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Some remarks on the Marcinkiewicz convexity theorem in the upper triangle

by

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Let (X, μ) and (Y, ν) be two measure spaces, and for each function $h(y)$ defined on Y let

$$E(\lambda) = \{y \in Y, |h(y)| > \lambda\}, \qquad h_*(\lambda) = \nu(E(\lambda)),$$
$$\|h\|_{q\infty} = \sup_{\lambda>0} \lambda(h_*(\lambda))^{1/q}.$$

Let

$$(*) \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad 0 < \theta < 1,$$

so that the point $P = (1/p, 1/q)$ is interior to the segment P_0P_1, $P_j = (1/p_j, 1/q_j)$, $j = 0, 1$, and let $L^p = L^p(X, \mu)$, $L^q = L^q(Y, \nu)$. The Marcinkiewicz-Zygmund convexity theorem says that if T is a sublinear operator, which assigns to each function $f \in L^{p_0} + L^{p_1}$ a measurable function Tf defined on Y, such that

$$(1) \qquad \|Tf\|_{q_j\infty} \leqslant M_j \|f\|_{p_j}, \qquad f \in L^{p_j}, \qquad j = 0, 1,$$

then the following convexity inequality is true:

$$(2) \qquad \|Tf\|_q \leqslant c M_0^{1-\theta} M_1^{\theta} \|f\|_p \quad \text{if} \quad f \in L^p,$$

where $c = c(p_0 - p, p_1 - p)$; $c \to \infty$ if $p_0 - p \to 0$, or $p - p_1 \to 0$. In particular,

(2a) T is a bounded operator from L^p to L^q.

This theorem was originally proved by Zygmund [15] under the following hypothesis:

(a) The segment P_0P_1 belongs to the lower triangle $p_j \leqslant q_j$, $1 \leqslant p_j, q_j$, and $q_0 \neq q_1$.

Zygmund did not consider the case where one of the points P, P_0, P_1 belongs to the upper tiangle $(p \geqslant q)$. Later a general theory of interpolation was developed, and from a theorem of Calderón concerning interpolation of Lorentz spaces (Calderón [3], Lions-Peetre [8], O'Neil [11], Hunt [5] and Krée [6]) the following complement to (a) was obtained:

(b) Inequality (2) is true if the point $P = (1/p, 1/q)$ belongs to the diagonal, that is if $p = q$.

In [5], Hunt gave a simple proof of (b) which uses only an inequality of Hardy, and indicated a counter-example for the case where P lies strictly in the upper triangle. However, in [5] only (2a) is proved, but not (2), and the counter-example there proves only that

(b_1) Theorem (2a) is not true if $q < p$ and P_0P_1 is parallel to the diagonal.

Theorem (b) was also proved independently by M. Cotlar (cf. [14] and [12], p. 97) in the following formally more general form:

(c) If (2) holds for a given segment P_0P_1 and for any measure space (Y, ν) (and sublinear operators), then it also holds for any other segment obtained by rotating P_0P_1 around P (provided none of the segments is horizontal nor vertical).

He also considered the case where P is in the upper triangle but μ and ν are finite measures and the slope of P_0P_1 is negative, and indicated (for linear operators) the following property:

(d) If $\mu(X) = \nu(Y) = 1$ if $p_0 > p_1, q_0 < q_1$, and if $q \leqslant p$, then (with θ as in (*)) we have

(3) $$\|Tf\|_q \leqslant C_{p_0-p, p_1-p, \varepsilon} M_0^{1-\bar{\theta}} M_1^{\bar{\theta}} \|f\|_p, \qquad 0 < \bar{\theta} < 1,$$

where

(3a) $$\bar{\theta} = \theta + \varepsilon\left(\frac{1}{p} - \frac{1}{q}\right), \qquad \bar{\theta} \to \theta \quad \text{if} \quad \frac{1}{p} - \frac{1}{q} \to 0.$$

Here $C_{p_0-p, p_1-p, \varepsilon} \to \infty$ if $\varepsilon \to 0$, but does not depend on $1/p - 1/q$. Hence for $q = p$ inequality (3) reduces to (2).

P. Krée observed that (d) is easily deduced (and extended to Lorentz spaces) using general facts from interpolation theory (see theorem 3); in particular, it follows from theorem 3 that

(e) Proposition (d) holds even if T is quasilinear and if (1) merely holds for characteristic functions.

Since inequality (2) (unlike (3) if $\bar{\theta} \neq \theta$) is invariant under the change of measure of the form $d\nu' = k d\nu$, it follows that if (2) holds for $\mu(X) = \nu(Y) = 1$, then it holds for any μ, ν, and by (c) we obtain

(c_1) Assume that, at a given point P, (2) holds if $\mu(X) = \nu(Y) = 1$ and P_0P_1 has negative slope, then (2) holds for any segment P_0P_1, $P \in P_0P_1$, and any measures μ and ν.

If in (3) we fix ε and let $q \to p$, then we have $\bar{\theta} \to \theta$ and (3) reduces to (2); and from (e) and (c_1) we obtain (b). Hence (b) *is contained in* (e) *and* (c), *so that* (e) *and* (c) *give a generalization of Marcinkiewicz's convexity theorem for the upper triangle.* From (c_1) it follows also that (3) cannot be improved: *if* $q < p$, *then* (3) *cannot hold with* $\bar{\theta} = \theta$, *that is* (2) *cannot hold*, for otherwise by (c_1) we shall obtain a contradiction with (b_1).

In (b_1) the angle between P_0P_1 and the diagonal is zero, in (d) this angle is greater than 45°. Berenstein and Kerzman considered the remaining cases and proved that

(f) If the slope of the segment P_0P_1 is positive, whichever this may be, (2a) does not hold for any point $P \in P_0P_1$ lying in the interior of the upper triangle, even if μ and ν are finite and T linear. Besides, if the measures are allowed to be infinite, this also applies to negative slopes.

