Some theorems on orthogonal functions (1),

by

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1. Let $c_0, c_1, c_2, \dots$ denote a bounded set of numbers, and let

$$c_0^* \geqslant c_1^* \geqslant c_2^* \geqslant \dots .$$

denote the set $|c_0|, |c_1|, |c_2|, \dots$ rearranged in descending order of magnitude. Hardy and Littlewood[1]) have proved the following theorems:

Theorem A. *If the series* $\Sigma c_n^* n^{q-2}$ *is convergent, where* $q \geqslant 2$, *then the series*

$$\tfrac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n \cos n\theta$$

is the Fourier series of a function $f(\theta)$ *of class* L^q, *and*

$$\int_0^{2\pi} |f(\theta)|^q \, d\theta \leqslant A_q \sum_{n=0}^{\infty} c_n^{*q} (n+1)^{q-2},$$

where A_q *is a constant depending only on* q.

Theorem B. *Let*

$$\tfrac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n \cos n\theta$$

be the Fourier series of a function $f(\theta)$ *of class* L^p, *where* $1 < p \leqslant 2$. *Then*

$$\sum_{n=0}^{\infty} c_n^{*p} (n+1)^{p-2} \leqslant A_p \int_0^{2\pi} |f(\theta)|^p \, d\theta,$$

where A_p *is a constant depending only on* p.

[1]) Hardy and Littlewood [1], [2].

The object of this paper is to extend the above results to the general case of bounded orthogonal functions.

2. We denote by $\vartheta_0(t), \vartheta_1(t), \vartheta_2(t), \dots$ a set of real normalised orthogonal functions in the interval $(0,1)$, so that we have

$$\int_0^1 \vartheta_n(t)\vartheta_m(t)\, dt = 0 \quad (n \neq m), \qquad = 1 \quad (n = m),$$

and we suppose that the set $\vartheta_n(t)$ are uniformly bounded in the interval, so that we have

$$|\vartheta_n(t)| \leqslant B \quad (n = 0, 1, 2) \quad (0 \leqslant t \leqslant 1).$$

We suppose throughout that $c_0^*, c_1^*, c_2^*, \dots$ denote the set $|c_0|, |c_1|, |c_2|, \dots$ rearranged in descending order of magnitude. With this convention we state the following theorem:

Theorem I. *If the series* $\Sigma c_n^{*q} n^{q-2}$ *is convergent, where* $q \geqslant 2$, *then*

$$f(t) = \sum_{n=0}^{\infty} c_n \vartheta_n(t) \tag{2.1}$$

is of class L^q, *and*

$$\int_0^1 |f(t)|^q \, dt \leqslant A_q \sum_{n=0}^{\infty} c_n^{*q} (n+1)^{q-2},$$

where A_q *depends only*[1]) *on* q *and* B.

We observe first that

$$\sum c_n^{*2} \leqslant \left(\sum c_n^{*q} (n+1)^{q-2}\right)^{\frac{2}{q}} \cdot \left(\sum (n+1)^{-2}\right)^{\frac{q-2}{q}} < \infty,$$

so that the series (2.1) does in fact represent some function (of class L^2). We observe also that it is legitimate to rearrange the functions $\vartheta_n(t)$ in any order we please, and so we may assume without loss of generality that the numbers $|c_n|$ are already in descending order of magnitude, and thus it is sufficient to prove that

$$\int_0^1 |f(t)|^q \, dt \leqslant A_q \sum_{n=0}^{\infty} |c_n|^q (n+1)^{q-2}. \tag{2.2}$$

[2]) It is not difficult to see that the constant is of the form $A_q B^{q-2}$, where A_q depends only on q.

We first need the following lemma due to M. RIESZ [3]).

