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# On certain actions of semi-groups on $L$-spaces*

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**§ 0. Introduction.** Given a semi-group $S$ of continuous linear transformations of $E$ into itself and an element $x$ of $E$, one can ask (a) whether the closed convex hull of the orbit of $x$ contains a common fixed point under $S$ and (b) whether such a fixed point is unique. The answer to (a) is affirmative if the closed convex hull of the orbit of $x$ is compact and the semi-group $S$ is left amenable (see [2]). In order to answer (b) affirmatively, usually one assumes, among others, that $S$ be both left and right amenable (see for example [3]). In the present paper we shall study a situation in which the left amenability of $S$ is sufficient to conclude (b) affirmatively. More specifically, we shall postulate a certain (right) action of a semi-group $S$ on $C(X)$, where $X$ is a compact Hausdorff space, and we shall study the resulting (left) action of $S$ on the dual $C(X)^*$. Throughout the paper, we prefer to speak of abstract $M$-spaces with units rather than $C(X)$. There are two reasons for this. First, not all $M$-spaces which arise naturally in this paper (such as $l_\infty(S)$, $UC_l(S)$ and $C(X)^{**}$) come neatly in the form of $C(X)$. Secondly, whenever possible, we favor order arguments over measure theoretic ones.

The basic facts on vector lattices, $M$-spaces and $L$-spaces can be found in [8].

The following is the summary of the contents. In § 1, we introduce the space $UC_l(S)$ of left uniformly continuous functions on a topological semi-group $S$ with separately continuous multiplication, and state basic properties of $UC_l(S)$ needed in the sequel. Section § 2 is devoted to a proposition concerning certain projections in $L$-spaces. This proposition is crucial for the proof of the main theorem. In § 3, we define an "action" of a topological semi-group $S$ on an $M$-space with unit, and the main theorem concerning this type of action is established. In § 4, we come back to $UC_l(S)$ where $S$ is a left amenable topological semi-group. The dual $UC_l(S)^*$ is a Banach algebra, and the results of § 3 give some information about the multiplication on $UC_l(S)^*$. Next we introduce a sub-

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space $J_0$ of $UC_l(S)$, and prove many interesting properties of the members of $J_0$. In particular, any left invariant mean on $UC_l(S)$ is automatically right invariant on $J_0$. In case $S$ is an amenable topological group, $J_0$ contains all continuous almost periodic functions on $S$.

At this point, we record our obligation to S. P. Lloyd. While pondering on geometric significances of some of his results in [9], we arrived at the formulation of our main theorem which would imply those results. Our use of projections on $L$-spaces is also inspired by him. We are deeply grateful for his making a preprint copy of [9] available to us.

**§ 1. Preliminaries.** *A semi-group* is a set with an associative binary operation $(s, t) \to s\cdot t$, which will be called the *multiplication* of the semi-group. A semi-group $S$ is a *topological semi-group* if $S$ is provided with a topology making the multiplication *separately continuous*; that is, for a fixed $s$ in $S$, the maps $t \to t\cdot s$ and $t \to s\cdot t$ are continuous on $S$ into itself. A semi-group $S$ can always be made into a topological semi-group by endowing $S$ with the discrete topology.

Let $S$ be a topological semi-group, and let $C(S)$ be the Banach algebra of all bounded continuous real-valued functions on $S$ with the usual supremum norm. For $s$ in $S$ and $f$ in $C(S)$, define the *left translate of $f$ by $s$* (or simply ${}_sf$) by ${}_sf(t) = f(s\cdot t)$. The *right translate of $f$ by $s$* (or simply $f_s$) is similarly defined by $f_s(t) = f(t\cdot s)$. Clearly $\|{}_sf\| \leqslant \|f\|$ and $\|f_s\| \leqslant \|f\|$.

A function $f$ in $C(S)$ is called *left uniformly continuous* if the map $s \to {}_sf$ is continuous on $S$ into $C(S)$. Let $UC_l(S)$ denote the space of all left uniformly continuous functions on $S$.

1.1. LEMMA. *Let $S$ be a topological semi-group; then $UC_l(S)$ is a closed subalgebra of $C(S)$. The space $UC_l(S)$ is also closed under the lattice operations.*

Proof. Given $f$ and $g$ in $UC_l(S)$, the map $s \to {}_s(f\vee g)$ can be "factored" as: $s \to ({}_sf, {}_sg) \to ({}_sf) \vee ({}_sg) = {}_s(f\vee g)$. Hence $f \vee g \in UC_l(S)$. Similarly, one can prove that $f+g, f\cdot g, f \wedge g \in UC_l(S)$. It is also easy to check that $UC_l(S)$ is closed in $C(S)$.

Lemma 1.1 implies that $UC_l(S)$ is an $M$-space with unit, the unit being the function $\mathbf{1}$ identically equal to 1 on $S$.

1.2. LEMMA. *If $f \in UC_l(S)$ and $t \in S$, then ${}_tf$ and $f_t$ are also in $UC_l(S)$.*

Proof. Since the composition of the maps: $s \to t\cdot s \to {}_{(t\cdot s)}f = {}_s({}_tf)$ is continuous, it follows that ${}_tf \in UC_l(S)$. To see that $f_t \in UC_l(S)$, it is enough to observe that $\|{}_s(f_t) - {}_{s'}(f_t)\| = \|({}_sf)_t - ({}_{s'}f)_t\| \leqslant \|{}_sf - {}_{s'}f\|$.

If $S$ is discrete, then $UC_l(S)$ is identical with the usual $l_\infty(S)$. If $S$ is a topological group, then $UC_l(S)$ is indeed the space of all bounded left uniformly continuous functions(1) in the usual sense. We record here the following generalization of a well-known fact.

(1) Some authors call these functions *right uniformly continuous.*



1.3. LEMMA. *If $S$ is a compact semi-group with jointly continuous multiplication, then $C(S) = UC_l(S)$.*

Proof. Let $f \in C(S)$, $s_0 \in S$ and $\varepsilon > 0$, and let $U$ be the subset of $S\times S$ given by $U = \{(s, t)\colon |f(s_0\cdot t) - f(s\cdot t)| < \varepsilon\}$. Then $U$ is open and $\{s_0\}\times S \subset U$. Hence there is a neighborhood $V$ of $s_0$ such that $V\times S \subset U$, and it follows that $\|{}_{s_0}f - {}_sf\| < \varepsilon$ whenever $s \in V$. Therefore $f \in UC_l(S)$.

