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On the boundary behavior in the metric L^p of subharmonic functions

by

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1. It is a classical result that if $u(x) = u(r, \theta)$ is harmonic in the open unit sphere $|x| < 1$ in n-dimensional Euclidean space and if

$$\int_{|x|=1} u^+(r, \theta)\, d\theta < M < \infty \quad \text{for} \quad r < 1, \tag{1.1}$$

then the function u has a non-tangential limit at almost all points on the surface $|x| = 1$ of the unit sphere.

The natural question whether this result can be extended to a subharmonic function v satisfying condition (1.1) was answered by Littlewood [2] who showed that for $n = 2$, v has a radial limit almost everywhere on $|x| = 1$. Littlewood's argument and result extend without difficulty to $n > 2$ [3]. On the other hand, simple examples show that a non-tangential limit may exist almost nowhere for the subharmonic funtion v, even when v is bounded [4].

In this note we obtain a result on the existence almost everywhere of non-tangential limits defined in a somewhat different way. Let $x \in E_n$ $(n \geqslant 2)$. We say that the function $v(x)$, $|x| < 1$, has a *non-tangential limit* λ *at a point* θ, $|\theta| = 1$, in the metric L^p $(1 \leqslant p < \infty)$ if for every $0 < \nu < \pi/2$ we have

$$\frac{1}{|T_\delta|} \int_{T_\delta} |v(x) - \lambda|^p\, dx \to 0 \quad \text{as} \quad \delta \to 0,$$

where $T_\delta = T_\delta(\theta, \nu)$ is the conical region formed by the points x, $|x| < 1$, whose distance from θ is less than δ, and such that the angle between the line segment $x\theta$ and the radius $O\theta$ is less than ν. A prerequisite for the applicability of this definition is that $v \in L^p(T_\delta)$. The ordinary non-tangential limit may be considered as the limiting case $p = \infty$ of this definition (1).

(1) The definition of non-tangential limit in L^p used here was suggested by the work of Calderon and Zygmund [1] on derivatives in L^p. The author gratefully acknowledges the guidance of Professor A. Zygmund in the preparation of this paper.

The main result of this note is

THEOREM 1. *Suppose that* $v(x) = v(r, \theta)$ *is subharmonic in* $|x| < 1$, *and that*

$$\int_{|x|=1} v^+(r, \theta)\, d\theta$$

is uniformly bounded for $r < 1$. *Then the function* v *has at almost every point on* $|x| = 1$ *a non-tangential limit in the metric* L^p, *provided* $1 \leqslant p < n/(n-2)$. *This limit coincides almost everywhere with the (ordinary) radial limit of* v.

For $n = 2$, the p of the theorem may assume any finite value $\geqslant 1$; but as mentioned above, the result fails for $p = \infty$. We note that the hypothesis of the theorem is satisfied if the function $v(x)$, $|x| < 1$, is subharmonic and bounded from above.

The argument that follows uses much of the technique of Littlewood [2] for $n = 2$, and Privaloff [3] for $n \geqslant 2$. Those parts of our proof which follow closely [2] or [3] we present in outline only.

2. We use the following notation: x is a point in the n-dimensional Euclidean space E_n $(n \geqslant 2)$; the open unit sphere K is the set of all points $x = (x_1, x_2, \ldots, x_n)$ such that $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} < 1$; Σ, the surface of the unit sphere, is the set $|x| = 1$; $x^* = x/|x|^2$ is the conjugate of x with respect to Σ; we use the representation $x = (r, \theta)$ where $r = |x|$, and $\theta = x/|x|$ is a point on Σ (we also use other representations for $n = 2$ and $n = 3$); for $x = (r, \theta)$ and $\xi = (\varrho, \varphi)$ in K, $s = 1 - r$ (the distance from x to Σ), $\sigma = 1 - \varrho$; for v, a function with domain in E_n, we may write equivalently $v(x)$, $v(r, \theta)$ etc.; $v^+ = \max(v, 0)$; $|E|$ is the n-dimensional volume of a region $E \subset E_n$; $|\Sigma|$ is the area of Σ; c, c_a etc., are suitably chosen positive constants, not always the same (even when repeated in the same expression), and depending upon the parameters indicated. (However, dependence upon the dimension n will not be indicated. Thus c denotes absolute constants, or constants depending only on n.) Limits are understood to be finite.