They also made some simplifications and extensions which appear in the proof of theorem 3. The proof of (f) is in section 6. Though (d) and (c) are already included in (e) and (f), we reproduce the original proofs in sections 2 and 3 since they do not use the general interpolation theory and may be of use in some other situation.

Remark. Let us observe that, as Guido Weiss pointed out, in our case, of negative slope and finite measures, it is evident that the continuity property (2a) holds and that

(4) $$\|Tf\|_q \leqslant c' \, M_1 \|f\|_p,$$

where c' depends on μ, ν and $p - p_1$.

In fact, in our case

$$\|f\|_{p_1} \leqslant c \|f\|_p$$

and

$$(\|Tf\|_q)^q = \int_0^1 |(Tf)^*(t)|^q dt \leqslant \int_0^1 (M_1 t^{-1/q_1} \|f\|_{p_1})^q dt \leqslant \left(c'(q_1 - q)^{-1} M_1 \|f\|_{p_1}\right)^q.$$

Thus if T is of weak type at P_1, then it is of strong type in the rectangle $\{1/p < 1/p_1, 1/q > 1/q_1\}$; and one can obtain other analogous results for other types of rectangles.

However (4) does not contain (b), even if combined with property (III) of § 1. Hence, though the proof *via* (4) is simpler than that of (3), it does not lead to (2) for $q = p$, and does not yield a proper generalization of (a) and (b) for the upper triangle as (e) and (c) do.

1. Notation

Let (X, μ) be a measure space, and f a (real or complex) measurable function on X; its *distribution function* is defined by

$$f_*(\lambda) = \mu\{x: |f(x)| > \lambda\}, \qquad \lambda > 0.$$

If $\mathscr{E} = \{f\}$ is a linear set of μ-measurable functions, and T is an operator which assigns to each $f \in \mathscr{E}$ a ν-measurable function defined on (Y, ν), then T is said to be *of (strong) type* (p, q) if $\|Tf\|_q \leqslant C\|f\|_p, f \in \mathscr{E}$. T is *of weak type* (p, q), $0 < p \leqslant \infty$, $0 < q < \infty$ if

(I) $$\sup_{0<\lambda} \lambda [f_*(\lambda)]^{1/q} \leqslant C\|f\|_p, \quad f \in \mathscr{E},$$

and *weak type* (p, ∞) is the same that strong type (p, ∞).

If f^* is the *non-increasing rearrangement* of $f, t > 0, f^*(t) = \inf\{\lambda: f_*(\lambda) \leqslant t\}$, then (see [10])

$$\sup_{0<\lambda} \lambda [f_*(\lambda)]^{1/q} = \sup_{0<t} t^{1/q} f^*(t), \quad 0 < q < \infty,$$

and if

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \quad \text{and} \quad 1 < q < \infty, \ \frac{1}{q} + \frac{1}{q'} = 1,$$

then

(II) $$\sup_{0<\lambda} \lambda [f_*(\lambda)]^{1/q} \leqslant \sup_{0<t} t^{1/q} f^{**}(t) \leqslant q' \sup_{0<\lambda} \lambda [f_*(\lambda)]^{1/q}.$$

The *square of types* is the plane set of those $(1/p, 1/q)$ such that $1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant \infty$.

A linear space $\mathscr{E}$ of μ-measurable functions is *closed by cuts* if $f \in \mathscr{E}$ implies that $f_\lambda \in \mathscr{E}$, where

$$f_\lambda(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \leqslant \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Let $P_j = (1/p_j, 1/q_j)$, $P = (1/p, 1/q)$; $P = (1-\theta)P_0 + \theta P_1$, $0 < \theta < 1$.

We shall say that Marcinkiewicz's theorem holds with *data at* P_0, P_1, and *thesis at* $P = (1/p, 1/q)$ if whenever (X, μ), (Y, ν) are measure spaces, $\mathscr{E}$ a space closed by cuts, and T a sublinear operator defined on $\mathscr{E}$ of *weak type* (p_j, q_j), $j = 0, 1$, with constants M_j, then T is of *strong type* (p, q) and

$$\|Tf\|_q \leqslant CM_0^{1-\theta} M_1^\theta \|f\|_p, \quad f \in \mathscr{E},$$

where C is a constant independent of T and M_j.

(III) It is a known fact ([4], p. 193) that if T is of weak type (p_j, q_j), $j = 0, 1$, then it is weak type (p, q) with constant M,

$$M \leqslant CM_0^{1-\theta} M_1^\theta,$$

and that in this case C is an absolute constant (for example $C = 6$).

(IV) We remark that if P_0, P_1 are two points of the real plane, and $P = (1-\theta)P_0 + \theta P_1$, $0 < \theta < 1$, we can find $\tilde{P}_0, \tilde{P}_1$ lying in the segment P_0P_1, as near P as we want and such that

$$\tilde{P}_j = (1-\theta_j)P_0 + \theta_j P_1 \quad \text{and} \quad P = (1-\theta)\tilde{P}_0 + \theta \tilde{P}_1.$$

We get in this case that if $\tilde{M}_j \leqslant C_j M_0^{(1-\theta_j)} M_1^{\theta_j}$, $j = 0, 1$, and $M \leqslant C_2 \tilde{M}_0^{(1-\theta)} \tilde{M}_1^\theta$, then $M \leqslant CM_0^{1-\theta} M_1^\theta$.

