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STUDIA MATHEMATICA, T. XXIX. (1967)

The spectrum of an infinite product measure

by

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The infinite product of certain probability measures on compact abelian groups was discussed by Varopoulus [4]; the measures he considered are included in the present description. We are mainly interested in showing how the orthogonality criterion of Kakutani [1] may be used in place of the almost-everywhere-convergence calculations of [4]. Besides this we give some elementary facts which aid in constructing examples in the harmonic analysis of measure algebras.

0. Let $G_1, G_2, G_3, \ldots$ be compact abelian groups $\neq 0$, e_n the unit measure at 0 in G_n, m_n the normalized Haar measure of G_n. Let $0 < a_n < 1$ for $1 \leqslant n$ and $\mu_n = a_n e_n + (1-a_n) m_n$. Finally,

$$G = \prod_1^\infty G_n, \qquad \mu = \prod_1^\infty \mu_n.$$

THEOREM 1. *The Fourier transform $\hat{\mu}$ vanishes at infinity in $\Gamma = \hat{G}$, if and only if $a_n \to 0$* (Varopoulos [4], p. 3806).

THEOREM 2. *The measure μ, as an element of the complex Banach algebra $M(G)$, has purely real spectrum if and only if*

(i) $\sum_{a_n > 1/2} (1-a_n) < \infty$.

(ii) *For some integer $k \geqslant 1$, $\sum_{a_n \leqslant 1/2} a_n^k < \infty$.*

1. The proofs are divided into one paragraph for the first theorem, and two for the second.

Proof of Theorem 1. It is well-known that a continuous character of G is composed in an obvious way from a *finite* number of characters $\gamma_1, \gamma_2, \ldots, \gamma_s$ on $G_1, G_2, \ldots, G_s$ respectively, and that $\hat{\mu}$ takes the value $\prod_1^s \hat{\mu}(\gamma_s)$ on the composite character. This degenerates to $\Pi' a_n$, Π' being the product over non-trivial components γ_n. It is clear that if

$$a < \limsup_{n\to\infty} |a_n| < b,$$

then $|\hat{\mu}(\gamma)| < b, \gamma \in \Gamma$, with a finite exceptional set, and $|\hat{\mu}(\gamma)| > a$ in an infinite subset of Γ. Γ being discrete, the theorem is proved.

Proof of Theorem 2. We begin by proving that μ^k and μ^{k+1} have purely real spectrum. In any case we require the formula

$$\mu_n^k = a_n^k e_n + (1-a_n^k) m_n, \quad 1 \leqslant n, 1 \leqslant k.$$

Write $\lambda_n = e_n$ if $a_n > \frac{1}{2}$, $\lambda_n = m_n$ if $a_n \leqslant \frac{1}{2}$. Then, for $1 \leqslant p$,

$$\Big\| \prod_1^p \mu_n^k \prod_{p+1}^\infty \lambda_n - \prod_1^{p+2} \mu_n^k \prod_{p+2}^\infty \lambda_n \Big\| = \|\lambda_{p+1} - \mu_{p+1}^k\|.$$

When $a_p > \frac{1}{2}$, the last norm $\leqslant (1-a_p^k) \leqslant k(1-a_p)$; when $a_p \leqslant \frac{1}{2}$, it is $\leqslant a_p^k$. We see then that $\prod_1^p \mu_n^k \prod_{p+1}^\infty \lambda_n$ converges in norm to $\prod_1^\infty \mu_n^k = \mu^k$. Exactly the same argument holds for μ^{k+1}. Multiplying out the products $\prod_1^p \mu_n^k \prod_{p+1}^\infty \lambda_n$, we obtain a convex combination of idempotents. The proof is easily completed from the elementary properties of the Gelfand transform ([2], Chapter 3).

Proof of Theorem 2b.