3. Proof of theorem 1. Let the dimension n $(n \geqslant 2)$ and the power p $\big(1 \leqslant p < n/(n-2)\big)$ be fixed. Assume as known the basic result (see [3]) that if $v = v(x) = v(r, \theta)$ is subharmonic in K and satisfies the condition

$$\int_{\Sigma} v^+(r, \theta)\, d\theta < M < \infty, \quad r < 1, \tag{3.1}$$

then

$$v = u - w, \tag{3.2}$$

where u is the least harmonic majorant of v, and w is a non-negative superharmonic function. Specifically, u is the limit as $R \to 1$ of the Poisson

integral u_R of the function $v(R, \theta)$. It is fundamental here that w is given by

$$w(x) = \int_K g(x, \xi)\, dF(\xi) \tag{3.3}$$

where

$$g(x, \xi) = \begin{cases} \log\left\{|x| \dfrac{|x^* - \xi|}{|x - \xi|}\right\}, & n = 2, \\ |x - \xi|^{2-n} - |x|^{2-n} |x^* - \xi|^{2-n}, & n > 2, \end{cases} \tag{3.4}$$

is Green's function for the unit sphere, and $F(\xi)$ is a non-negative mass distribution satisfying the condition

$$\int_K (1 - |\xi|)\, dF(\xi) < \infty. \tag{3.5}$$

It can be shown [3] that the harmonic function u in (3.2) satisfies an inequality of the form (3.1) and that consequently u has, at almost every point θ on Σ, an ordinary non-tangential limit $\lambda(\theta)$. Also ([2], [3]), v has at almost every point θ on Σ an ordinary radial limit equal to the same limit $\lambda(\theta)$. Thus the theorem will be established upon showing that the function w in (3.3) has a non-tangential limit in the metric L^p equal to zero at almost every point of Σ.

Without loss of generality, we may replace by zero the masses in $|\xi| \leqslant \varrho_0 < 1$, where ϱ_0 is arbitrarily close to 1; and we may assume that $\int_K \sigma dF(\xi)$ is arbitrarily small.

For $\theta \in \Sigma$ and $0 \leqslant h \leqslant \pi$, let $E = E(\theta, h)$ denote the sector of K consisting of those points x of K for which the angle between the radial segment Ox and the radius $O\theta$ is less than h. Then

$$\Phi(\theta, h) = \int_E \sigma dF(\xi)$$

may be considered as the mass of the open spherical cap on Σ with center θ and angular radius h, induced by the mass distribution $\sigma dF(\xi)$ in E. Let

$$(D\Phi)(\theta) = \limsup_{h \to 0} \frac{\Phi(\theta, h)}{h^{n-1}}.$$

Given $\varepsilon > 0$, we may assume that the total mass $\int_K \sigma dF(\xi)$ is small enough so that (see [3]) the derived function $(D\Phi)(\theta)$ is less than ε except on a set of points on Σ of measure less than ε. Choose $\theta_0 \in \Sigma$, so that setting $\Phi(h) = \Phi(\theta_0, h)$ we have,

$$\limsup_{h \to 0} \frac{\Phi(h)}{h^{n-1}} < \varepsilon. \tag{3.6}$$

Assume further that there is zero mass on the diameter of K terminating at θ_0.

4. Consider the point $\theta_0 \in \Sigma$ chosen in Section 3. Let $0 < r_0 < 1$, $s_0 = 1 - r_0$. Let $K_\alpha(s_0)$, $0 < \alpha < 1$, denote the sphere with center (r_0, θ_0) and radius αs_0. Let $K_\beta(s_0)$ be the sphere concentric with $K_\alpha(s_0)$ and having radius $\beta = \frac{1}{2}(1+\alpha)$.