Besides we shall also use the following properties of f^* (see [10]):

(V) $$\int_X f(x) g(x)\, d\mu \leqslant \int_0^\infty f^*(t) g^*(t)\, dt;$$

(VI) $$[f^*(t)]^a = (|f|^a)^*(t) \quad \text{if} \quad 0 < a < \infty;$$

(VII) if $\mu(X) < \infty$, then $f^*(t) = 0$ for $t \geqslant \mu(X)$.

Finally we shall assume known the following theorem of Marcinkiewicz-Zygmund [15]:

(VIII) The Marcinkiewicz theorem holds whenever the segment P_0P_1 lies in the lower triangle.

2. Invariance of the Marcinkiewicz theorem under rotations

THEOREM 1. *Let $l = P_0P_1$, $l' = P_0'P_1'$ be two segments (contained in the interior of the square of types) which intersect at P, such that they are neither horizontal nor vertical. If the thesis of Marcinkiewicz's theorem holds in P when the data are given on l, then it holds in P when the data are given on l'.*

Proof. We may assume that the slopes of l and l' have opposite signs, since otherwise the thesis will follow by applying the argument twice. By (III) and (IV) of § 1 we can also assume that l' is so small that the parallels to the ordinate axis through P_j intersect l.

We define the operator S by

(1) $$Sf(t) = t^c (Tf)^{**}(t), \quad t \in (0, \infty),$$

(2) $$c > \frac{1}{q_j}, \quad j = 0, 1;$$

then S is sublinear; we define in $R_+ = (0, \infty)$ the measure

$$d\omega = t^{-cq}\, dt,$$

where dt is Lebesgue measure, and let

$$\|h\|_{q,\omega} = \|h\|_{L^q(R_+, \omega)}.$$

We have

$$\|Sf\|_{q,\omega}^q = \int_0^\infty (Sf)^q(t)\, d\omega = \int_0^\infty t^{cq} [(Tf)^{**}(t)]^q t^{-cq}\, dt \geqslant \|Tf\|_q^q.$$

Now we intend to prove that S is of strong type (p, q). Let $E_\lambda = \{t > 0: (Sf)(t) > \lambda\}$; if $t \in E_\lambda$, then, by (II) of § 1,

$$\lambda < t^c (Tf)^{**}(t) = t^{c-1/q_j} t^{1/q_j} (Tf)^{**}(t) \leqslant t^{c-1/q_j} q_j' M_j \|f\|_{p_j}.$$

Then if

$$a = \left[\frac{\lambda}{q_j' M_j \|f\|_{p_j}}\right]^{1/(c-1/q_j)}$$

we see by (2) that

$$\omega(E_\lambda) \leqslant \int_a^\infty d\omega = \frac{a^{-cq+1}}{cq-1}$$

so that

$$(Sf)_*(\lambda) \leqslant \frac{1}{cq-1}\left[\frac{q_j' M_j \|f\|_{p_j}}{\lambda}\right]^{a_j p_j}$$

where

$$a_j = \frac{cq-1}{cq_j-1}.$$

This shows that S is weak type $(p_j, a_j q_j)$ and after an easy computation we get

$$\frac{1}{q} = \frac{1-\theta}{a_0 q_0} + \frac{\theta}{a_1 q_1}. \tag{3}$$

Let s_j be such that $(1/p_j, 1/s_j) \in l$, then taking

$$c = \frac{1}{q_0}\frac{s_0 - q_0}{s_0 - q} = \frac{1}{q_1}\frac{s_1 - q_1}{s_1 - q}$$

we have $a_j q_j = s_j$ and (2) holds, since l and l' have opposite slopes.

Our hypothesis says that S is strong type (p, q) with

$$\|Sf\|_{q,\omega} \leqslant C_0 C(l)\, M_0^{1-\theta} M_1^\theta \|f\|_p.$$

As $\|Tf\|_q \leqslant \|Sf\|_{q,\omega}$ the theorem is proved.

From (VIII) of § 1 it follows that

COROLLARY. *Marcinkiewicz's theorem holds for P in the lower triangle even if one datum is in the upper one.*

3. Invariance of Marcinkiewicz's theorem under vertical translations

In this section we shall prove proposition (d) stated in the Introduction, so that the measures μ and ν will be assumed finite and the slope of the segment P_0P_1 negative.

The aim of the proof is *to obtain a generalization of the convexity inequality* (2) of the Introduction. Instead the proof of (2a) is immediate now since, when the measures μ and ν are finite, weak type (p, q) implies strong type (r, s) for $r < p$, $s < q$ (this follows also from (d)).

It is necessary to observe also that theorems (d), and that of § 2, cannot be joined together to get a proof of Marcinkiewicz's theorem in the whole upper triangle for finite measure (which would contradict known counter-examples, see § 6), because (d) does not apply to infinite measure spaces as required in § 2.

If

$$f = \sum_{i=1}^n c_i \chi_i$$

is a simple function, then $\mathscr{E}(f)$ will denote the linear space spanned by the n characteristic functions χ_i. Taking into account that the simple functions are dense in $L^p, p < \infty$, in proving (d) it will be enough to merely consider the case $\mathscr{E} = \mathscr{E}(f)$, which is a finite-dimensional vector space closed by cuts and powers, provided the constant obtained in the proof is independent of $\mathscr{E}(f)$.

The idea of the proof is to show that, in case of finite measures and negative slopes, the Marcinkiewicz theorem is invariant under vertical translations of the segment P_0P_1.