Lemma 1. *Let $T = T(f)$ be a linear functional transformation of $L^a\varphi$ into $L^c\psi$, i. e.*

(i) *the transformation is distributive, so that for arbitrary constants λ_1, λ_2,*

$$T(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 T(f_1) + \lambda_2 T(f_2),$$

(ii) *there exists a constant M^*, such that*

$$\left(\int |T(f)|^c d\psi\right)^{1/c} \leqslant M^* \left(\int |f|^a d\varphi\right)^{1/a}.$$

Let $M^*_{\alpha,\gamma}$ denote the upper bound of the ratio

$$\left(\int |T(f)|^c d\psi\right)^{1/c} \Big/ \left(\int |f|^a d\varphi\right)^{1/a},$$

where $a\alpha = c\gamma = 1$. Then $\log M^*_{\alpha,\gamma}$ *is a convex function of the variables α, γ in the triangle* [4])

$$0 \leqslant \gamma \leqslant \alpha \leqslant 1.$$

We observe that (2.2) may be written

$$\int_0^1 |f(t)|^q dt \leqslant A_q \sum_{n=0}^{\infty} \left((n+1) c_n\right)^q (n+1)^{-2},$$

and that

$$f(t) = \sum_{n=0}^{\infty} (n+1) c_n \{\vartheta_n(t) / (n+1)\}$$

is obtained by a linear transformation from the numbers $(n+1)c_n$. Thus it is legitimate to interpolate by means of the last lemma, and it follows that it is sufficient to prove (2.2) in the case when q is an even integer.

3. To fix the ideas we assume that $q=4$. For $q=2$ the theorem is well known, and for other even integers the proof is similar to that in the case $q=4$. We write

$$f(t) = \sum_{m=0}^{\infty} f_m(t),$$

where

$$f_0(t) = c_0 \vartheta_0(t), \qquad f_1(t) = c_1 \vartheta_1(t),$$

[3]) M. Riesz [1], Theorem V.

[4]) Or any segment in the triangle for which the conditions (i), (ii) are satisfied.

$$f_m(t) = \sum_{n=2^{m-1}}^{2^m-1} c_n \vartheta_n(t) \qquad (m \geqslant 2).$$

We write

$$\varepsilon_0 = c_0^4, \qquad \varepsilon_1 = c_1^4 \,.\, 2^2 = 4 c_1^4,$$

$$\varepsilon_m = \sum_{n=2^{m-1}}^{2^m-1} c_n^4 (n+1)^2 \qquad (m \geqslant 2).$$

Let $0 < \mu \leqslant \nu$. Then

$$\int_0^1 f_\mu^2(t) f_\nu^2(t)\, dt$$

$$\leqslant \left(\int_0^1 f_\nu^2(t)\, dt\right) \operatorname*{Max}_{0 \leqslant t \leqslant 1} f_\mu^2(t)$$

$$\leqslant \left(\sum_{n=2^{\nu-1}}^{2^\nu-1} c_n^2\right) . \left(B \sum_{n=2^{\mu-1}}^{2^\mu-1} |c_n|\right)^2$$

$$\leqslant B^2 \left(\sum_{n=2^{\nu-1}}^{2^\nu-1} c_n^4 (n+1)^2\right)^{\frac{1}{2}} . \left(\sum_{n=2^{\nu-1}}^{2^\nu-1} (n+1)^{-2}\right)^{\frac{1}{2}}$$

$$\times \left(\sum_{n=2^{\mu-1}}^{2^\mu-1} c_n^4 (n+1)^2\right)^{\frac{1}{2}} . \left(\sum_{n=2^{\mu-1}}^{2^\mu-1} (n+1)^{-\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$\leqslant A \varepsilon_\nu^{\frac{1}{2}} \varepsilon_\mu^{\frac{1}{2}} 2^{\frac{1}{2}(\mu-\nu)} \leqslant A(\varepsilon_\nu + \varepsilon_\mu) 2^{\frac{1}{2}(\mu-\nu)},$$

where A, here and in the sequel, denotes an absolute constant (not the same constant in different contexts). It follows from the above equation that if m_1, m_2, m_3, m_4 are arbitrary integers (all greater than zero) then

