and thus $M^*_q$ is bounded. But in virtue of Lemma 1, $\log M^*_q$ is a convex function of $1/q$ ($1 \leq q < \infty$), and it follows that $M^*_q$ is bounded (independently of $N$ and the choice of $n(t)$ and $\lambda(n)$) for all $q$ in the range $2 < q < \infty$. Thus

\[ \int_0^1 |s_n(t)|^q dt \leq A_q \sum_{n=0}^{\infty} |c_n|^q (\lambda(n) + 1)^{q-2}. \tag{9.1} \]

We can choose $\lambda(n)$ so that the right hand side of (9.1) is identically

\[ \sum_{n=0}^{\infty} c_n^\ast (n+1)^{q-2}, \]

and we have

\[ \int_0^1 S^\ast(t) dt = \sup \int_0^1 |s_n(t)|^q dt \leq A_q \sum_{n=0}^{\infty} c_n^\ast (n+1)^{q-2}. \]

References.


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On a theorem of Privaloff

by

A. ZYGMUND (Wilno).

1. Fejér has proved the following theorem. If a trigonometric series

\[ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \]

is uniformly convergent ($0 \leq \theta \leq 2\pi$), the conjugate series

\[ \sum_{n=1}^{\infty} (a_n \sin n\theta - b_n \cos n\theta) \]

is convergent almost everywhere in $(0, 2\pi)$.

Fejér's result has been extended by Privaloff who has shown that, if the partial sums of the series (1) are uniformly bounded in $(0, 2\pi)$ and the series itself is convergent in a set $E$ of positive measure, the series (2) is convergent almost everywhere in $E$.

We are going to prove a little more general theorem.

Theorem. If the partial sums $s_n$ of the series (1) satisfy an inequality

\[ s_n(\theta) \geq q(\theta) \quad (0 \leq \theta \leq 2\pi), \]

where $q$ is integrable $L^1$), then the series (1) is convergent in a set $E$ of positive measure, then (2) is convergent almost everywhere in $E$.

\[ 1) \text{L. Fejér, Uber konjugierte trigonometrische Reihen, Crelles Journal 144 (1915).} \]

\[ 2) \text{I. I. Privaloff, Sur la convergence des séries trigonométriques conjuguées (in russian, with french résumé), Recueil de la Société Math. de Moscou, 32 (1925) p. 357—363.} \]

\[ 3) \text{In particular if } s_n \geq 0. \]
2. Privaloff, in proving his theorem, worked with Poisson's formula. We find it more convenient to use instead a formula, due to F. Riesz, for the derivative of a trigonometrical polynomial. This formula, which has proved useful in similar problems, is

\[ s_n'(\theta) \left( \frac{t}{n+1} \right) = \frac{2}{\pi} \int_{-\pi}^{\pi} s_n(t + \theta) \sin (n + 1) t K_n(t) dt, \]

where \( K_n \) denotes Fejér's well known kernel. It may be obtained by considering the obvious equality

\[ \frac{s_n'(\theta)}{n+1} = \frac{1}{\pi(n+1)} \int_{-\pi}^{\pi} s_n(t + \theta) \sin (n + 1) t K_n(t) dt \]

and adding to the sum in the brackets the expression

\[(n+1) \sin (n+1) t + n \sin (n+2) t + \ldots + \sin (2n+1) t.\]

In the same way we get the formula for the conjugate polynomial \( \overline{s}_n \):

\[ \frac{\overline{s}_n'(\theta)}{n+1} = \frac{2}{\pi} \int_{-\pi}^{\pi} \overline{s_n}(t + \theta) \cos (n + 1) t K_n(t) dt \]

due to G. Szegö.