THEOREM 2. *Let (X, μ) and (Y, ν) be totally finite measure spaces, $\mathscr{E}$ a vector space of simple functions on X, closed by cuts and powers, T a linear operator defined on $\mathscr{E}$ of weak types (p_j, q_j) with norms M_j, $j = 0, 1$, where*

$$1 < p_1 < p_0 < \infty, \quad 1 < q_0 < q_1 < \infty$$

and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 < \theta < 1.$$

The we can choose $\tilde\theta$, $0 < \tilde\theta < 1$, such that $\tilde\theta$ is as near θ as we like and

$$\|Tf\|_q \leqslant C M_0^{1-\tilde\theta} M_1^{\tilde\theta} \|f\|_p \quad \text{for all } f \in \mathscr{E}.$$

C depends only upon $p_j, q_j, \theta, \tilde\theta, \mu(X)$ and $\nu(Y)$. In general $\tilde\theta \neq \theta$.

Proof. As was observed, we may assume that $\mathscr{E}$ is a finite-dimensional vector space.

$\mathscr{E}$ being finite-dimensional and T linear, T is of strong type (p, q) and there exists $f_0 \in \mathscr{E}$ such that $\|f_0\|_p = 1$ and $\|Tf_0\|_q = \|T\|$. If $g \in \mathscr{E}$, the function $\|T(f_0+sg)\|_q / \|f_0 + sg\|_p$ attains its maximum at $s = 0$, and it follows (cf. [13], ch. IX, § 2) that if $1 < p, q < \infty$, then

$$\frac{\int_Y |Tf_0|^q d\nu}{\int_X |f_0|^p d\mu} = \frac{\int_Y |Tf_0|^{q-1} (\operatorname{sgn} Tf_0)\, Tg\, d\nu}{\int_X |f_0|^{p-1} (\operatorname{sgn} f_0)\, g\, d\mu}.$$

Taking $g = |f_0|^\lambda \operatorname{sgn} f_0$, we obtain

$$\text{(1)} \qquad \frac{\|Tf_0\|_q^q}{\|f_0\|_p^p} \leqslant \frac{\int_Y |Tf_0|^{q-1}|Tg|\,d\nu}{\|f_0\|_{p_2}^{p-1}\|g\|_{p_3}}$$

with

$$\text{(2)} \qquad p_2 = p+\lambda-1, \qquad p_3 = \frac{p+\lambda-1}{\lambda}.$$

Because of the corollary of § 2, we only consider the case when $P = (1/p, 1/q)$ is strictly in the upper triangle above diagonal. Calling l the segment P_0P_1, $P_j = (1/p_j, 1/q_j)$, we can suppose by (III), (IV) of § 1 that l is contained in the interior of the upper triangle and that there exists $\gamma > 0$ such that

$$l_\gamma = \left\{\left(\frac{1}{r}, \frac{1}{s}-\gamma\right) : \left(\frac{1}{r}, \frac{1}{s}\right) \in l\right\}$$

is contained in the interior of the lower triangle.

Define a new operator T_γ on $\mathscr{E}$ by

$$(T_\gamma f)(t) = t^\gamma (Tf)^{**}(t), \qquad 0 < t < \infty.$$

Using the hypothesis and (II) of § 1

$$(T_\gamma f)(t) \leqslant q_j' M_j t^{-1/q_j} t^\gamma \|f\|_{p_j};$$

but $\gamma < 1/q_j$, then

$$\text{(3)} \qquad (T_\gamma f)^{**}(t) \leqslant q_j'^2 M_j t^{\gamma-1/q_j}\|f\|_{p_j},$$

which means that T_γ is weak type $(p_j, q_j/(1-\gamma q_j))$.

If in (2) we choose λ near to 1, $\lambda > 1$, we can find p_2 and p_3 such that

$$\text{(4)} \qquad p_1 < p_3 < p < p_2 < p_0.$$

Let q_2, q_3 be such that $(1/p_2, 1/q_2) \in l$, $(1/p_3, 1/q_3) \in l$; then by the theorem of Marcinkiewicz (applied in the lower triangle), we have for all $f \in \mathscr{E}$

$$\text{(5)} \qquad \|T_\gamma f\|_{q_2/(1-\gamma q_2)} \leqslant K(q_0'^2 M_0)^{1-\theta_2}(q_1'^2 M_1)^{\theta_2}\|f\|_{p_2},$$

$$\text{(6)} \qquad \|T_\gamma f\|_{q_3/(1-\gamma q_3)} \leqslant K(q_0'^2 M_0)^{1-\theta_3}(q_1'^2 M_1)^{\theta_3}\|f\|_{p_3}$$

with

$$\text{(7)} \qquad \frac{1-\theta_2}{p_0} + \frac{\theta_2}{p_1} = \frac{1}{p_2}, \qquad \frac{1-\theta_3}{p_0} + \frac{\theta_3}{p_1} = \frac{1}{p_3}$$

and K is the geometrical constant given by the theorem.

Let $f_0 \in \mathscr{E}$ as above, giving $\|T\|$, and $g \in \mathscr{E}$, using (V), (VI), (VII) of § 1

$$\int_Y |Tf_0|^{q-1}|Tg|\,d\nu \leqslant \int_0^\infty [(Tf_0)^*(t)]^{q-1}(Tg)^*(t)\,dt = \int_0^{\nu(Y)} [(Tf_0)^*(t)]^{q-1}(Tg)^*(t)\,dt$$
$$\leqslant \int_0^{\nu(Y)} [(Tf_0)^{**}(t)]^{q-1}(Tg)^{**}(t)\,dt = \int_0^{\nu(Y)} [T_\gamma f_0(t)]^{q-1} T_\gamma g(t) t^{-\gamma q}\,dt.$$

In order to apply Hölder's inequality we introduce

$$k_1 = \frac{q_2}{(1-\gamma q_2)(q-1)}, \qquad k_2 = \frac{q_3}{1-\gamma q_3}, \qquad \frac{1}{k_3} = 1 - \frac{1}{k_1} - \frac{1}{k_2}.$$