$$\int_0^1 |f_{m_1}(t) f_{m_2}(t) f_{m_3}(t) f_{m_4}(t)|\, dt$$

$$\leqslant \left(\int_0^1 f_{m_1}^2 f_{m_2}^2\right)^{\frac{1}{6}} \left(\int_0^1 f_{m_1}^2 f_{m_3}^2\right)^{\frac{1}{6}} \left(\int_0^1 f_{m_1}^2 f_{m_4}^2\right)^{\frac{1}{6}} \left(\int_0^1 f_{m_2}^2 f_{m_3}^2\right)^{\frac{1}{6}} \left(\int_0^1 f_{m_2}^2 f_{m_4}^2\right)^{\frac{1}{6}} \left(\int_0^1 f_{m_3}^2 f_{m_4}^2\right)^{\frac{1}{6}}$$

$$\leqslant (\varepsilon_{m_1} + \varepsilon_{m_2} + \varepsilon_{m_3} + \varepsilon_{m_4}) \,.\, 2^{-1/12(|m_1-m_2|+|m_1-m_3|+|m_1-m_4|+|m_2-m_3|+|m_2-m_4|+|m_3-m_4|)}.$$

Thus

$$\int_0^1 (|f_1|+|f_2|+\ldots+|f_m|+\ldots)^4\,dt$$

$$\leqslant 6\sum\int_0^1 |f_{m_1}(t)\,f_{m_2}(t)\,f_{m_3}(t)\,f_{m_4}(t)|\,dt$$

$$\leqslant A\sum(\varepsilon_{m_1}+\varepsilon_{m_2}+\varepsilon_{m_3}+\varepsilon_{m_4})\,.\,2^{-1/_{12}(|m_1-m_2|+|m_1-m_3|+|m_1-m_4|+|m_2-m_3|+|m_2-m_4|+|m_3-m_4|)}.$$

The coefficient of ε_m in the above sum is

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}2^{-1/_{12}(|m-m_1|+|m-m_2|+|m-m_3|+|m_1-m_2|+|m_1-m_3|+|m_2-m_3|)}\leqslant A.$$

It follows that

$$\int_0^1 (|f_1|+|f_2|+\ldots+|f_m|+\ldots)^4\,dt=A\sum_{m=1}^{\infty}\varepsilon_m.$$

Since

$$\int_0^1 f_0^4(t)\,dt\leqslant B^2\,c_0^4=B^2\,\varepsilon_0,$$

we have

$$\int_0^1\left(\sum_{m=0}^{\infty}|f_m(t)|\right)^4 dt\leqslant A\sum_{m=0}^{\infty}\varepsilon_m=A\sum_{n=0}^{\infty}c_n^4\,(n+1)^2, \tag{3.1}$$

from which the result (2.2) follows for $q=4$, and the theorem follows in virtue of what has already been said.

4. Theorem II. *Let*

$$f(t)=\sum_{n=0}^{\infty}c_n\vartheta_n(t)\in L^p\qquad(1<p\leqslant 2).$$

Then

$$\sum_{n=0}^{\infty}c_n^{*p}\,(n+1)^{p-2}\leqslant A_p\int_0^1|f(t)|^p dt,$$

where A_p depends only on p and B.

We observe that, in virtue of the remark made above that it is legitimate to rearrange the functions $\vartheta_n(t)$ in any desired order, it is sufficient to prove that

$$\sum_{n=0}^{\infty}|c_n|^p\,(n+1)^{p-2}\leqslant A_p\int_0^1|f(t)|^p\,dt.$$

We write

$$d_n=|c_n|^{p-1}(n+1)^{p-2}\operatorname{sgn}c_n,$$

so that

$$c_n=|d_n|^{p'-1}(n+1)^{p'-2}\operatorname{sgn}d_n,$$

where p' is defined by the equation

$$\frac{1}{p}+\frac{1}{p'}=1.$$

We observe that $p'\geqslant 2$, and that

$$\sum_{n=0}^{N}|c_n|^p\,(n+1)^{p-2}=\sum_{n=0}^{N}c_n d_n=\sum_{n=0}^{N}|d_n|^{p'}\,(n+1)^{p'-2}.$$