From (3) it follows that the integrals of \(| s_n(\theta) | \) in \((0, 2\pi)\) form a bounded sequence, and so the series (1), when integrated, is, without its linear term, the Fourier series of a function of bounded variation. Consequently the series (1) and (2) are summable \((C, 1)\) almost everywhere in \((0, 2\pi)\). Denoting the partial sums and the arithmetical means of the series \((2)\) respectively by \( s_n(\theta), \overline{s}_n(\theta) \), we have the equality

\[ \overline{s}_n(\theta) - \overline{s}_n(\theta) = \frac{s_n'(\theta)}{n+1} \]

and so our problem reduces to showing that \( s_n'(\theta) = o(n) \) almost everywhere in \( E \).

3. We begin by proving a result, which although more general than Privaloff's, is a special case of the theorem enunciated. We assume namely that (1) is the Fourier series of a function \( f \in L^p \), and that \( \psi \in L^p \) \((p > 1)\). In that case there exists a function \( \psi \in L^p \) (and even \( \psi \in L^2 \)), such that

\[ |s_n(\theta)| < \psi(\theta) \quad (0 \leq \theta < 2\pi). \]

Lemma. If the partial sums of the series (1) satisfy the inequality (6) with \( \psi \in L^p \), and if in a set \( \mathcal{E}, m(\mathcal{E}) > 0 \), we have

\[ |s_n(\theta)| < \varepsilon \quad (n \geq n_0), \]

then

\[ \lim_{n \to \infty} \frac{\overline{s}_n(\theta)}{n+1} \leq 2\varepsilon \]

almost everywhere in \( \mathcal{E} \).

In fact, we may suppose without loss of generality that \( \varphi(\theta) > 0 \). Let \( \chi(\theta \equiv 1 \in \mathcal{E} \) and to 0 elsewhere \((\mod 2\pi)\) and \( \psi^p(\theta) = \chi(\theta \equiv \varphi(\theta) > 0) \). Let \( \theta_0 \) be an arbitrary point of \( \mathcal{E} \) where the integral of \( \psi^p(\theta) \) has 0 for its derivative. Then, decomposing in the inequality

\[ \frac{s_n'(\theta)}{n+1} = \frac{2}{\pi} \int_{-\pi}^{\pi} s_n(t) \sin (n + 1) (t - \theta_0) K_n(t - \theta_0) dt \]

the last integral into two, extended over the sets \( \mathcal{E} \) and \( \mathcal{C} \), and denoting them by \( A_n \) and \( B_n \), we have (for \( n \geq n_0)\)

\[ |A_n| \leq \frac{2}{\pi} \varepsilon \int_{\mathcal{E}} K_n(t - \theta_0) dt \leq \frac{2}{\pi} \varepsilon \int_{-\pi}^{\pi} K_n(t) dt = 2\varepsilon, \]

\[ |B_n| \leq \frac{2}{\pi} \int_{\mathcal{C}} \psi(\theta) K_n(t - \theta_0) dt = \frac{2}{\pi} \int_{-\pi}^{\pi} \psi^p(t) K_n(t - \theta_0) dt. \]

\[ \text{Note:} \] See R. E. A. C. Paley and A. Zygmund, loc. cit.

\[ \text{Note:} \] Considering such points is suggested by Privaloff's paper.
Using Lebesgue's well known test for the summability (C, 1) of Fourier series we see that \( B_n \to 0 \) and (8) follows at once.

4. Now, if (1) converges in a set \( E \), it may be found, by the theorem of Egoroff, a subset \( G \) of \( E \), its measure differing as little as we like from \( m(E) \), in which (1) would converge uniformly. It follows, that, for a given \( \varepsilon > 0 \), there exists an integer \( n_0 \) such that

\[
|s_n(\theta) - s_n(\theta)| < \varepsilon \quad (\theta \in G, \quad n \geq n_0).
\]