It is easy to verify, using that the slope of l is negative, that $1 \leqslant k_1$, $1 \leqslant k_2$, and for k_3 we have

$$\text{(8)} \qquad \frac{1}{k_3} > q\gamma > 0$$

because

$$\frac{1}{k_3} = 1 - \frac{1}{k_1} - \frac{1}{k_2} = 1 - \frac{q}{q_2} + \frac{1}{q_2} - \frac{1}{q_3} + \gamma q$$

and by (2)

$$\frac{1/q_2 - 1/q_3}{1/q_2 - 1/q} = \frac{1/p_3 - 1/p_2}{1/p - 1/p_2}$$
$$= \frac{p+\lambda-1-(p+\lambda-1)/\lambda}{p_2 p_3} \cdot \frac{p p_2}{p+(\lambda-1)-p} = p;$$

then

$$1 - \frac{q}{q_2} + \frac{1}{q_2} - \frac{1}{q_3} = q\left(\frac{1}{q} - \frac{1}{q_2}\right) + \frac{1}{q_2} - \frac{1}{q_3} > 0$$

because $p > q$.

Then, by Hölder inequality, we obtain from (8), (6) and (5)

$$\int_Y |Tf_0|^{q-1}|Tg|\,d\nu \leqslant \|T_\gamma f_0\|_{q_2/(1-\gamma q_2)}^{q-1} \|T_\gamma g\|_{q_3/(1-\gamma q_3)} \frac{[\nu(Y)]^{1-\gamma q k_3}}{1-\gamma q k_3}$$
$$\leqslant C_0 M_0^{(1-\theta_2)(q-1)+(1-\theta_3)} M_1^{\theta_2(q-1)+\theta_3} \|f_0\|_{p_2}^{q-1}\|g\|_{p_3}.$$

Applying (1), and since $\|f_0\|_p = 1$, we have

$$\|T\|^q = \|Tf_0\|_q^q \leqslant C_0 M_0^{(1-\theta_2)(q-1)+(1-\theta_3)} M_1^{\theta_2(q-1)+\theta_3} \|f_0\|_{p_2}^{q-p}.$$

As by (4) we have $p < p_2$, it follows

$$\|f_0\|_p \leqslant \mu(X)^{(p_2-p)/p_2 p}\|f_0\|_{p_2}$$

and then $(q-p < 0)$ we obtain

$$\|T\| \leqslant CM_0^{1-\tilde\theta}M_1^{\tilde\theta}$$

with $\tilde\theta = \theta_2(1-1/q)+\theta_3/q$ by (2), if $\lambda \to 1$, $\tilde\theta \to 0$ (although C may increase when $\tilde\theta \to \theta$). Also $\tilde\theta \to \theta$ if $q \to p$ since $\theta = \theta_2(1-1/p)+\theta_3/p$, q. e. d.

4. Review of some notions of interpolation theory

Let (X, μ) be a measure space, and $\mathscr{M}(X)$ the family of (equivalence classes) of measurable functions (complex-valued for example). If f^* is the non-increasing rearrangement of f (see § 1) the Lorentz spaces $L^{pq}(X)$ (see [3], [9], [10] for $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ and [5], [6] for $0 < p, q \leqslant \infty$) are defined by

$$L^{pq}(X) = \left\{f \in \mathscr{M}(X) \colon \|f\|_{pq} = \left\{\int_0^\infty [t^{1/p}f^*(t)]^q \frac{dt}{t}\right\}^{1/q} < \infty\right\} \text{ if } 0 < p, q < \infty,$$

$$L^{p\infty}(X) = \{f \in \mathscr{M}(X) \colon \|f\|_{p\infty} = \sup_{0<t} t^{1/p}f^*(t) < \infty\} \text{ if } 0 < p \leqslant \infty,$$

$$L^{\infty q}(X) = L^\infty(X) \quad \text{for all } q > 0.$$

They are naturally quasinormed spaces, i. e. $\|\cdot\|_{pq}$ is a quasinorm in the sense that:

1. if $\|f\|_{pq} = 0$, then $f = 0$, $\|f\|_{pq} \geqslant 0$;
2. $\|\lambda f\|_{pq} = |\lambda|\,\|f\|_{pq}$ if λ is a scalar;
3. there exists $C \geqslant 1$ such that $\|f+g\|_{pq} \leqslant C\{\|f\|_{pq}+\|g\|_{pq}\}$.

Therefore these spaces are metrizable topological vector spaces (but not locally convex in general if $p \leqslant 1$) and it can be proved that they are complete [2], so that the closed graph theorem applies to them (see [1]).

If $p > 1$, $q \geqslant 1$, they are Banach spaces, $L^{pp}(X) = L^p(X)$ if $0 < p \leqslant \infty$ and

(I) $L^{pq}(X) \overset{\text{id}}{\to} L^{pr}(X)$ holds and is continous if $r \geqslant q$. A sublinear operator T is of weak type (p, q) (see § 1) if $T \colon L^p(X) \to L^{q\infty}(Y)$ is bounded. T is said to be of *restrained weak type* (p, q) if $T \colon L^{p1}(X) \to L^{q\infty}(Y)$ is bounded (this is equivalent to be weak type (p, q) only for characteristic functions, see [10]).

Let (A_j), $j = 0, 1$, be a pair of interpolation, i.e., a pair of quasinormed spaces, continuously contained in a separated topological vector space A, and $(A_0, A_1)_{\theta p}$, $0 < \theta < 1$, $0 < p \leqslant \infty$, the naturally quasinormed space of those $a \in A_0+A_1$ such that

$$\left\{\int_0^\infty [t^{-\theta}K(t, a)]^p \frac{dt}{t}\right\}^{1/p} < \infty \quad \text{if} \quad 0 < p < \infty,$$

with the obvious modification for $p = \infty$, where

$$K(t, a) = \inf\{\|a_0\|_{A_0}+t\|a_1\|_{A_1} \colon a = a_0+a_1,\ a_j \in A_j\}.$$

Then $A_0 \cap A_1 \subseteq (A_0, A_1)_{\theta p} \subseteq A_0+A_1$ holds.