We write

$$g_n(t)=\sum_{n=0}^{N}d_n\vartheta_n(t).$$

Then, using Theorem I, we have

$$\sum_{n=0}^{N}|c_n|^p\,(n+1)^{p-2}=\sum_{n=0}^{N}c_n d_n=\int_0^1 f(t)\,g_N(t)\,dt$$

$$\leqslant\left(\int_0^1|f(t)|^p\,dt\right)^{1/p}\left(\int_0^1|g_N(t)|^{p'}\,dt\right)^{1/p'}$$

$$\leqslant A_{p'}\left(\int_0^1|f(t)|^p\,dt\right)^{1/p}\left(\sum_{n=0}^{N}|d_n|^{p'}\,(n+1)^{p'-2}\right)^{1/p'}$$

$$=A_{p'}\left(\int_0^1|f(t)|^p\,dt\right)^{1/p}\left(\sum_{n=0}^{N}|c_n|^p\,(n+1)^{p-2}\right)^{1/p'}.$$

It follows that

$$\sum_{n=0}^{N}|c_n|^p\,(n+1)^{p-2}\leqslant A_p\int_0^1|f(t)|^p dt,$$

and since A_p is independent of N, the desired result follows by making N tend to infinity.

5. The form of (3.1) suggests that some stronger result than that of Theorem I may be true. We prove the following more general result:

Theorem III. *Let $S(t)$ denote the upper bound*

$$S(t)=\sup_{0\leq m<\infty}\left|\sum_{n=0}^{m} c_n \vartheta_n(t)\right|$$

Then, if $q>2$,

$$\int_0^1 S^q(t)\,dt \leqslant A_q \sum_{n=0}^{\infty} c_n^{*q}(n+1)^{q-2} \leqslant A_q\left(\sum_{n=0}^{\infty}|c_n|^{q'}\right)^{q-1}, \tag{5.1}$$

where q' is defined by the relation

$$\frac{1}{q}+\frac{1}{q'}=1,$$

and A_q depends only on q.

The second of the inequalities (5.1), which is semi-trivial, is due to Hardy and Littlewood. In fact

$$\begin{aligned}\sum_{n=0}^{\infty} c_n^{*q}(n+1)^{q-1} &\leqslant A_q \sum_{n=0}^{\infty} c_{2^n-1}^{*q}\, 2^{n(q-1)}\\ &= A_q \sum_{n=0}^{\infty}\left(c_{2^n-1}^{*q'}\, 2^n\right)^{q-1}\\ &\leqslant A_q\left(\sum_{n=0}^{\infty} c_{2^n-1}^{*q'}\, 2^n\right)^{q-1}\\ &\leqslant A_q\left(\sum_{n=0}^{\infty} c_n^{*q'}\right)^{q-1}\\ &= A_q\left(\sum_{n=0}^{\infty}|c_n|^{q'}\right)^{q-1},\end{aligned}$$

where A_q here and in the sequel denotes a constant which depends only on q.

As in Theorem I, we split up the series

$$\sum_{n=0}^{\infty} c_n \vartheta_n(t) \tag{5.2}$$

into a number of finite subsequences. Suppose that, in rearranging the moduli $|c_n|$ in decreasing order of magnitude[5]) (or star order), $|c_{\lambda(n)}|$ becomes c_n^*. We write

$$f_0(t)=c_{\lambda(0)}\vartheta_{\lambda(0)}(t), \qquad f_1(t)=c_{\lambda(1)}\vartheta_{\lambda(1)}(t),$$

[5]) Where the number of the moduli $|c_n|$ are equal, we may suppose that they are rearranged in order of increasing index.