**§ 2. Projections in $L$-spaces.** Let $E$ be an $L$-space. We consider operators $P\colon E \to E$ satisfying:

(L.1) $$P^2 = P,$$
(L.2) $$P \geqslant 0,$$
(L.3) $$\|P(x)\| = \|x\| \quad \text{for} \quad x \geqslant 0.$$

Note that (L.3) implies that $\|P\| = 1$.

2.1. THEOREM. *Let $E$ be an $L$-space, and let $P$ and $Q$ be operators in $E$ satisfying* (L.1)-(L.3). *If $P$ and $Q$ have the same range, say $F$, then $Px = Qx$ whenever $x \in (I(F))^-$, where $I(F) = \{u\colon |u| \leqslant v$ for some $v$ in $F\}$.*

2.2. Remark. A linear subspace $I$ of a vector lattice $E$ is called an *order ideal* if $v \in I$ and $|u| \leqslant |v|$ imply $u \in I$. The space $I(F)$ in the theorem is clearly an order ideal in $E$. We assert that $F \subset I(F)$. In fact, if $x \in F$, then $-|x| \leqslant x \leqslant |x|$. Hence, from the fact that $P \geqslant 0$ and $Px = x$, we see that $-P(|x|) \leqslant x \leqslant P(|x|)$ or $|x| \leqslant P(|x|) \in F$. Therefore $x \in I(F)$. Thus $I(F)$ is the order ideal generated by $F$, and $I(F)^-$ is the closed order ideal generated by $F$.

Proof of 2.1. If $U$ is an operator in an $M$-space $M$ with unit $e$ such that

(M.1) $$U^2 = U,$$
(M.2) $$U \geqslant 0,$$
(M.3) $$U(e) = e,$$

then it is clear that the adjoint map $U^*\colon M^* \to M^*$ satisfies (L.1)-(L.3). Conversely, if $P$ satisfies (L.1)-(L.3), then the adjoint map $P^*$ satisfies (M.1)-(M.3). Now $P$ and $Q$ in the theorem have the common range if and only if $PQ = Q$ and $QP = P$. Hence $P^{**}$ and $Q^{**}$ are operators in $E^{**}$ satisfying all the conditions of theorem 2.1. Therefore by embedding $E$ into $E^{**}$, we see that it is sufficient to establish the conclusion of 2.1 for $P^{**}$ and $Q^{**}$. Therefore, without loss of generality, we may assume that $E = M^*$ for some $M$-space $M$ with unit $e$ and $P = U^*, Q = V^*$, where $U$ and $V$ are operators in $M$ satisfying (M.1)-(M.3) such that $UV = U$ and $VU = V$.

Let
$$K = \{u\colon u \in F, u \geqslant 0, \|u\| = 1\}$$
$$= \{u\colon u \in F, u \geqslant 0, \langle u, e\rangle = 1\}.$$

If $I$ denotes the identity operator on $E$, then the image of $P$ coincides with the null space of $P-I = U^*-I$. It follows that $F$ is weak* closed, and therefore $K$ is weak* compact and convex. By a familiar argument(²), it is easy to see that a point $u$ in $K$ is extreme in $K$ if and only if $v \in [0, u] \cap F$ implies $v = \|v\| \cdot u$, where $[0, u]$ denotes the order interval $\{w: w \in E, 0 \leqslant w \leqslant u\}$. Let $A$ be the set of extreme points of $K$, and let $[0, A]$ denote the set $\bigcup \{[0, u]: u \in A\}$. If $v \in [0, A]$, then $0 \leqslant v \leqslant u$ for some $u$ in $A$, and therefore $0 \leqslant Pv \leqslant u$. Hence, by the above remark, $Pv = \|Pv\| \cdot u = \|v\| \cdot u$. Similarly, $Qv = \|v\| \cdot u$. Thus $P$ and $Q$ agree on $[0, A]$.

Let $\langle A \rangle$ denote the convex hull of $A$ and let $V \in [0, \langle A \rangle]$. Then $0 \leqslant v \leqslant \sum \lambda_i u_i$, where $\lambda_i > 0$, $\sum \lambda_i = 1$, $u_i \in A$. By the decomposition property of vector lattices (see, for instance, [8; 23.8]), $v$ can be written as $v = \Sigma v_i$, $0 \leqslant v_i \leqslant \lambda_i u_i$. Since we already know that $Pv_i = Qv_i$ for each $i$, it follows that $Pv = Qv$. Hence $P$ and $Q$ agree on $[0, \langle A \rangle]$. Since both $P$ and $Q$ are weak* continuous, they agree on the set $[0, \langle A \rangle]^{-w^*}$, where $-w^*$ indicates the weak* closure. But, by lemma 8.12 of [10], $[0, K] = [0, \langle A \rangle^{-w^*}] \subset [0, \langle A \rangle]^{-w^*}$. Thus $P$ and $Q$ agree on $[0, K]$.

Finally assume that $v \in I(F)$; that is $-u \leqslant v \leqslant u$ or $0 \leqslant v+u \leqslant 2u$ for some $u$ in $F$. Hence $v+u \in 2\|u\|[0, K]$, and it follows that $P(v+u) = Q(v+u)$. Since $u \in F$, $Pu = u = Qu$. Therefore $Pv = Qv$, and, by the continuity of $P$ and $Q$, it follows that $P$ and $Q$ agree on $I(F)^-$.

**§ 3. Actions of semi-groups.** Let $S$ be a topological semi-group (cf. § 1), and let $M$ be an $M$-space with unit $e$. An *action* (or, more precisely, *right action*) of $S$ on $M$ is a separately continuous map $S \times M \to M$ (denoted by $(s, x) \to s \cdot x$) satisfying:

A.1. For each $s$ in $S$, $x \to s \cdot x$ is a positive linear operator in $M$.

A.2. $s \cdot e = e$ for each $s$.