In (3.3) we split K, the domain of integration, into three regions R_1, R_2 and R_3. Let Γ denote the smaller sector of K whose lateral boundary is formed by the rays tangent to the sphere $K_\beta(s_0)$. We define R_1 to be the complement of Γ in the open unit sphere K, R_2 to be the complement of the sphere $K_\beta(s_0)$ in the sector Γ, and R_3 to be the sphere $K_\beta(s_0)$. Thus (3.3) can be written

$$(4.1) \qquad w(x) = \Big(\int_{R_1} + \int_{R_2} + \int_{R_3}\Big) g(x, \xi)\, dF(\xi) = w_1(x) + w_2(x) + w_3(x).$$

We will estimate w by estimating separately the $w_i(x)$ $(i = 1, 2, 3)$, with x confined to the sphere $K_\alpha(s_0)$ and $s_0 \to 0$. In estimating w_1 and w_2 we follow [2] and [3]. The main novelty here will be the treatment of w_3. Unlike w_1 and w_2, w_3 will be found to be small not at individual points, but on the average only.

For x, ξ in the open unit sphere K, let γ $(0 \leqslant \gamma \leqslant \pi)$ denote the angle between the radial segments Ox and $O\xi$. We obtain from (3.4) the estimates (see [2], [3])

$$(4.2) \qquad g(x, \xi) \leqslant \log\left\{1 + \frac{cs\sigma}{|x-\xi|^2}\right\} \qquad (n = 2),$$

$$(4.3) \qquad g(x, \xi) \leqslant cs\sigma[(s-\sigma)^2 + \gamma^2]^{-n/2} \qquad (s < 1/2,\ n \geqslant 2).$$

For x in $K_\alpha(s_0)$ we have $s \leqslant 2s_0$. Denote now by $h = h(\xi)$ $(0 \leqslant h \leqslant \pi)$ the angle between the radius $O\theta_0$ and the radial segment $O\xi$. Let $s_0 \leqslant 1/4$.

To estimate w, we begin with w_1. Let $x \in K_\alpha(s_0)$ and $\xi \in R_1$. We obtain, using (4.3),

$$g(x, \xi) \leqslant c_\alpha s_0 \sigma h^{-n}.$$

Now R_1 is contained in a region $H = \{\xi\colon \xi \in K,\ c_\alpha s_0 < h \leqslant \pi\}$. Hence

$$w_1(x) \leqslant c_\alpha s_0 \int_H h^{-n} \sigma\, dF(\xi) \leqslant c_\alpha s_0 \int_{c_\alpha s_0}^{\pi} h^{-n}\, d\Phi(h).$$

Integrating by parts and using (3.6), we obtain

$$(4.4) \qquad \limsup w_1(x) \leqslant c_\alpha \varepsilon \quad \text{as} \quad s_0 \to 0 \text{ and } x \in K_\alpha(s_0).$$

We now estimate w_2. Let $x \in K_\alpha(s_0)$ and $\xi \in R_2$. We now have $(s-\sigma)^2 + \gamma^2 \geqslant c_\alpha s_0^2$, $s \leqslant 2s_0$, $h \leqslant c_\alpha s_0$. Using (4.3) we obtain

$$w_2(x) \leqslant \frac{c_\alpha}{s_0^{n-1}} \int_{h \leqslant c_\alpha s_0} \sigma\, dF(\xi) \leqslant c_\alpha \frac{\Phi(c_\alpha s_0)}{s_0^{n-1}}.$$

It follows from (3.6) that

$$(4.5) \qquad \limsup w_2(x) \leqslant c_\alpha \varepsilon \quad \text{as} \quad s_0 \to 0 \text{ and } x \in K_\alpha(s_0).$$

5. Finally we estimate w_3. Set $|x - \xi| = t$. Let $x \in K_\alpha(s_0)$ and $\xi \in K_\beta(s_0)$. Then $s, \sigma, t \leqslant 2s_0$; and we obtain from (4.2)

$$g(x, \xi) \leqslant \log(cs_0^2 t^{-2}) \qquad (n = 2).$$

From (3.4) we directly obtain

$$g(x, \xi) \leqslant t^{2-n} \qquad (n > 2).$$

Letting

$$G(t) = \begin{cases} \log(cs_0^2 t^{-2}), & n = 2, \\ t^{2-n}, & n > 2, \end{cases}$$

we obtain, since $R_3 = K_\beta(s_0)$,

$$(5.0) \qquad w_3(x) \leqslant \int_{K_\beta(s_0)} G(t)\, dF(\xi).$$

It follows from Minkowski's inequality for integrals that

$$(5.1) \qquad \Big\{\int_{K_\alpha(s_0)} w_3^p(x)\, dx\Big\}^{1/p} \leqslant \int_{K_\beta(s_0)} \Big[\int_{K_\alpha(s_0)} G^p(t)\, dx\Big]^{1/p} dF(\xi).$$