$$f_m(t)=\sum_{n=0}^{\infty} c_n \vartheta_n(t)\,\psi_m(n) \qquad (m\geqslant 2),$$

where $\psi_m(n)$ is equal to 1 if $n=\lambda(j)$, $2^{m-1}\leqslant j\leqslant 2^m-1$, and vanishes otherwise. We write

$$S_0(t)=c_0^*|\vartheta_{\lambda(0)}(t)|, \qquad S_1(t)=c_1^*|\vartheta_{\lambda(1)}(t)|,$$

$$S_m(t)=\max_{0\leq N<\infty}\left|\sum_{n=0}^{N} c_n\vartheta_n(t)\,\psi_m(n)\right| \qquad (m\geqslant 2),$$

so that

$$S(\theta)\leqslant S_0(\theta)+S_1(\theta)+S_2(\theta)+\dots$$

As in Theorem I we write

$$\varepsilon_0=c_0^{*q}, \quad \varepsilon_1=c_1^{*q}\,2^{q-2},$$

$$\varepsilon_m=\sum_{n=2^{m-1}}^{2^m-1} c_n^{*q}(n+1)^{q-2} \qquad (m\geqslant 2).$$

5. We first need the following lemma.

Lemma 2. *Let $G(t)$ denote the maximum*

$$G(t)=\operatorname{Max}_{0\leq m\leq 2^{\mu}-1}\left|\sum_{n=0}^{m} d_n\vartheta_n(t)\right|.$$

Then, if $2<k<\infty$,

$$\int_0^1 G^k(t)\,dt\leqslant A_k 2^{\mu(k-2)}\sum_{n=0}^{2^{\mu}-1}|d_n|^k,$$

where A_k depends only on k.

We write

$$\varphi_{0,1}(t)=\sum_{n=0}^{2^{\mu}-1} d_n\vartheta_n(t),$$

$$\varphi_{1,1}(t)=\sum_{n=0}^{2^{\mu-1}-1} d_n\vartheta_n(t), \qquad \varphi_{1,2}(t)=\sum_{n=2^{\mu-1}}^{2^{\mu}-1} d_n\vartheta_n(t),$$

and generally

$$\varphi_{\lambda,m}(t)=\sum_{n=(m-1)\,2^{\mu-\lambda}}^{m2^{\mu-\lambda}-1} d_n\vartheta_n(t) \qquad (0\leqslant\lambda\leqslant\mu,\ 1\leqslant m\leqslant 2^{\lambda}).$$

We write

$$\Phi_\lambda(t) = \operatorname*{Max}_{1 \le m \le 2^\lambda} |\varphi_{\lambda, m}(t)|,$$

so that

$$G(t) \leqslant \sum_{\lambda=0}^{\mu} \Phi_\lambda(t).$$

Now

$$\int_0^1 \Phi_\lambda^k(t)\,dt \leqslant \sum_{m=1}^{2^\lambda} \int_0^1 |\varphi_{\lambda, m}(t)|^k\,dt$$

$$\leqslant A_k \sum_{m=1}^{2^\lambda} 2^{(\mu-\lambda)(k-2)} \sum_{n=(m-1)2^{\mu-\lambda}}^{m 2^{\mu-\lambda}-1} |d_n|^k$$

$$\leqslant A_k 2^{(\mu-\lambda)(k-2)} \sum_{n=0}^{2^\mu - 1} |d_n|^k \qquad (0 \leqslant \lambda \leqslant \mu),$$

using a simplified form of Theorem I. It follows, by MINKOWSKI's inequality, that

$$\left(\int_0^1 G^k(t)\,dt\right)^{1/k} \leqslant \sum_{\lambda=0}^{\mu} \left(\int_0^1 \Phi_\lambda^k(t)\,dt\right)^{1/k}$$

$$\leqslant A_k \sum_{\lambda=0}^{\mu} 2^{(\mu-\lambda)(k-2)/k} \left(\sum_{n=0}^{2^\mu-1} |d_n|^k\right)^{1/k}$$

$$\leqslant A_k \left(2^{\mu(k-2)} \sum_{n=0}^{2^\mu-1} |d_n|^k\right)^{1/k},$$

from which the lemma follows.