A.3. $s \cdot (t \cdot x) = (t \cdot s) \cdot x$ for $s, t$ in $S$ and $x$ in $M$.

It is immediate from A.1 and A.2 that $\|s \cdot x\| \leqslant \|x\|$. The following is an important example.

(²) Here is the argument: Assume that $u$ is extreme in $K$ and that $v \in [0, u] \cap F$. If $v \neq 0$ and $v \neq u$, then we may express

$$u = \|u-v\| \frac{u-v}{\|u-v\|} + \|v\| \frac{v}{\|v\|},$$

where $\|u-v\| + \|v\| = \langle u-v, e\rangle + \langle v, e\rangle = \langle u, e\rangle = 1$. Therefore $u = v/\|v\|$ or $v = \|v\| \cdot u$. Conversely, assume that $u$ is an element in $K$ such that $v \in [0, u] \cap F$ implies $v = \|v\| \cdot u$. If $u = \lambda v_1 + (1-\lambda) v_2$, where $v_1, v_2 \in K$ and $0 < \lambda < 1$, then $\lambda v_1 \in [0, u] \cap F$. Hence, by the assumption, $\lambda v_1 = \|\lambda v_1\| \cdot u = \lambda u$ or $v_1 = u$. Similarly $v_2 = u$. Therefore $u$ is extreme.



3.1. Example. Let $S$ be a topological semi-group. Then we can define a map on $S \times UC_l(S)$ into $UC_l(S)$ by $(s, f) \to {}_s f$ (cf. lemma 1.2). As noted in § 1, $UC_l(S)$ is an $M$-space with unit, and the definition of $UC_l(S)$ implies that the map $(s, f) \to {}_s f$ is separately continuous. The properties A.1-A.3 can be verified easily.

Assume that a right action of $S$ on $M$ is given; then it induces a left action of $S$ on the adjoint $M^*$. Specifically: given $s$ in $S$ and $f$ in $M^*$, define an element $s \cdot f$ in $M^*$ by $(s \cdot f)(x) = f(s \cdot x)$; then it is easy to deduce from A.1-A.3 that:

A*.1. For each $s$ in $S$, $f \to s \cdot f$ is a positive linear operator in $M^*$.

A*.2. $\|s \cdot f\| = \|f\|$ for $f \geqslant 0$.

A*.3. $s \cdot (t \cdot f) = (s \cdot t) \cdot f$ for $s, t$ in $S$.

Note also that $\|s \cdot f\| \leqslant \|f\|$. An element $f$ of $M^*$ is called *$S$-invariant* if $s \cdot f = f$ for all $s$ in $S$.

3.2. Proposition. *Let an action of a topological semi-group $S$ on an $M$-space $M$ with unit $e$ be given. Then the set of $S$-invariant elements in $M^*$ is a weak* closed linear subspace of $M^*$ which is closed under the lattice operations; in particular, it is an $L$-space.*

Proof. The fact that the set of $S$-invariant elements form a weak* closed subspace of $M^*$ is trivial. To prove the assertion concerning the lattice operations, by the usual translation arguments, it is sufficient to prove that if $f$ is $S$-invariant so is $f^+$. Since $f^+ \geqslant f$ and $f^+ \geqslant 0$, for an arbitrary $s$ in $S$, we have $s \cdot (f^+) \geqslant s \cdot f = f$ and $s \cdot (f^+) \geqslant 0$. Hence $s \cdot (f^+) \geqslant f^+$ or $s \cdot (f^+) - f^+ \geqslant 0$. Therefore $\|s \cdot (f^+) - f^+\| = \langle s \cdot (f^+) - f^+, e\rangle = \langle s \cdot (f^+), e\rangle - \langle f^+, e\rangle = \langle f^+, e\rangle - \langle f^+, e\rangle = 0$. It follows that $s \cdot f^+ = f^+$ or $f^+$ is $S$-invariant.

Using the action of $S$ on $UC_l(S)$ described in example 3.1, we can speak of $S$-invariant elements in $UC_l(S)^*$. A topological semi-group $S$ is called *left amenable* if there is a non-trivial $S$-invariant element in $UC_l(S)^*$. In view of proposition 3.2, $S$ is left amenable if and only if there is an $S$-invariant positive linear functional of unit norm on $UC_l(S)$; we shall call such a functional a *left invariant mean* on $UC_l(S)$. If $S$ is discrete, our definition of amenability agrees with the usual one (for example the one given in Day [1]), since, as noted in § 1, $UC_l(S) = l_\infty(S)$. If $S$ is a locally compact topological group, then the left amenability in our sense has recently been shown to be equivalent to other important properties of $S$, which were investigated more or less independently in the past (see [13], [12](³), [5] and also [6] for a detailed survey on this subject).

(³) In [12], we actually proved that, for a locally compact topological group $G$, the existence of a left invariant mean over $UC_l(G)$ implies property (J) and hence $(P_1)$ for $G$.

In particular, a compact topological group is (left) amenable. Let $S$ be a topological semi-group such that $S$ is left emanable as a discrete semi-group; then $S$ is left amenable. In particular, an abelian topological semi-group is left amenable.

Now we consider a (right) action of a topological semi-group $S$ on an $M$-space with unit in general (satisfying, of course, A.1-A.3). For $f$ in $M^*$, let $C_f$ be the convex hull of the orbit of $f$ under $S$ i.e. the convex hull of $\{s\cdot f : s\in S\}$. Then the weak* closure $C_f^{-w^*}$ of $C_f$ is weak* compact. Hence if $S$ is left amenable, it follows from a modification of the fixed point theorem in [2] that $C_f^{-w^*}$ contains an $S$-invariant element. However, we will construct an $S$-invariant element in $C_f^{-w^*}$ explicitly.