We may represent the series (1) as the sum of two, the first being the polynomial \( s_n \). Let \( t_n = s_n - s_n \) (\( n > n_0 \)) denote the partial sums of the second. We have then

\[
|t_n(\theta)| < \varepsilon \quad (n > n_0, \quad \theta \in G) ; \quad |t_n(\theta)| \leq \psi(\theta) + M = \psi(\theta) \quad (0 \leq \theta \leq 2\pi),
\]

where \( M = \max |s_n(\theta)| \). Hence, by our lemma,

\[
\lim_{n \to \infty} \frac{t_n(\theta)}{n+1} \leq 2\varepsilon, \quad \lim_{n \to \infty} \frac{s_n(\theta)}{n+1} \leq 2\varepsilon, \quad \lim_{n \to \infty} s_n(\theta) = 0,
\]

almost everywhere in \( G \). Evidently the last equality holds almost everywhere in \( E \).

5. Now we pass to the general case (3), where \( \psi \) is integrable. In this case it may be shown \(^7\) that we have still (6), but with \( \psi \in L^{1-\varepsilon} \) and so our previous argument fails. In the subsequent proof we shall not use any result of the paper just referred to.

As we have already mentioned, the series (1), under the condition (3), is the differentiated Fourier series of a function of bounded variation. In other words, (1) is, what may be called, the Fourier-Stieltjes series, i.e. there exists a function \( F(\theta) \) of bounded variation, such that

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n \theta \, dF(\theta), \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin n \theta \, dF(\theta)
\]

\[(n = 0, 1, 2, \ldots).\]

\(^7\) See Studia Mathematica, loc. cit.

A theorem of Privaloff.

Generally \( F(\pi) \neq F(-\pi) \), but if \( a_0 = 0 \), then \( F(\pi) = F(-\pi) \) and we may define \( F(\theta) \) for all \( \theta \), setting \( F(\theta + 2\pi) = F(\theta) \). The first arithmetical means \( a_n(\theta) \) of the series (1) may then be written in the form

\[
a_n(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t-\theta) \, dF(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t) \, dF(t+\theta)
\]

\[= \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t) \, d(F(\theta + t) - F(\theta - t)),
\]

where the symbol \( d_t \) means that the variation is taken with respect to \( t \). As in the general case \( F(\theta) \) is the sum of \( \frac{1}{2} a_0 t \) and of a periodic function, it will be readily seen that (10) remains true even if \( a_0 \neq 0 \).

6. Let \( \beta \) be any positive integer such that \( \beta|n < \pi \). Taking into account that

\[
K_n(t) < \frac{\pi^2}{8} \quad (0 \leq t \leq \beta|n),
\]

(11)

\[
K_n(t) < \frac{\pi^2}{4n^2} \quad (\beta|n \leq t \leq \pi),
\]

we get from (10) that

\[
|a_n(\theta)| \leq \frac{\beta|n}{\pi} \int_{-\pi}^{\pi} \frac{|d_t[F(\theta + t) - F(\theta - t)]|}{t} \, dt + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{|d_t[F(\theta + t) - F(\theta - t)]|}{t^2} \, dt.
\]

Replacing in (5) \( s_n(\theta + t) \) by \( (s_n(\theta + t) + \varphi(\theta + t) - \varphi(\theta + t)) + \varphi(\theta + t)K_n(t) dt \), we have

\[
\frac{|s_n'(\theta)|}{n+1} \leq \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{(s_n(\theta + t) + \varphi(\theta + t))K_n(t) dt}{n+1} + \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{\varphi(\theta + t)K_n(t) dt}{n+1} = 2a_n(\theta) + \frac{4}{\pi} \int_{-\pi}^{\pi} \varphi(\theta + t)K_n(t) dt.
\]