7. Suppose now that $q = 4$. Let $0 < \mu \leqslant \nu$. Then using Lemma 2, with $k = 3$, we have

$$\int_0^1 S_\nu^3(t) S_\mu(t)\,dt$$

$$\leqslant \left(\int_0^1 S_\nu^3(t)\,dt\right) \operatorname*{Max}_{0 \le t \le 1} S_\mu(t)$$

$$\leqslant \left|A \cdot 2^\nu \sum_{n=2^{\nu-1}}^{2^\nu-1} c_n^{*3}\right| \cdot \left|B^a \cdot \sum_{n=2^{\mu-1}}^{2^\mu-1} c_n^*\right| \leqslant$$

$$\leqslant A \cdot 2^\nu \left(\sum_{n=2^{\nu-1}}^{2^\nu-1} c_n^{*4}(n+1)^2\right)^{3/4} \cdot \left(\sum_{n=2^{\nu-1}}^{2^\nu-1} (n+1)^{-6}\right)^{1/4}$$

$$\times \left(\sum_{n=2^{\mu-1}}^{2^\mu-1} c_n^{*4}(n+1)^2\right)^{1/4} \cdot \left(\sum_{n=2^{\mu-1}}^{2^\mu-1} (n+1)^{-2/3}\right)^{3/4}$$

$$\leqslant A \varepsilon_\nu^{3/4} \varepsilon_\mu^{1/4} 2^{1/4(\mu-\nu)} \leqslant A(\varepsilon_\nu + \varepsilon_\mu) 2^{1/4(\mu-\nu)}.$$

Using HÖLDER's inequality again, we get

$$\int_0^1 S_\nu^2(t) S_\mu^2(t)\,dt$$

$$\leqslant \left(\int_0^1 S_\nu^3(t) S_\mu(t)\,dt\right)^{2/3} \left(\int_0^1 S_\mu^4(t)\,dt\right)^{1/3}$$

$$\leqslant A(\varepsilon_\nu + \varepsilon_\mu) 2^{-1/6|\mu-\nu|}.$$

Proceeding as in Theorem I we may prove that

$$\int_0^1 \left(\sum_{m=0}^{\infty} S_m(t)\right)^4 dt \leqslant A \sum_{m=0}^{\infty} \varepsilon_m \leqslant A \sum_{n=0}^{\infty} c_n^{*4}(n+1)^2.$$

This establishes the theorem for $q = 4$. For $q \geqslant 4$ and even integer the proof is similar.

8. Now suppose that $q \leqslant 3$. Then we have

$$\int_0^1 S^q(t)\,dt \leqslant \int_0^1 \left(\sum_{m=0}^{\infty} S_m(t)\right)^q dt$$

$$= \int_0^1 \left(\sum_{m=0}^{\infty} S_m(t)\right)^2 \left(\sum_{m=0}^{\infty} S_m(t)\right)^{q-2} dt$$

$$\leqslant \int_0^1 \left(\sum_{m=0}^{\infty} S_m(t)\right)^2 \left(\sum_{m=0}^{\infty} S_m^{q-2}(t)\right) dt$$

$$\leqslant 2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \int_0^1 S_{m_1}(t) S_{m_2}(t) S_{m_3}^{q-2}(t)\,dt. \tag{8.1}$$

We have

$$\int_0^1 S_{m_1}(t) S_{m_2}(t) S_{m_3}^{q-2}(t)\,dt \leqslant$$

$$\leqslant \int_0^1 \left(S_{m_1}^{\frac{1}{2}q}(t)\, S_{m_2}^{\frac{1}{2}q}(t)\, dt \right)^{\frac{4-q}{q}} \left(\int_0^1 S_{m_1}^{\frac{1}{2}q}(t)\, S_{m_3}^{\frac{1}{2}q}(t)\, dt \right)^{\frac{q-2}{q}}$$

$$\times \left(\int_0^1 S_{m_2}^{\frac{1}{2}q}(t)\, S_{m_3}^{\frac{1}{2}q}(t)\, dt \right)^{\frac{q-2}{q}},$$

and, if $0 < m_1 \leqslant m_2$,

$$\int_0^1 S_{m_1}^{\frac{1}{2}q}(t)\, S_{m_2}^{\frac{1}{2}q}(t)\, dt$$

$$\leqslant \left(\int_0^1 S_{m_1}^{q}(t)\, dt \right)^{\frac{2}{q+2}} \left(\int_0^1 S_{m_1}^{\frac{1}{2}q-1}(t)\, S_{m_2}^{\frac{1}{2}q+1}(t)\, dt \right)^{\frac{q}{q+2}}.$$