Fix $f$ in $M^*$, and define, for each $x$ in $M$, a real valued function $T_f(x)$ on $S$ by

$$T_f(x)(t) = f(t\cdot x).$$

Clearly $T_f(x)$ is continuous and bounded. We claim that $T_f(x)\in UC_l(S)$; in fact, for $s, s'$ in $S$, we have

$$\left|{}_s\big(T_f(x)\big)(t) - {}_{s'}\big(T_f(x)\big)(t)\right| = \left|f\big((s\cdot t)\cdot x - (s'\cdot t)\cdot x\big)\right| \leqslant \|f\|\cdot\|t\cdot(s\cdot x - s'\cdot x)\| \leqslant \|f\|\cdot\|s\cdot x - s'\cdot x\|.$$

Therefore the continuity of $s\to {}_s\big(T_f(x)\big)$ follows from the continuity of $s\to s\cdot x$. Therefore $T_f$ is a linear map of $M$ into $UC_l(S)$ with $\|T_f\|\leqslant\|f\|$. Let $m$ be an arbitrary member of the adjoint space $UC_l(S)^*$, and define $Q_m(f) = T_f^*(m)\in M^*$. Then, for $x$ in $M$, we can write

$$Q_m(f)(x) = m\big(T_f(x)\big) = m_s\big(f(s\cdot x)\big) = m_s\big((s\cdot f)(x)\big).$$

It is clear that $Q_m$ is a linear operator on $M^*$ into itself and $\|Q_m\|\leqslant\|m\|$. The most interesting case is the case where $m$ is a left invariant mean on $UC_l(S)$.

3.3. PROPOSITION. *Let an action of a left amenable topological semi-group $S$ on an $M$-space with unit be given, let $m$ be a left invariant mean on $UC_l(S)$, and let $Q_m$ be as above. Then, for each $f$ in $M^*$, $Q_m(f)$ is an $S$-invariant element in $C_f^{-w^*}$. In addition, $Q_m(f) = f$ if and only if $f$ is $S$-invariant.*

Proof. Let $a\colon S\to UC_l(S)^*$ be the evaluation map; then, by a standard separation theorem, we see easily that $m$ is in the weak* closed convex hull of $a[S]$. For $s$ in $S$ and $x$ in $M$, $T_f^*\big(a(s)\big)(x) = \langle T_f(x), s\rangle = f(s\cdot x) = (s\cdot f)(x)$. Hence $T_f^*\big(a(s)\big) = s\cdot f\in C_f$. Since $T_f^*$ is weak* continuous, $T_f^*$ maps the weak* closed convex hull of $a[S]$ into $C_f^{-w^*}$. In particular, $Q_m(f) = T_f^* m\in C_f^{-w^*}$. Next, $\big(s\cdot Q_m(f)\big)(x) = Q_m(f)(s\cdot x) = m_t[f(t\cdot(s\cdot x))] = m_t[f((s\cdot t)\cdot x)] = m_t[f(t\cdot x)] = Q_m(f)(x)$, since $m$ is left invariant. Hence $Q_m(f)$ is $S$-invariant for any $f$. If $f$ is $S$-invariant, then the function $s\to f(s\cdot x)$ is the constant function $s\to f(x)$. Hence, in this case, $Q_m(f)(x) = f(x)$ or $Q_m(f) = f$. This completes the proof.



Suppose for a moment that $m$ is left *and* right invariant; then an easy calculation shows that $Q_m(s\cdot f) = Q_m(f)$ for each $s$ in $S$ and each $f$ in $M^*$. However, without $m$ being right invariant, one can still assert the following:

3.4. THEOREM. *Let $S$, $M$ and $m$ be as in proposition 3.3, and let $f$ be an element in the (norm) closed order ideal generated by all the $S$-invariant elements in $M^*$* (cf. remark 2.2). *Then, for each $s$ in $S$, $Q_m(s\cdot f) = Q_m(f)$.*

Proof. Let $F$ be the space of all $S$-invariant elements in $M^*$; then $F$ is a vector sublattice of $M^*$ by proposition 3.2. Let $I(F) = \{g : |g|\leqslant h$ *for some $h$ in $F$*$\}$; then the norm closure $I(F)^-$ is the closed order ideal generated by $F$. Now by proposition 3.3, $Q_m$ is an operator such that $Q_m^2 = Q_m$ having the range $F$. Obviously $Q_m\geqslant 0$. Let $e$ be the unit in $M$. Then, for $f\geqslant 0$, we have

$$\|Q_m(f)\| = Q_m(f)(e) = m_s\big(f(s\cdot e)\big) = f(e) = \|f\|.$$

Therefore $Q_m$ satisfies (L.1)-(L.3) of § 2. Fix $s$ in $S$, and we define an operator $P\colon M^*\to M^*$ by $P(f) = Q_m(s\cdot f)$. Then by A*.1,2, $P$ satisfies (L.2) and (L.3). In addition, from what we proved for $Q_m$, it follows that $P^2 = P$ and that the range of $P$ is precisely $F$. Hence, by theorem 2.1, $Q_m(s\cdot f) = P(f) = Q_m(f)$ whenever $f\in I(F)^-$.

In the proof of the next theorem, we shall use the following important fact: In an $L$-space $L$, an order interval is weakly compact. This follows immediately from the theorem which asserts that the image of $L$ in $L^{**}$ under the evaluation map is an order ideal in $L^{**}$. For a proof of the latter, consult [14] or [8].

3.5. THE MAIN THEOREM. *Let an action of a left amenable topological semi-group $S$ on an $M$-space with unit be given, and let $f$ be an element in the norm closed order ideal generated by the $S$-invariant elements in $M^*$. Then the norm closure $\bar{C}_f$ of $C_f$ is weakly compact and $\bar{C}_f$ contains a unique $S$-invariant element.*

Proof. Let $F$ and $I(F)$ be as in the proof of theorem 3.5. Let $g\in I(F)$; then for some $h$ in $F$, $h\geqslant 0$, we have $g\in[-h, h]$. Since $h$ is $S$-invariant $C_g\subset[-h, h]$. Since $C_g$ is convex, $\bar{C}_g$ is weakly closed, and as remarked above $[-h, h]$ is weakly compact. Therefore $\bar{C}_g$ is weakly compact for each $g\in I(F)$. To prove that $\bar{C}_f$ is weakly compact for an arbitrary $f$ in $I(F)^-$, we can invoke Eberlein's argument [4]. For convenience of the readers, we shall give a proof based on the double limit theorem ([8], theorem 17.12). Let $\{s_i\}$ be a sequence in $S$ and let $\{\theta_j\}$ be a sequence in the unit ball of $M^{**}$ such that

$$\lim_i\lim_j\langle s_i\cdot f, \theta_j\rangle = a \quad\text{and}\quad \lim_j\lim_i\langle s_i\cdot f, \theta_j\rangle = b.$$