If (B_j), $j = 0, 1$, is another pair of interpolation and

$$T \colon A_0+A_1 \to B_0+B_1,$$

we say that T is *bounded quasilinear* if there are two constants $M_j > 0$ such that for every (a_j), $j = 0, 1$, $a_j \in A_j$, $a_0+a_1 = a$ there exists $b_j \in B_j$, $j = 0, 1$, with the property $b_0+b_1 = Ta$, $\|b_j\|_{B_j} \leqslant M_j\|a_j\|_{A_j}$.

In that case the interpolation theorem (see [8], [6]) says that

$$T \colon (A_0, A_1)_{\theta p} \to (B_0, B_1)_{\theta p},$$

$$\|Ta\|_{(B_0,B_1)_{\theta p}} \leqslant CM_0^{1-\theta}M_1^{\theta}\|a\|_{(A_0,A_1)_{\theta p}},$$

where C is a constant.

We remark that the bounded linear or sublinear operators are bounded quasilinear.

The following generalization of the Calderón's identities given in [6] and [7] is true for the Lorentz spaces: if $p_0 \neq p_1$, then

$$(L^{p_0q_0}, L^{p_1q_1})_{\theta r} = L^{pr} \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \quad 0 < p_j, q_j, r \leqslant \infty$$

and with equivalent quasinorms. Therefore it follows that if T is bounded quasilinear

$$T \colon L^{p_0r_0}(X)+L^{p_1r_1}(X) \to L^{q_0s_0}(Y)+L^{q_1s_1}(Y),$$

then $T \colon L^{pr}(X) \to L^{qs}(Y)$ is bounded for $r \leqslant s$,

$$\frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0}+\frac{\theta}{q_1},$$

and also

$$\|Tf\|_{qs} \leqslant CM_0^{1-\theta}M_1^{\theta}\|f\|_{pr}$$

with $C = C(p_j, q_j, \theta)$.

5. The theorem of Marcinkiewicz in the upper triangle. An interpolation approach

THEOREM 3. *Let (X, μ) and (Y, ν) be totally finite measure spaces; T a bounded quasilinear operator from $L^{p_0r_0}(X)+L^{p_1r_1}(X)$ into $L^{q_0s_0}(Y)+ +L^{q_1s_1}(Y)$, $0 < r_j, p_j, s_j, q_j \leqslant \infty$, the constants being M_0 and M_1; and the slope of the segment P_0P_1 be negative, where $P_j = (1/p_j, 1/q_j)$. Then T is bounded from $L^{pr}(X)$ into $L^{qs}(Y)$,*

$$\frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0}+\frac{\theta}{q_1}, \qquad 0<\theta<1,\ 0<r,s\leqslant\infty;$$

moreover, its norm M satisfies $M \leqslant CM_0^{1-\tilde\theta}M_1^{\tilde\theta}$ with C a geometrical constant, $C = C\big(p_j, r_j, q_j, s_j, \mu(X), \nu(Y), \theta, \tilde\theta\big)$ and $\tilde\theta$ may be chosen as near θ as we like (but in general $\tilde\theta \neq \theta$).

The proof is based on the following remark.

If (Z, σ) is a totally finite measure space, and $1/t < 1/v$, $0 < t, u, v, w \leqslant \infty$, then $L^{tu}(Z) \overset{\mathrm{id}}{\to} L^{vw}(Z)$ is bounded.

For $L^{tu}(Z) \subseteq L^{zu}(Z)$ if $1/t < 1/z$, $0 < z \leqslant \infty$, choosing z such that $1/v < 1/z$, we obtain

$$L^{tu}(Z) \subseteq L^{tu}(Z) \cap L^{zu}(Z) \subseteq \big(L^{tu}(Z), L^{zu}(Z)\big)_{\alpha w} = L^{vw}$$

with an α such that $0 < \alpha < 1$,

$$\frac{1-\alpha}{t}+\frac{\alpha}{z} = \frac{1}{v}.$$

As id: $L^{tu}(Z) \to L^{vw}(Z)$ has closed graph, both spaces being continuously contained in the same separated linear topological space, it is continuous.

Proof of the theorem. As mentioned in § 4, if $r \leqslant s$, the thesis is a known result which holds without any restriction. So we study $r > s$; we may suppose $p_1 < p_0$.

Take p_2 such that $1/p < 1/p_2 < 1/p_1$, and $\tilde\theta$ which fulfils $0 < \tilde\theta < 1$, $1/p_2 = (1-\tilde\theta)/p_0+\tilde\theta/p_1$.

If $1/q_2 = (1-\tilde\theta)/q_0+\tilde\theta/q_1$, then

$$L^{pr}(X) \overset{\mathrm{id}}{\to} L^{p_2s}(X) = \big(L^{p_0r_0}(X), L^{p_1r_1}(X)\big)_{\tilde\theta s} \overset{T}{\to} \big(L^{q_0s_0}(Y), L^{q_1s_1}(Y)\big)_{\tilde\theta s}$$
$$= L^{q_2s}(Y) \overset{\mathrm{id}}{\to} L^{qs}(Y)$$

is bounded by the above remark, the interpolation theorem and taking into account that the slope is negative, $1/q_2 < 1/q$. Moreover

$$\|Tf\|_{qs} \leqslant CM_0^{1-\tilde\theta}M_1^{\tilde\theta}\|f\|_{pr}, \qquad \text{q. e. d.}$$

The remarks of § 4 show that theorem 2 follows from this, even if we have only weak restrained type as the data. Further, it is not necessary to have T defined in whole of $L^{p_0q_0}+L^{p_1q_1}$, the proof applies if T is defined in a vector space $\mathscr{E}$ closed by cuts, $\mathscr{E} \subseteq L^{p_0q_0} \cap L^{p_1q_1}$, because Calderón's identies also hold in this case.