An application of Lemma 2 with $k = \frac{1}{2}q + 1$, gives

$$\int_0^1 S_{m_1}^{\frac{1}{2}q-1}(t)\, S_{m_2}^{\frac{1}{2}q+1}(t)\, dt$$

$$\leqslant \left(\int_0^1 S_{m_2}^{\frac{1}{2}q+1}(t)\, dt \right) \operatorname*{Max}_{0 \leq t \leq 1} S_{m_1}^{\frac{1}{2}q-1}(t)$$

$$\leqslant A_q\, 2^{m_2(\frac{1}{2}q-1)} \left| \sum_{n=2^{m_2-1}}^{2^{m_2}-1} c_n^{*\,\frac{1}{2}q+1} \right| \cdot \left| \sum_{n=2^{m_1-1}}^{2^{m_1}-1} c_n^{*} \right|^{\frac{1}{2}q-1}$$

$$\leqslant A_q\, 2^{m_2(\frac{1}{2}q-1)} \left| \sum_{n=2^{m_2-1}}^{2^{m_2}-1} c_n^{*\,q}(n+1)^{q-2} \right|^{\frac{q+2}{2q}} \cdot \left| \sum_{n=2^{m_2-1}}^{2^{m_2}-1} (n+1)^{-q-2} \right|^{\frac{q-2}{2q}}$$

$$\times \left| \sum_{n=2^{m_1-1}}^{2^{m_1}-1} c_n^{*\,q}(n+1)^{q-2} \right|^{\frac{q-2}{2q}} \cdot \left| \sum_{n=2^{m_1-1}}^{2^{m_1}-1} (n+1)^{-\frac{q-2}{q-1}} \right|^{\frac{(q-2)(q-1)}{2q}}$$

$$\leqslant A_q(\varepsilon_{m_1} + \varepsilon_{m_2})\, 2^{-|m_1-m_2|(q-2)/2q}.$$

It follows that, if none of the numbers m_1, m_2, m_3 is zero,

$$\int_0^1 S_{m_1}(t)\, S_{m_2}(t)\, S_{m_3}^{q-2}(t)\, dt \tag{8.2}$$

$$\leqslant A_q(\varepsilon_{m_1} + \varepsilon_{m_2} + \varepsilon_{m_3})\, 2^{-\lambda_q(|m_1-m_2|+|m_2-m_3|+|m_3-m_1|)},$$

where

$$\lambda_q = (q-2)^2/2q(q+2) \geqslant 0.$$

The equation (8.2) can be extended easily to the case where one or all of m_1, m_2, m_3 is zero. Substitution in (8.1) now gives

$$\int_0^1 S^q(t)\, dt$$

$$\leqslant A_q \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} (\varepsilon_{m_1} + \varepsilon_{m_2} + \varepsilon_{m_3})\, 2^{-\lambda_q(|m_1-m_2|+|m_2-m_3|+|m_3-m_1|)}$$

$$\leqslant A_q \sum_{m=0}^{\infty} \varepsilon_m = A_q \sum_{n=0}^{\infty} c_n^{*\,q}(n+1)^{q-2}.$$