We must show that $a = b$. Let $\varepsilon > 0$ and choose $g \in I(F)$ so that $\|f-g\| \leqslant \varepsilon$; then, for all $i$, $\|s_i\cdot f - s_i\cdot g\| \leqslant \varepsilon$. By taking subsequences of $\{s_i\}$ and $\{\theta_j\}$, we can assume that $\lim_i \lim_j \langle s_i\cdot g, \theta_j\rangle$ and $\lim_j \lim_i \langle s_i\cdot g, \theta_j\rangle$ exist, but then those limits are equal because $\bar{C}_g$ is weakly compact. Moreover, for each $i$ and $j$, $|\langle s_i\cdot g, \theta_j\rangle - \langle s_i\cdot f, \theta_j\rangle| \leqslant \varepsilon$. Therefore $|a-b| \leqslant \varepsilon$, and, since $\varepsilon$ is arbitrary, we have $a = b$.

Since $\bar{C}_f$ is weakly compact, $\bar{C}_f = C_f^{-w^*}$, and, by proposition 3.3, $\bar{C}_f$ contains at least one $S$-invariant element, say $h$. For $\varepsilon > 0$, there is a convex combination $\sum_{i=1}^{n} \tau_i(s_i\cdot f)$ such that

$$\Big\| h - \sum_{i=1}^{n} \tau_i(s_i\cdot f)\Big\| \leqslant \varepsilon.$$

Let $m$ be an arbitrary left invariant mean on $UC_l(S)$. Then by theorem 3.4,

$$\varepsilon \geqslant \Big\| Q_m\Big(h - \sum_{i=1}^{n} \tau_i(s_i\cdot f)\Big)\Big\|$$

$$= \Big\| h - \sum_{i=1}^{n} \tau_i Q_m(s_i\cdot f)\Big\| = \|h - Q_m(f)\|.$$

Since $\varepsilon$ is arbitrary, $Q_m(f) = h$. Hence $h$ is the unique $S$-invariant element in $\bar{C}_f$.

3.6. Remark. It is implicit in theorem 3.5 that, if $f \in I(F)^-$, then $Q_m(f)$ is independent of the left invariant mean $m$ on $UC_l(S)$, i.e. whenever $m$ and $m'$ are left invariant means, we have $Q_m(f) = Q_{m'}(f)$. This fact can be established directly by applying theorem 2.1 to $Q_m$ and $Q_{m'}$.

3.7. Corollary. *Let $f$ be as in theorem 3.5, and, for an $x$ in $M$, define a function $u$ in $UC_l(S)$ by $u(s) = f(s\cdot x)$; that is $u = T_f(x)$. Then*

(a) *$m(u)$ is independent of left invariant means $m$ on $UC_l(S)$. Let this value be $k$.*

(b) *$m(u_t) = m(u)$ for each $t$ in $S$, where $u_t(s) = u(s\cdot t)$.*

(c) *The convex combinations of right translates of $u$ approximate the constant function $k\cdot\mathbf{1}$ uniformly; i.e., for a given $\varepsilon > 0$, there is a convex combination $\sum_{i=1}^{n} \tau_i u_{s_i}$ such that*

$$\Big\| \sum_{i=1}^{n} \tau_i u_{s_i} - k\cdot\mathbf{1}\Big\| \leqslant \varepsilon.$$

Proof. In our earlier notation $m(u) = Q_m(f)(x)$. Hence in view of remark 3.6, (a) is clear and $k = Q_m(f)(x)$. Now

$$u_t(s) = u(s\cdot t) = f\big((s\cdot t)\cdot x\big) = f\big(t\cdot(s\cdot x)\big) = (t\cdot f)(s\cdot x).$$



Hence $m(u_t) = Q_m(t\cdot f)(x)$. By theorem 3.4, $Q_m(t\cdot f) = Q_m(f)$; consequently $m(u_t) = m(u)$. Finally, let $\varepsilon > 0$ be given; then, since $Q_m(f) \in \bar{C}_f$, there is a convex combination $\sum_{i=1}^{n} \tau_i(s_i\cdot f)$ such that

$$\Big\| Q_m(f) - \sum_{i=1}^{n} \tau_i(s_i\cdot f)\Big\| \leqslant \varepsilon.$$

Since, for all $t$, $\|t\cdot x\| \leqslant \|x\|$, we have

$$\varepsilon\|x\| \geqslant \Big| < Q_m(f) - \sum_{i=1}^{n} \tau_i(s_i\cdot f), t\cdot x > \Big|$$

$$= \Big| k - \sum_{i=1}^{n} \tau_i u_{s_i}(t)\Big| \quad \text{for all } t \text{ in } S.$$

Therefore

$$\Big\| \sum_{i=1}^{n} \tau_i u_{s_i} - k\cdot\mathbf{1}\Big\| \leqslant \varepsilon\|x\|,$$

and (c) is proved.

3.8. Remark. If $S$ is a left amenable discrete semi-group, then (a) of corollary 3.7 is equivalent to

(a′) *The convex combinations of left translates of $u$ approximate the constant function $k\cdot\mathbf{1}$ uniformly.*

We shall only sketch a proof of (a) $\Leftrightarrow$ (a′). Let $E$ be the linear subspace of $l_\infty(S) = UC_l(S)$ generated by the set $\{s\cdot v - v\colon s \in S \text{ and } v \in l_\infty(S)\}$. Now (a) is equivalent to that $u - k\cdot\mathbf{1}$ is annihilated by all $S$-invariant functionals on $l_\infty(S)$; hence, by the double polar theorem (see, for instance [8; 16.3]), (a) is equivalent to $u - k\cdot\mathbf{1} \in \bar{E}$. Using the strong left amenability of $S$ (cf. [1] or [11]), one easily sees that $v \in \bar{E}$ if and only if the convex combinations of left translates of $v$ approximate 0 uniformly. Therefore (a) $\Leftrightarrow$ (a′). Incidentally, the argument given above can be used to prove theorem 11 of [9] equally well.

