}{u}\right) du < \delta$$

for α sufficiently large. Hence.

$$A_n^{2,2} \leqslant P_n + Q_n \leqslant o(1) + 2\delta.$$

Consequently the integral (20) is less than

$$o(1) + o(1) + o(1) + \delta + o(1) + 2\delta = o(1) + 3\delta.$$

As δ is arbitrarily small the truth of the lemma follows. To deduce the theorem we proceed as in the case dealt with previously.

9. It is evident that if (3) is true in an interval $\alpha < \theta < \beta$, and if $E \subset (\alpha, \beta)$, then the conclusions of the theorem are still valid.

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[10]) (Added 29. 12. 31). Supposing, as we may, that $G(t)$ is either even or odd.

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