Fix $\xi \in K_\beta(s_0)$. Since $K_\alpha(s_0)$ is contained in the sphere with center ξ and radius $2s_0$, we have

$$\int_{K_\alpha(s_0)} G^p(t)\, dx \leqslant |\Sigma| \int_0^{2s_0} G^p(t)\, t^{n-1}\, dt.$$

Thus, for $n = 2$

$$\int_{K_\alpha(s_0)} G^p(t)\, dx \leqslant 2\pi \int_0^{2s_0} \log^p\left(\frac{cs_0^2}{t^2}\right) t\, dt \leqslant c_p s_0 \int_0^2 \log^p\left(\frac{c}{t}\right) t\, dt \leqslant c_p s_0^2\ (^2);$$

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(2) If $F(\xi)$ is absolutely continuous and $F'(\xi) = o[(1-\varrho)^{-2}]$ then the left hand side of (5.0) is

$$\sigma(s_0^{-2}) \int_{K_\beta(s_0)} \log \frac{cs_0^2}{t^2}\, t\, dt = o(1) \int_0^2 \log\left(c\frac{s_0^2}{t^2}\right) t\, dt = o(1)$$

and $w_3 \to 0$ as $\delta \to o$ for $x \in K_\alpha(s_0)$. Thus under this hypothesis it will follow as in the sequel, that v has a nontangential limit in the classical sense almost everywhere. In this connection see M. G. Arsove and Alfred Huber, Notices A. M. S., 634-30, April, 1966.

and for $n>2$, since $p\leqslant n/(n-2)$,

$$\int_{K_\alpha(s_0)} G^p(t)\,dx \leqslant |\Sigma| \int_{\infty}^{2s_0} t^{p(2-n)+n-1}\,dt \leqslant c_p\, s_0^{n+p(2-n)}.$$

It follows from (5.1) that (for $n\geqslant 2$)

$$\int_{K_\alpha(s_0)} w_3^p(x)\,dx \leqslant c_p\, s_0^{n+p(2-n)} \Big(\int_{K_\beta(s_0)} dF(\xi)\Big)^p.$$

Since $\xi\in K_\beta(s_0)$, we have $s_0\leqslant c_\alpha \sigma$, and consequently

$$\int_{K_\alpha(s_0)} w_3^p(x)\,dx \leqslant c_{\alpha,p}\, s_0^n \Big(\int_{K_\beta(s_0)} \sigma^{2-n}\,dF(\xi)\Big)^p. \tag{5.2}$$

For $\theta\in\Sigma$, and $0<\eta<1$, let $\Omega(\theta)\subset K$ denote the conical region with vertex at θ, with lateral boundary formed by the tangents to the sphere K_η: $|x|\leqslant\eta$ and with base the smaller spherical cap of K_η. Let

$$I(\theta) = I_\eta(\theta) = \int_{\Omega_\eta(\theta)} \sigma^{2-n}\,dF(\xi). \tag{5.3}$$

Then

$$\int_{\Sigma} I(\theta)\,d\theta \leqslant c_\eta \int_K \sigma\,dF(\xi).$$

(See [5], vol. II, p. 209, where the proof is for Lebesgue integrals and $n=2$. However the proof is readily adapted to Lebesgue-Stieltjes integrals and $n\geqslant 2$.) By hypothesis (3.5), $\int_K \sigma\,dF(\xi)<\infty$. Hence $I(\theta)$ is finite for almost all θ. Suppose that $I(\theta_0)$ is finite. Having chosen $\eta=\eta(\alpha)$ in (5.3) sufficiently close to 1 so that the sphere $K_\beta(s_0)$ is contained in $\Omega_\eta(\theta_0)$ for s_0 sufficiently small, we have

$$\int_{K_\beta(s_0)} \sigma^{2-n}\,dF(\xi) = o(1) \quad \text{as} \quad s_0\to 0.$$