As seen above, the functions in the order ideal generated by $S$-invariant elements in $M^*$ possess many special properties. We give an alternative characterization of those functions.

3.9. Theorem. *Let an action of a topological semi-group $S$ on an $M$-space $M$ with unit $e$ be given, and let $E$ be the linear subspace of $M$ generated by the set $\{s\cdot x - x\colon s \in S, x \in M\}$. Then an element $f$ of $M^*$ belongs to the order ideal generated by $S$-invariant elements in $M^*$ if and only if*

$$-\infty < \inf\{f(x)\colon x \in E, x \leqslant e\} \leqslant \sup\{f(x)\colon x \in E, x \leqslant e\} < \infty.$$

Proof. Let $F$ and $I(F)$ be as in the proof of theorem 3.4. If $f \in I(F)$, then $f \in [-h, h]$ for some $h$ in $F$ with $h \geqslant 0$. Hence $h - f \geqslant 0$, and, for

$x$ in $E$ such that $x \leqslant e$, we have

$$\|h-f\| = (h-f)(e) \geqslant (h-f)(x) = h(x)-f(x) = -f(x)$$

or

$$f(x) \geqslant -\|h-f\|.$$

Therefore $\inf\{f(x): x \in E, x \leqslant e\} \geqslant -\|h-f\| > -\infty$. Similarly, we have $\sup\{f(x): x \in E, x \leqslant e\} < \infty$.

Conversely assume that the inequality of the theorem is satisfied. Let $f|E = g$; then, by corollary 4.5 of [10],

$$\sup\{g(x): x \in E, x \leqslant e\} < \infty$$

implies that $g$ can be extended to a continuous positive linear functional $\bar{g}$ on $M$. Let $h_1 = f-\bar{g}$; then $h_1 \leqslant f$ and $h_1$ is $S$-invariant because $h_1|E = 0$. Repeating the argument for $-f$, we see that there is an $S$-invariant element $h_2$ such that $f \leqslant h_2$. Hence $f \in [h_1, h_2] \subset I(F)$.

Let $X$ be a compact Hausdorff space and let $M = C(X)$. Suppose that an action of a left amenable topological semi-group $S$ on $M$ is given. Then $S$ acts on the space of finite signed Baire measures on $X$. Let $\mu$ be an $S$-invariant Baire measure, and let $\nu$ be a signed Baire measure such that $\nu \ll \mu$. Then $\nu$ is in the closed order ideal generated by $\mu$; hence all the assertions for $f$ in 3.4-3.7 are valid for $\nu$. In particular, for a $g$ in $C(X)$, the real-valued function $s \to \int (s\cdot g)\,d\nu$ on $S$ has properties (a)-(c) of corollary 3.7. As a special case, we can take $\nu = h\cdot\mu$ for some $h \in C(X)$; that is $\int k\,d\nu = \int k\cdot h\,d\mu$ for all $k$ in $C(X)$. Then property (a) becomes the first part of theorem 10 in [9].

**§ 4. $UC_l(S)$ and $UC_l(S)^*$.** Let $S$ be a topological semi-group. Then $UC_l(S)$ is a Banach algebra and there is a natural action of $S$ on $UC_l(S)$ (example 3.1). Being the adjoint of an $M$-space, $UC_l(S)^*$ is an $L$-space; furthermore, it is possible to define a multiplication on $UC_l(S)^*$ as follows: For $m, n$ in $UC_l(S)^*$ and $f$ in $UC_l(S)$, $(m*n)(f) = m_s(n({}_sf))$. If we use the notation of § 3, we can write $m*n = Q_m(n)$.

4.1. PROPOSITION. *Under the multiplication defined above $UC_l(S)^*$ is a Banach algebra.*

Proof. We check only the associativity of the multiplication. Let $m, n, l \in UC_l(S)^*$ and $f \in UC_l(S)$; then,

$$[m*(n*l)](f) = m_t[n*l({}_tf)] = m_t[n_s l({}_s({}_tf))] = m_t[n_s l({}_{t\cdot s}f)],$$

$$[(m*n)*l](f) = (m*n)_s l({}_sf) = m_t[n_s l({}_{t\cdot s}f)],$$

and therefore $m*(n*l) = (m*n)*l$.



Let $a: S \to UC_l(S)^*$ be the evaluation map; that is $a(s)(f) = f(s)$ for $s$ in $S$ and $f$ in $UC_l(S)$. We can easily check the following for $s, t$ in $S$, $m$ in $UC_l(S)^*$ and $f$ in $UC_l(S)$:

(a) $$a(s\cdot t) = a(s)*a(t).$$

(b) $$\big(a(s)*m\big)(f) = m({}_sf).$$

(c) $$\big(m*a(s)\big)(f) = m(f_s).$$

Let $\mathbf{1}$ be the function on $S$ identically equal to 1; then $m*n(\mathbf{1}) = m(\mathbf{1})\cdot n(\mathbf{1})$.

Now assume that $S$ is left amenable and let $F$ be the space of all $S$-invariant elements in $UC_l(S)^*$, i.e.

$$F = \{m: a(s)*m = m \text{ for all } s \text{ in } S\}.$$

As in § 3, let $I(F)$ be the order ideal generated by $F$, and let $I(F)^-$ be the norm closure of $I(F)$.