\[16^*\]
From the equality
\[ s_n(\theta) = \sigma_n(\theta) + \frac{s'_n(\theta)}{n+1} \]
we get
\[ s_n(\theta) + \varphi(\theta) = |s_n(\theta) + \varphi(\theta)| \leq |\sigma_n(\theta) + \varphi(\theta)| + \frac{|s'_n(\theta)|}{n+1} \]
\[ = \sigma_n(\theta) + \varphi(\theta) + \frac{s'_n(\theta)}{n+1} \]
Consequently, using (13),
\[ s_n'(\theta) \leq s_n(\theta) + \frac{s'_n(\theta)}{n+1} \leq \varphi_n(\theta), \]
where \( \varphi_n \) denotes the arithmetical means of the series we get by multiplying (1) by 3 and adding the fourfold Fourier series of \( \varphi(\theta) \).
This new series is also a Fourier-Stieltjes series and so for \( \varphi_n(\theta) \) we have the inequality analogous to (12)
\[ |\varphi_n(\theta)| \leq n \int_{0}^{\pi} |d_1(G(\theta + t) - G(\theta - t))| dt \]
\[ + \frac{1}{n} \int_{0}^{\pi} |d_1(G(\theta + t) - G(\theta - t))| dt \]
where \( G \) is a function of bounded variation. From (3) and (14) it follows that
\[ \|s_n(\theta)\| \leq |\varphi_n(\theta)| + \varphi(\theta). \]
Now we are going to prove the following

\[ \int_{\theta_0}^{\theta_1} |d_1(G(\theta + t)| = O(h). \]
It is well known that (17) is true for almost all \( \theta \), integrating by part one deduces easily from (17) the inequality
\[ \int_{\theta_0}^{\theta_1} |d_1(G(\theta + t)| = O(h^{-1}). \]

Without loss of generality we may suppose that \( \theta_0 = 0 \). Let \( \mathcal{G}(\alpha, \beta) \) denote the portion of \( \mathcal{G} \) belonging to an interval \((\alpha, \beta)\).
From (4) we get (putting \( C \mathcal{G} = H \))
\[ \frac{s_n'(\theta)}{n+1} \leq \frac{2}{\pi} \int_{\mathcal{G}} K_n(\theta) dt + \frac{2}{\pi} \int_{H} |s_n(\theta)| K_n(\theta) dt \]
and it remains to be shown that the last integral tends to 0.
Inequality (16) shows that the integral considered is less than
\[ \frac{2}{\pi} \int_{H} |\varphi_n(\theta)| K_n(\theta) dt + \frac{2}{\pi} \int_{H} |\varphi_n(\theta)| K_n(\theta) dt. \]
Putting \( \rho_n(\theta) = \varphi(\theta) \chi(\theta) \), where \( \chi(\theta) \) is the characteristic function of the set \( \mathcal{G} \), and supposing that the integral of \( \rho_n(\theta) \) has for \( \theta = 0 \) the derivative vanishing, we prove, in the case of the preceding lemma, that the first of the integrals (19) tends to zero.
The second is equal to
\[ \frac{2}{\pi} \int_{H} |\varphi_n(\theta)| K_n(\theta) dt + \frac{2}{\pi} \int_{H} |\varphi_n(\theta)| K_n(\theta) dt. \]
We shall estimate only the first of these two terms, for the second may be dealt with in the same way.
8. Let us put \( a = \beta/2 \),
\[ \int_{\theta_n}^{\theta_{n+1}} |d_1(G(\theta + t)| = \int_{H} + \int_{H} = A_{\alpha}^1 + A_{\beta}^1. \]
Using (15) and (11) we have
\[ A_{\alpha}^1 \leq 2n \int_{H} |d_1(G(\theta + t)| + |d_1(G(\theta - t)| dt \]
\[ + 2 \int_{H} |d_1(G(\theta + t)| + |d_1(G(\theta - t)| dt \]
From (17) we deduce that
\[ A_{\alpha}^1 = 2n \int_{H} O \left( \frac{1}{\pi} \right) dt = o(1). \]
From (18) it follows that
\[ A_{\beta}^1 = 2 \int_{H} O(n) dt = o(1). \]
Further

\begin{align*}
A_2^2 & \leq 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta (|d_1 G(u + \theta)| + d_1 |G(u - \delta)|) \\
+ \frac{4}{n^2} \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta (|d_1 G(u + \theta)| + |d_1 G(u - \delta)|) = A_n^{2,1} + A_n^{2,2}.
\end{align*}