It follows from (5.2) that

$$\int_{K_\alpha(s_0)} w_3^p(x)\,dx = o(1) \quad \text{as} \quad s_0\to 0. \tag{5.4}$$

6. It is now easy to complete the proof of the theorem. We immediately deduce from the decomposition (4.1) and the estimates (4.4), (4.5) and (5.4) that (for $\varepsilon<1$)

$$\int_{K_\alpha(s_0)} w^p(x)\,dx \leqslant c_\alpha \varepsilon s_0^n, \tag{6.1}$$

provided s_0 is sufficiently close to zero.

Let T_δ be the conical region (of Section 1) with vertex θ_0. It is geometrically clear that if the number α in $K_\alpha(s_0)$ is sufficiently close to 1, and $\lambda<1$ is also sufficiently close to 1, then the family of spheres $K_\alpha(\delta\lambda^m)$, $m=0,1,\ldots$, covers T_δ. If δ is sufficiently small, we now obtain from (6.1)

$$\int_{T_\delta} w^p(x)\,dx \leqslant \sum_{m=0}^{\infty} \int_{K_\alpha(\delta\lambda^m)} w^p(x)\,dx \leqslant \sum_{m=0}^{\infty} c_\alpha \varepsilon(\delta\lambda^m)^n \leqslant c_{\alpha,\lambda}\varepsilon\delta^n.$$

Thus

$$\limsup_{\delta\to 0} \frac{1}{|T_\delta|} \int_{T_\delta} w^p(x)\,dx \leqslant c_{\alpha,\lambda}\varepsilon. \tag{6.2}$$

Since ε is arbitrarily small, and (6.2) holds on Σ except for a set of points on Σ of measure less than ε, it readily follows that there is a set $S\subset\Sigma$, having the same measure as Σ, such that at all points of S and for all p, $1\leqslant p<n/(n-2)$, w has a non-tangential limit equal to zero in the metric L^p. The theorem is established.

7. The conclusion of the theorem is false for $p=n/(n-2)$. For $n=2$ (and $p=\infty$) this is well known and was mentioned above. A counter example for that case (see [4]), suitably modified, extends without difficulty to $n>2$. The case $n=3$ (and $p=3$) is typical and we present a counter-example for it.

We use spherical coordinates $x=(r,\theta,\varphi)$, θ being the longitude and φ the polar angle. For each $m=2,3,\ldots$, let S_m denote the system of $\lambda_m=2m^2(m^2-1)$ points situated on the spherical surface Σ_m: $|x|=r_m=1-1/m$, and having coordinates (r_m,θ_i,φ_j), where $\theta_i=\pi i/m^2$, $i=0,1,\ldots,2m^2-1$, and $\Phi_j=\pi j/m^2$, $j=1,2,\ldots,m^2-1$.

(We may here think of the points of S_m as being uniformly distributed on Σ_m.) We now place a mass $\mu_m=2^{-m}$ at each of the points of S_m and consider the mass distribution F consisting of all the masses for all m. Now

$$\int_K \sigma\,dF(\xi) = \sum_{m=2}^{\infty} (1-r_m)\lambda_m\mu_m$$

is finite, and we obtain the subharmonic function

$$v(x) = -\int_K g(x,\xi)\,dF(\xi)\leqslant 0, \qquad x\in K.$$

Thus v satisfies the hypothesis of Theorem 1. Since $n=3$,

$$g(x,\xi) = |x-\xi|^{-1} - |x|^{-1}|x^*-\xi|^{-1},$$

and v has a pole of order 1 at each point of $S=\bigcup_{m=2}^{\infty} S_m$. Consequently, in the neighborhood of any $x\in S$, v^3 is not integrable. But every conical region T, situated except for its vertex entirely in the open unit sphere K, contains points of S. Thus v^3 is not integrable over T; and so, everywhere on Σ, v fails to have a non-tangential limit in the metric L^3.