4.2. THEOREM. *Let $S$ be a left amenable topological semi-group and let $F$ and $I(F)$ be as above. Then*

(i) *If $m \in F$ and $n \in UC_l(S)^*$, then $m*n \in F$.*

(ii) *If $m \in UC_l(S)^*$ and $n \in F$, then $m*n = m(\mathbf{1})\cdot n$.*

(iii) *There is a linear map $Q: I(F)^- \to F$ such that, for $m$ in $F$ and $n$ in $I(F)^-$, $m*n = m(\mathbf{1})\cdot Q(n)$.*

Proof. Assertions (i) and (ii) are direct consequences of the definition of $m*n$. By remark 3.6, if $m$ and $m'$ are left invariant means on $UC_l(S)$, then, for each $n$ in $I(F)^-$, $m*n = Q_m(n) = Q_{m'}(n) = m'*n$. Hence there is a linear map $Q: I(F)^- \to F$ such that $m*n = Q(n)$ for each left invariant mean $m$ on $UC_l(S)$ and each $n$ in $I(F)^-$. Since $F$ is a lattice (proposition 3.2), each element $m$ in $F$ is of the form $m = am_1 + bm_2$, where $m_1$ and $m_2$ are left invariant means; hence, for each $n$ in $I(F)^-$, we have

$$m*n = am_1*n + bm_2*n = aQ(n) + bQ(n) = (a+b)Q(n) = m(\mathbf{1})Q(n).$$

4.3. COROLLARY. *$I(F)$ and $I(F)^-$ are two-sided ideals in $UC_l(S)^*$.*

Proof. It suffices to prove that $I(F)$ is a two-sided ideal. Let $m \in I(F)$ and $n \in UC_l(S)^*$. In order to prove $m*n$, $n*m \in I(F)$, we may assume that $n \geqslant 0$. Since $m \in I(F)$, $-m_1 \leqslant m \leqslant m_1$ for some $m_1$ in $F$, $m_1 \geqslant 0$. Hence $-n*m_1 \leqslant n*m \leqslant n*m_1$ or $-n(\mathbf{1})m_1 \leqslant n*m \leqslant n(\mathbf{1})m_1$ by 4.2 (ii); it follows that $n*m \in I(F)$. Again $-m_1*n \leqslant m*n \leqslant m_1*n$, and $m_1*n \in F$ by 4.2 (i). Hence $m*n \in I(F)$.

4.4. THEOREM. *Let $S$ be a left amenable topological semi-group and let $\Phi$ be the convex hull of $a[S]$. Then there is a net $\{\varphi_\gamma\}$ in $\Phi$ such that, for each left invariant mean $m$ on $UC_l(S)$ and for each $n$ in $I(F)^-$,*

$$\lim_\gamma \|(\varphi_\gamma - m)*n\| = 0.$$

Proof. Let $m$ be a left invariant mean on $UC_l(S)$. Then certainly there is a net $\{\varphi_\beta\}$ in $\Phi$ such that $\varphi_\beta \to m$ relative to the weak* topology. Let $n \in I(F)^-$, then by the definition of $*$, $\varphi_\beta * n \to m*n$ relative to the weak* topology. Now both $\{\varphi_\beta * n\}$ and $m*n$ lie in $C_n^{-w^*}$. By theorem 3.5, $\overline{C}_n$ is weakly compact; hence, it is weak* compact. Therefore $C_n^{-w^*} = \overline{C}_n$, and weak and weak* topologies coincide on $\overline{C}_n$. It follows that $\varphi_\beta * n \to m*n$ relative to the weak topology whenever $n \in I(F)^-$.

The rest of the proof is similar to the proof of 2.2 in [11]. Let $E$ be the product space $\big(UC_l(S)^*\big)^{I(F)^-}$, and define a linear map $T\colon UC_l(S)^* \to E$ by $T(l)(n) = l*n$. Since the weak topology on $E$ is the product of weak topologies of the factors, we see from the foregoing that $T(m)$ is in the weak closure of $T[\Phi]$. Since $E$ is locally convex and $T[\Phi]$ is convex, $T(m)$ is in the closure of $T[\Phi]$ relative to the original topology, i.e. the product norm topologies. Hence there is a net $\{\varphi_\gamma\}$ in $\Phi$ such that, for each $n$ in $I(F)^-$, $\varphi_\gamma * n = T(\varphi_\gamma)(n) \to m*n$ in the norm. Finally, note that $m*n$ is independent of the left invariant means $m$.

Let $J$ be the closed two sided ideal in $UC_l(S)^*$ defined by $J = \{m\colon m*n = 0 \textit{ for all } n \textit{ in } I(F)^-\} = \{m\colon m*n = 0 \textit{ for all } n \textit{ in } I(F)\}$. Then the functions in $UC_l(S)$ which annihilate $J$ have interesting properties.

4.5. THEOREM. *Let $S$ be a left amenable topological semi-group, and let $J$ be as defined above. Then the polar $J_0$ of $J$ in $UC_l(S)$ is the closed linear subspace of $UC_l(S)$ generated by the functions of the form $s \to n({}_sf)$, $n \in I(F)$, $f \in UC_l(S)$. Each function $u$ in $J_0$ enjoys properties* (a)-(c) *of corollary* 3.7 (*and* (a′) *of remark* 3.8 *in case $S$ is discrete*).

Proof. Let $n \in I(F)$, $f \in UC_l(S)$ and $g(s) = n({}_sf)$; then, for each $m$ in $UC_l(S)^*$, $m(g) = m*n(f)$. Hence $m$ annihilates all the functions of the form $s \to n({}_sf)$ $\big(n \in I(F), f \in UC_l(S)\big)$ if and only if $m*n = 0$ for all $n \in I(F)$ or $m \in J$. Hence by the double polar theorem, we see that $J_0$ is the closed linear subspace generated by the functions of the form $s \to n({}_sf)$ $\big(n \in I(F), f \in UC_l(S)\big)$. By corollary 3.7, each function of the form $s \to n({}_sf)$ satisfies properties (a) and (b); therefore, functions $u$ in $J_0$ satisfy (a) and (b) of 3.7. If $S$ is discrete, (a) is equivalent to (a′) as remarked in 3.8.