9. We have thus established the theorem in the cases $2 < q \leqslant 3$, and when q is an even integer greater than or equal to 4. For other values of q we may either obtain the result by an argument analogous to that used in the last paragraph, observing for instance that for $3 < q \leqslant 4$

$$\int_0^1 S^q(t)\, dt \leqslant A \sum \int_0^1 S_{m_1}(t)\, S_{m_2}(t)\, S_{m_3}(t)\, S_{m_4}^{q-3}(t)\, dt,$$

or we may interpolate by means of Lemma 1. The latter argument runs as follows. Let $n(t)$ denote an arbitrary integer which varies with t (but which we suppose to be measurable and bounded above by some large number N). We denote by $s_{n(t)}(t)$ the $n(t)$-th partial sum of the series (5.2). Let $\lambda(n)$ define an operation which gives a (1,1) transformation of the positive integers $(n = 0, 1, 2 \ldots\ldots)$ again into the same set. We denote by M^*_q the maximum

$$\operatorname{Max} \frac{\left(\int_0^1 |s_{n(t)}(t)|^q\, dt \right)^{1/q}}{\left(\sum_{n=0}^{\infty} |c_n|^q (\lambda(n)+1)^{q-2} \right)^{1/q}}$$

for arbitrary variation of the numbers c_n. If $q \geqslant 4$ is an even integer or if $2 < q \leqslant 3$, we have

$$\int_0^1 |S_{n(t)}(t)|^q\, dt \leqslant \int_0^1 S^q(t)\, dt \leqslant A_q \sum_{n=0}^{\infty} c_n^{*\,q}(n+1)^{q-2}$$

$$\leqslant A_q \sum_{n=0}^{\infty} |c_n|^q (\lambda(n)+1)^{q-2}$$

and thus M^*_q is bounded. But in virtue of Lemma 1, $\log M^*_q$ is a convex function of $1/q$ $(1 \leqslant q \leqslant \infty)$, and it follows that M^*_q is bounded (independently of N and the choice of $n(t)$ and $\lambda(n)$) for all q in the range $2 < q < \infty$. Thus

$$(9.1) \qquad \int_0^1 | s_{n(t)}(t) |^q \, dt \leqslant A_q \sum_{n=0}^{\infty} | c_n |^q (\lambda(n) + 1)^{q-2}.$$

We can choose $\lambda(n)$ so that the right hand side of (9.1) is identically

$$\sum_{n=0}^{\infty} c_n^{*q} (n+1)^{q-2},$$

and we have

$$\int_0^1 S^q(t)\, dt = \sup \int_0^1 | s_{n(t)}(t) |^q \, dt \leqslant A_q \sum_{n=0}^{\infty} c_n^{*q} (n+1)^{q-2}.$$

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[2], Notes on the theory of series (XIII): Some new properties of Fourier constants, Journal of the London Math. Soc. 6 (1931) p. 3—9.

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On a theorem of Privaloff

by

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1. Fejér has proved the following theorem. If a trigonometrical series

$$(1) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

is uniformly convergent $(0 \leqslant \theta \leqslant 2\pi)$, the conjugate series

$$(2) \qquad \sum_{n=1}^{\infty} (a_n \sin n\theta - b_n \cos n\theta)$$

is convergent almost everywhere in $(0, 2\pi)$[1]).

Fejér's result has been extended by Privaloff who has shown that, if the partial sums of the series (1) are uniformly bounded in $(0, 2\pi)$ and the series itself is convergent in a set E of positive measure, the series (2) is convergent almost everywhere in E[2]). We are going to prove a little more general theorem.

Theorem. *If the partial sums s_n of the series* (1) *1° satisfy an inequality*

$$(3) \qquad s_n(\theta) > -\varphi(\theta) \qquad (0 \leqslant \theta \leqslant 2\pi),$$

where φ is integrable L[3]), *2° the series* (1) *is convergent in a set E of positive measure, then* (2) *is convergent almost everywhere in E.*

[1]) L. Fejér, Über konjugierte trigonometrische Reihen, Crelles Journal 144 (1913).

[2]) I. I. Privaloff, Sur la convergence des séries trigonométriques conjuguées (in russian, with french résumé), Recueil de la Societé Math. de Moscou, 32 (1925) p. 357—363.

[3]) In particular if $s_n \geqslant 0$.