Let \( \gamma = \beta^* > \beta \). Then

\begin{align*}
A_n^{2,1} &= 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta (|d_1 G(u + \theta)| + |d_1 G(u - \theta)|) \\
+ \frac{4}{n^2} \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta (|d_1 G(u + \theta)| + |d_1 G(u - \theta)|) = I_n + K_n,
\end{align*}

where

\begin{align*}
I_n &= 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} = o(1), \\
K_n &= 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta |d_1 G(u + \theta)| \leq o(1). \\
\end{align*}

If we set (for \( h > 0 \))

\begin{align*}
\int_0^h \frac{dG(t)}{n} = L(h),
\end{align*}

we may write

\begin{align*}
K_n &= 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta |d_1 G(u + \theta)| \\
&= 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \left( L(u + \frac{\beta}{n}) - L(u - \frac{\beta}{n}) \right) \\
&= o(1) + 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta} \frac{L(u)}{n} \frac{du}{(u^2 - \beta^2)^{\frac{1}{2}}} - 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta} \frac{L(u)}{n} \frac{du}{(u^2 + \beta^2)^{\frac{1}{2}}} \\
&\leq o(1) + 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta} \frac{L(u)}{n} \left( \frac{1}{(u^2 - \beta^2)^{\frac{1}{2}}} + \frac{1}{(u^2 + \beta^2)^{\frac{1}{2}}} \right) du \\
&= o(1) + \frac{6\beta^2}{n} \int_{\mathbb{H}(\infty, \infty)}^{\beta} \frac{du}{n} \left( \frac{u}{(u^2 - \beta^2)^{\frac{1}{2}}} + \frac{u}{(u^2 + \beta^2)^{\frac{1}{2}}} \right).
\end{align*}

As \( L(u) < \text{const.} \), it will be readily seen that the last integral is less than a given number \( \delta > 0 \), provided \( \beta \) is sufficiently large. Hence

\begin{align*}
A_n^{2,1} \leq o(1) + \delta.
\end{align*}

Now

\begin{align*}
A_n^{2,1} &\leq 4 \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta (|d_1 G(u + \theta)| + |d_1 G(u - \theta)|) \\
&= 8 \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta (|d_1 G(u + \theta)| + |d_1 G(u - \theta)|) \\
&= o(1) + \frac{8}{n^2} \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \int_\delta (|d_1 G(u + \theta)| + |d_1 G(u - \theta)|) \\
&\leq o(1) + P_n + Q_n.
\end{align*}

If \( \alpha (\approx \beta / 2) \) is large enough, the last expression is less than \( \delta \). From (18) (with \( \theta_0 = 0 \)) we deduce that

\begin{align*}
Q_n &= \frac{8}{n^2} \int_{\mathbb{H}(\infty, \infty)}^{\beta_0} \frac{du}{u^2} \left( o\left( \frac{1}{u} \right) \right) du < \delta
\end{align*}

for \( \alpha \) sufficiently large. Hence

\begin{align*}
A_n^{2,1} \leq P_n + Q_n \leq o(1) + 2\delta.
\end{align*}

Consequently the integral (20) is less than

\begin{align*}
o(1) + o(1) + o(1) + \delta + o(1) + 2\delta = o(1) + 3\delta.
\end{align*}

As \( \delta \) is arbitrarily small the truth of the lemma follows. To deduce the theorem we proceed as in the case dealt with previously.

9. It is evident that if (3) is true in an interval \( \alpha < \theta < \beta \), and if \( E \subset (\alpha, \beta) \), then the conclusions of the theorem are still valid.

\[ \text{[Reçu par la Rédaction le 30. 6. 1931.]} \]