Let $m$ be a left invariant mean on $UC_l(S)$, and let $\Phi$ be the convex hull of $a[S]$. Then, by theorem 4.4, there is a net $\{\varphi_\gamma\}$ in $\Phi$ such that

$$\lim_\gamma \|(\varphi_\gamma - m)*n\| = 0$$

for each $n$ in $I(F)$. Since

$$\big\|\big(a(s)*\varphi_\gamma - m\big)*n\big\| = \|a(s)*(\varphi_\gamma - m)*n\| \leqslant \|(\varphi_\gamma - m)*n\|,$$

it follows that, for a fixed $n$ in $I(F)$,

$$\lim_\gamma \big\|\big(a(s)*\varphi_\gamma - m\big)*n\big\| = 0$$



uniformly for $s$ in $S$. Now let $E$ be the space of all $g$ in $UC_l(S)$ such that

$$\lim_\gamma \big(a(s)*\varphi_\gamma - m\big)(g) = 0$$

uniformly for $s$ in $S$. Then it is easy to see that $E$ is a closed linear subspace of $UC_l(S)$, and the discussion above reveals that functions of the form $s \to n({}_sf)$ $\big(n \in I(F), f \in UC_l(S)\big)$ belong to $E$. Hence, by the first part of theorem 4.5, $J_0 \subset E$. Since a function $s \to \big(a(s)*\varphi_\gamma\big)(g)$ is simply a convex combination of *right* translates of $g$, it follows that, if $g \in J_0$, the convex combinations of right translates of $g$ approximate the constant function $m(g)\cdot 1$ uniformly.

4.6. Remark. What we have proved above is a stronger version of corollary 3.7 (c) in case $M = UC_l(S)$. Namely, we produced a net of convex combinations of right translates which works for all $g$ in $J_0$. A similar strengthening of corollary 3.7 (c) in general is possible by the same method.

In view of theorem 4.5, it is of interest to know what sort of functions belong to $J_0$. It will be proved below (corollary 4.8) that, if $S$ is a (left) amenable topological group, each continuous almost periodic function belongs to $J_0$. In particular, if $S$ is a compact group, then $J_0 = C(S) = UC_l(S)$.

Let $S$ be a left amenable topological semi-group. A subset $B$ of $S$ is called *substantial* if there is a member $g$ of $UC_l(S)$ such that $g\,|\,S \sim B \equiv 0$ and $m(g) \neq 0$ for some left invariant mean $m$ on $UC_l(S)$. For $f$ in $UC_l(S)$ and $\varepsilon > 0$, the *right $\varepsilon$-period* of $f$ is the set $\{t\colon |f(s\cdot t) - f(s)| \leqslant \varepsilon \textit{ for all } s \textit{ in } S\}$.

4.7. THEOREM. *Let $S$ be a left amenable topological semi-group, and let $f$ be a member of $UC_l(S)$ with substantial right $\varepsilon$-period for each $\varepsilon > 0$. Then $f \in J_0$, where $J_0$ is as in theorem* 4.5.

Proof. Let $\varepsilon > 0$, and let $B$ be the $\varepsilon$-period of $f$. Since $B$ is substantial, there is $g$ in $UC_l(S)$ such that $g\,|\,S \sim B \equiv 0$ and $m(g) \neq 0$ for some left invariant mean $m$ on $UC_l(S)$. We can assume that $g \geqslant 0$ and $m(g) = 1$. If $t \in B$,

$$|f(s\cdot t)g(t) - f(s)g(t)| \leqslant \varepsilon\cdot g(t) \quad \text{for all } s.$$

If $t \in S \sim B$, then

$$|f(s\cdot t)g(t) - f(s)g(t)| = 0.$$

Therefore,

$$|f(s\cdot t)g(t) - f(s)g(t)| \leqslant \varepsilon g(t) \quad \text{for all } t, s \text{ in } S.$$

Applying $m_t$ to the both sides, we see that

$$|m({}_sf\cdot g) - f(s)| \leqslant \varepsilon \quad \text{for all } s \text{ in } S.$$

Now let $n$ be the continuous linear functional on $UC_l(S)$ defined by $n(k) = m(k\cdot g)$ for all $k$ in $UC_l(S)$. Then $-\|g\|\cdot m \leqslant n \leqslant \|g\|\cdot m$; hence, $n \in I(F)$. If we let $u(s) = n({}_sf) = m({}_sf\cdot g)$, then, by theorem 4.5, $u \in J_0$, and the above inequality becomes $\|f-u\| \leqslant \varepsilon$. Since $\varepsilon$ is arbitrary and $J_0$ is closed, $f \in J_0$.

4.8. COROLLARY. *Let $G$ be a (left) amenable topological group. Then a continuous almost periodic function $f$ on $G$ has the substantial right $\varepsilon$-period for each $\varepsilon > 0$; hence $f \in J_0$.*

Proof. By a standard theorem (see for example 7T in [7]), there are a continuous homomorphism $\varphi$ from $G$ into a compact topological group $H$ and a continuous function $h$ on $H$ such that $f = h\circ\varphi$. Since $h$ is left and right uniformly continuous, so is $f$. Given $\varepsilon > 0$, there is an open neighborhood $U$ of the identity $e$ in $H$ such that $|h(x\cdot y) - h(x)| \leqslant \varepsilon$ for all $x$ in $H$ and $y$ in $U$. Then $\varphi^{-1}[U]$ is contained in the right $\varepsilon$-period of $f$. It remains to prove that $\varphi^{-1}[U]$ is substantial.

Since $H$ is completely regular, there is a continuous function $k: H \to [0,1]$ such that $k|H \sim U \equiv 0$ and $k(e) = 1$. Let $g = k\circ\varphi$; then $g \in UC_l(G)$ and $g|G \sim \varphi^{-1}[U] \equiv 0$. Let $V = \{y: k(y) > \frac{1}{2}\}$; then there are points $x_1, \ldots, x_n$ in $G$ such that

$$\bigcup \{\varphi(x_i^{-1})\cdot V: i = 1, \ldots, n\} \supset \varphi[G].$$

Then

$$\sum_{i=1}^{n} {}_{x_i}g \geqslant \frac{1}{2}\cdot\mathbf{1}.$$

If $m$ is an arbitrary left invariant mean on $UC_l(G)$, then

$$\frac{1}{2} \leqslant \sum_{i=1}^{n} m({}_{x_i}g) = n\cdot m(g).$$

Hence $m(g) > 0$, which proves that $\varphi^{-1}[U]$ is substantial.

4.9. Remark. It follows from 4.8 and 4.5 that continuous almost periodic functions on an amenable topological group $G$ have properties (a)-(c) of corollary 3.7. This fact is well-known and can be proved directly by applying standard facts on compact groups to the almost periodic compactification of $G$.

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