To deduce (e) from theorem 3 it is sufficient to take on the end of the proof of this theorem, $L^{pr} = L^{pp}$, $L^{qs} = L^{qq}$, and observe that if $q < p$, $p_2 < p$, $\mu(X) = 1$, then by Hölder's inequality

$$\|f\|_{L^{p_2q}} = \Big\{\int_0^1 \big(f^*(t)t^{1/p_2}\big)^q \frac{dt}{t}\Big\}^{1/q}$$
$$\leqslant \varkappa\Big\{\int_0^1 \big(f^*(t)t^{1/p}\big)^p \frac{dt}{t}\Big\}^{1/p} = \varkappa\|f\|_{L^p},$$

where $\varkappa = (1/\varepsilon)^{1/q-1/p}(1/q-1/p)^{1/q-1/p}$, $\varepsilon = 1/p_2-1/p$, and p_2 may be allowed to tend to p if q tends to p; so that if in theorem 3 we put $\tilde\theta = \theta+ +\varepsilon(1/p_1-1/p)$, we shall obtain $M \leqslant \mathrm{const}(1/\varepsilon)^{1/q-1/p}M_0^{1-\tilde\theta}M_1^{\tilde\theta}$, and for $p = q$ we shall obtain (2).

6. Counter-examples

It is known that Marcinkiewicz's theorem always holds for vertical segments and that it does not on horizontal ones (see [4]).

It can be shown that if $(1/p_0, 1/q_0)$ belongs to the interior of the upper triangle then:

A. *The Marcinkiewicz theorem can never hold at $(1/p_0, 1/q_0)$ with data on l, if the segment l has positive slope even when the spaces are of totally finite measure.*

B. *If the measures are allowed to be not totally finite, counter-examples may be given for $(1/p_0, 1/q_0)$ and any finite slope.*

This is shown by introducing the *linear* operators

$$T = T^m_{\alpha\beta}, \qquad 0 \leqslant \beta < 1, \qquad 0 \leqslant \alpha, \qquad m \in R,$$

$$Tf(x) = \frac{1}{x^\alpha}\int_0^{x^m} t^{-\beta}f(t)\,dt.$$

The spaces (X, μ) and (Y, ν) shall be: $X = Y = (0, 1)$ in case A and $X = Y = (0, \infty)$ in case B, both with Lebesgue measure. As a matter of fact in case B we can adopt $Y = (0, 1)$; the remaining case $\mu(X) < \infty$, $\nu(Y) = \infty$ follows from § 2 and case A.

To make it short, the idea is the following:

If $P_0 = (1/p_0, 1/q_0)$ is in the interior of the upper triangle and $m \neq 0$ is a given slope, there exist α and β such that $T^m_{\alpha\beta}$ is of restrained weak type in $L \cap S$, where L is the line of slope m through P_0 and $S = \{(1/u, 1/v): 1/u \leqslant 1/v, 1 \leqslant u, v \leqslant \infty\}$ (so that it is of weak type in $L \cap \mathring{S}$ as follows, for example, from Calderón's identities (see [7])); but T is not of strong type (p_0, q_0), in fact it is not at any point of $L \cap \mathring{S}$.

The computations run as follows.

6.1. Proof of A. 1st case: $m \geqslant 1$, and the ordinate to the origin of L is $k \geqslant 0$.

Let $f = \chi_A$ be the characteristic function of the measurable set $A \subseteq (0, 1)$; then

$$Tf(x) \leqslant \frac{1}{x^\alpha} \frac{[\mu(A \cap (0, x^m))]^{1-\beta}}{1-\beta} \leqslant \frac{1}{1-\beta} \inf\left\{\frac{[\mu(A)]^{1-\beta}}{x^\alpha}, x^{m(1-\beta)-\alpha}\right\}.$$

If

(i) $$m(1-\beta) \leqslant \alpha,$$

then

$$x^{1/q}(Tf)^*(x) \leqslant \frac{1}{1-\beta} \inf\left\{\frac{\mu(A)^{1-\beta}}{x^{\alpha-1/q}}, x^{m(1-\beta)-\alpha+1/q}\right\} \quad \text{for } 0 < x < 1.$$

$(Tf)^*(x) = 0$ for $x \geqslant 1$ because $\nu(Y) = 1$. But

$$\inf\{,\} = \begin{cases} x^{1/q-\alpha}\mu(A)^{1-\beta} & \text{for } x \geqslant \mu(A)^{1/m}, \\ x^{m(1-\beta)-\alpha+1/q} & \text{for } 0 < x \leqslant \mu(A)^{1/m}. \end{cases}$$

If

(ii) $$1/q - \alpha \leqslant 0,$$

(iii) $$1/q + m(1-\beta) - \alpha \geqslant 0,$$

we obtain

$$\|Tf\|_{q\infty} \leqslant \frac{1}{1-\beta} \|f\|_p$$

with

$$\frac{1}{q} = m\frac{1}{p} + k, \quad k = \alpha - m(1-\beta).$$

(i) holds in this case because $k \geqslant 0$, and if we choose $\alpha = 1$ also (ii) holds in the square and as (iii) is equivalent to $1/q - k \geqslant 0$, it also holds, the slope of L being positive. We obtain $\beta = 1 + (k-\alpha)/m$ and so $0 \leqslant \beta < 1$ (using $m \geqslant 1$, $k < 1$).