8. In the preceding sections we considered functions subharmonic in a sphere of any number of dimensions. We shall now restrict ourselves to the case $n = 2$ and prove a result concerning functions subharmonic in any domain bounded by a simple closed rectifiable curve. Presumably the analogue of the theorem that follows holds for any n, but the proof we give uses conformal mapping and is therefore valid for $n = 2$ only.

THEOREM 2. *Let $u(z)$ be a function subharmonic in a domain D bounded by a simple closed rectifiable curve C. Suppose that there is a set E on C of positive measure (length) with the following property: with each point $z_0 \in E$ we can associate an open triangle $\Delta = \Delta(z_0) \subset D$ with vertex at z_0, such that $u(z)$ is bounded from above in Δ. Then at almost all points $z_0 \in E$, the function $u(z)$ has a non-tangential limit in the metric L^p, $1 \leqslant p < \infty$.*

We say that $u(z)$ has a *non-tangential limit equal to* λ in the metric L^p at the point $z_0 \in C$ if for every family of homothetic open triangles $\Delta \subset D$ with common vertex z_0, shrinking to z_0, we have

$$\frac{1}{|\Delta|}\int_{\Delta} |u(z)-\lambda|^p \, dA(z) = o(1),$$

where $dA(z)$ denotes the element of area in Δ. Since C has a tangent at almost all points of E, it follows that the angle of Δ at z_0 may, for almost all $z_0 \in E$, be as close to π as we wish. The present definition is consistent with the one given in Section 1.

Proof of Theorem 2. The proof resembles an argument from the theory of analytic functions ([5], Vol. II, p. 199-201) and we may be brief. We first prove the theorem for the special case where D is the unit circle $|z| < 1$.

Let z_0 vary over E. We may assume (perhaps after a denumerable decomposition) that E is closed; that the triangles $\Delta(z_0)$ are all congruent and small; that each $\Delta(z_0)$ is symmetric with respect to the radius of D terminating at z_0; and that the upper bound of $u(z)$ in all the $\Delta(z_0)$ is less than a fixed constant. The union of the $\Delta(z_0)$ suitably extended inward in D yields a star-shaped domain $D_1 \subset D$ with a rectifiable boundary $C_1 \supset E$, such that the function $u(z)$ is bounded from above in D_1.

Let $z = \Phi(\zeta)$ be a conformal mapping of the unit circle $|\zeta| < 1$ onto D_1, extended so as to be bicontinuous from $|\zeta| \leqslant 1$ onto $D_1 \cup C_1$; and let $u_1(\zeta) = u[\Phi(\zeta)] = u(z)$. The function $u_1(\zeta)$ is subharmonic and bounded from above for $|\zeta| < 1$. For a given θ $(0 \leqslant \theta < 2\pi)$ denote by Δ_θ members of a family of homothetic open triangles situated in D, with common vertex $e^{i\theta}$, and shrinking to $e^{i\theta}$. By Theorem 1, u_1 has a non-tangential limit at almost all points $e^{i\theta}$ of $|\zeta| = 1$. Hence for almost all $e^{i\theta} \in \Phi^{-1}(E)$ there is a number $\lambda = \lambda(e^{i\theta})$ such that

$$\frac{1}{|\Delta_\theta|}\int_{\Delta_\theta} |u_1(\zeta)-\lambda|^p \, dA(\zeta) \to 0 \tag{8.1}$$

as the sides of Δ_θ shrink to 0. At almost all points $e^{i\theta} \in \Phi^{-1}(E)$, $z = \Phi(\zeta)$ is conformal (with the derivative $\Phi'(\zeta)$ approaching a non-zero limit as ζ approaches $e^{i\theta}$ non-tangentially). Also, under Φ, the sets of measure zero on $|\zeta| = 1$ correspond to the sets of measure zero on C_1 ([5], Vol. I, p. 289-295). Using (8.1) and the specified properties of Φ, we obtain the conclusion of the theorem for the case where the domain D is the open unit circle.

Now let D be any domain bounded by a simple closed rectifiable curve. Using a conformal mapping of the open unit circle K onto D and an argument paralleling the one in the paragraph above, we see that the theorem holds for D, since it holds for K.

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