Thus we have restrained weak type in $L \cap S$. But taking the function $f(t) = t^\gamma$, $\gamma p + 1 > 0$, we see, after an easy computation that

$$\frac{\|Tf\|_{q_0}}{\|f\|_{p_0}} = \frac{1}{(mq_0/p_0)^{1/q_0}(\gamma-\beta+1)} (\gamma p_0 + 1)^{1/p_0 - 1/q_0}$$

which tends to infinite when $\gamma \to -1/p_0$ (using $1/p_0 - 1/q_0 < 0$).

2nd case: $m \geqslant 1$, $k < 0$.

This case differs from the 1st case since (i) is not fulfilled for $k < 0$. But if $m(1-\beta) - \alpha \geqslant 0$

$$Tf(x) \leqslant \frac{1}{1-\beta} \inf\left\{\frac{\mu(A)^{1-\beta}}{x^\alpha}, x^{m(1-\beta)-\alpha}\right\}$$

$$\leqslant \begin{cases} \dfrac{1}{1-\beta}\mu(A)^{1-\beta-\alpha/m} & \text{if } 0 < x \leqslant \mu(A)^{1/m} \\ \dfrac{\mu(A)^{1-\beta}}{(1-\beta)x^\alpha} & \text{if } x \geqslant \mu(A)^{1/m}. \end{cases}$$

Then if $0 < x < 1$ and $1/q - \alpha \leqslant 0$, we obtain

$$x^{1/q}(Tf)^*(x) \leqslant \frac{1}{1-\beta}[\mu(A)]^{1/mq - \alpha/m + 1 - \beta}$$

and $(Tf)^*(x) = 0$ for $x \geqslant 1$.

Taking $\alpha = 1$ we obtain $1/q - \alpha \leqslant 0$ for all points of the square. And there is restrained weak type in $L \cap S$, because $0 < \beta < 1$, which follows from

$$1 - \beta = \frac{1 + |k|}{m} \quad \text{and} \quad m = \frac{|k| + 1/q_0}{1/p_0} > 1$$

(using $q_0 \geqslant 1$, $1/p_0 < 1/q_0$).

The rest of the computations are as in 1st case.

3rd case: $0 < m < 1$ (here always $0 < k < 1$ holds).

Now, as in the 1st case, (i) and (iii) are true, α must be chosen with caution. Two cases are to be distinguished: L intersects the diagonal outside or inside the square. As the intersection point is $1/p = 1/q = k/(1-m)$, then if $k/(1-m) \geqslant 1$ we choose $\alpha = 1$, and in the other case take $\alpha = k/(1-m)$ and proceed as in the previous cases.

6.2. Proof of B. In this case $m < 0$ (then also $0 < k$) If $0 < x < \infty$,

$$Tf(x) \leqslant \frac{1}{1-\beta} \inf\left\{\frac{\mu(A)^{1-\beta}}{x^\alpha}, x^{m(1-\beta)-\alpha}\right\}$$

but $m < 0$, so that

$$x^{1/q}(Tf)^*(x) \leqslant \frac{1}{1-\beta} \begin{cases} \dfrac{\mu(A)^{1-\beta}}{x^{\alpha-1/q}} & \text{if } x \leqslant \mu(A)^{1/m}, \\ x^{m(1-\beta)-\alpha+1/q} & \text{if } x \geqslant \mu(A)^{1/m}; \end{cases}$$

then if (i) $a \leqslant 1/q$, (ii) $m(1-\beta)-a+1/q \leqslant 0$,

$$x^{1/q}(Tf)^*(x) \leqslant \frac{1}{1-\beta}[\mu(A)]^{1-\beta-a/m+1/mq}.$$

As in the other cases T will be of restrained weak type (p, q) with $1/q = m/p+k$, $k = a-m(1-\beta)$ if (i) and (ii) are fulfilled. Choosing $a = k/(1-m) > 0$, we obtain (i) for the points which we are interested in. And as (ii) says that $1/q-k \leqslant 0$, this holds because the slope m is negative. As $k/(1-m) \leqslant 1$, it follows $0 \leqslant \beta < 1$.

As in the previous case we prove similarly that T is not of strong type (p_0, q_0), taking

$$f(t) = t^{-\gamma}\chi_{(1,\infty)}(t), \qquad \gamma p_0 > 1.$$

Then $f \in L^{p_0}(X)$. Observing that $Tf(x) = 0$ for $x \geqslant 1$ because $m < 0$, $f(t) = 0$ for $0 < t \leqslant 1$, and $\beta+1/p_0 < 1$ (this follows from the choosing of β) as we are only interested in γ near $1/p_0$, it follows

$$\|Tf\|_{q_0} \geqslant \frac{1}{1-\beta-\gamma}\left|\frac{1}{[q_0(m(1-\beta-\gamma)-a)+1]^{1/q_0}} - \frac{1}{(1-aq_0)^{1/q_0}}\right|.$$

(Remember that $a < 1/q_0$ since $(1/p_0, 1/q_0)$ is above the diagonal and $q_0(m(1-\beta-\gamma)-a)+1 = mq_0(1/p_0-\gamma) > 0$.) Then

$$\frac{\|Tf\|_{q_0}}{\|f\|_{p_0}} \geqslant \frac{1}{1-\beta-\gamma}\left|\frac{(\gamma p_0-1)^{1/p_0-1/q_0}}{(-mq_0p_0)^{1/q_0}} - \frac{(\gamma p_0-1)^{1/p_0}}{(1-aq_0)^{1/q_0}}\right| \xrightarrow[\gamma\to 1/p_0]{} \infty,$$

q. e. d.

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