LEMMA. *If a probability measure μ in G has only mutually singular powers, its spectrum in $M(G)$ contains the complex unit circle.*

This is a very slight variation of one in [3], p. 107. For $|z| > 1$, e the identity,

$$(ze - \mu)^{-1} = \sum_0^\infty \mu^k z^{-k-1}.$$

The powers of μ being singular,

$$\|(ze - \mu)^{-1}\| = \sum_0^\infty |z|^{-k-1}.$$

The inverse is thus unbounded near $\{|z| = 1\}$ so that $ze - \mu$ is singular for $|z| = 1$, [2].

A class of functions. We shall require some properties of functions $f(x, u)$ defined for integers $k > l \geqslant 1$, $\frac{1}{2} \geqslant u \geqslant 0$, $1 > x > 0$. Write

$$f(x, u) = (1-u)(1-x^k)^{1/2}(1-x^l)^{1/2} + \big(u(1-x^k)+x^k\big)^{1/2}\big(u(1-x^l)+x^l\big)^{1/2}.$$

The properties required are not difficult to verify:

(1) $$0 \leqslant f(x, u) < 1,$$

(2) $$\frac{\partial f}{\partial u} > 0,$$

(3) $$f\Big(x, \frac{1}{2}\Big) = 1 - \frac{1}{8} x^{2l} + o(x^{2l}) \quad \text{as} \quad x \to 0,$$

(4) $$f\Big(x, \frac{1}{2}\Big) = 1 + \Big[\frac{1}{2}(kl)^{1/2} - \frac{1}{4}(k+l)\Big](1-x) + o(1-x) \quad \text{as} \quad x \to 1.$$

Kakutani's criterion. We now state for later reference a part of Kakutani's theorem on orthogonality of infinite product measures ([4], p. 221). In somewhat imprecise terms, we consider probability measures P_n and Q_n on a space Ω_n, $n = 1, 2, 3, \ldots$ Supposing that Q_n is absolutely continuous with respect to P_n, we write

$$\varrho(P_n, Q_n) = \int \Big(\frac{dQ_n}{dP_n}\Big)^{1/2} dP_n, \quad 1 \leqslant n,$$

Then, writing

$$P = \prod_{n=1}^\infty Q_n, \quad Q = \prod_{n=1}^\infty Q_n,$$

Q and P are singular unless $\sum_{n=1}^\infty [1 - \varrho(P_n, Q_n)] < \infty$.

Completion of the proof. We apply the criterion just stated with $P_n = \mu_n^k$, $Q_n = \mu_n^l$, $\Omega_n = G_n$. Here we assume, on account of the lemma at the beginning of this paragraph, that μ^l and μ^k are not totally singular for certain integers $k > l \geqslant 1$.

Let us write $1/\lambda_n$ for the number of elements in G_n, so that $0 \leqslant \lambda_n \leqslant \frac{1}{2}$. The parts of the measures μ_n^k and μ_n^l which are concentrated in $\{0\}$, or concentrated in its complement $G_n \sim \{0\}$, are respectively proportional. The numerical masses assigned are

$$a_n^k + \lambda_n(1-a_n^k) \quad \text{and} \quad a_n^l + \lambda_n(1-a_n^l) \quad (\text{in } \{0\}),$$

$$(1-\lambda_n)(1-a_n^k) \quad \text{and} \quad (1-\lambda_n)(1-a_n^l) \quad (\text{in } G_n \sim \{0\}),$$

whence $\varrho(\mu_n^k, \mu_n^l) = f(a_n, \lambda_n) \leqslant f(a_n, \frac{1}{2})$, by (2). By Kakutani's theorem, $f(a_n, \frac{1}{2}) \to 1$, so that $\{a_n\}$ can accumulate only at 0 or 1, by (1). The convergence of the sums

$$\sum_{a_n > 1/2} (1 - a_n) \quad \text{and} \quad \sum_{a_n \leqslant 1/2} a_n^{2l}$$

is assured by (3) and (4), so that the necessity of the conditions in Theorem 2 is proved. In regard to the work of Varopoulos, this contains the theorem on p. 3807.

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STUDIA MATHEMATICA, T. XXIX. (1967)

On certain actions of semi-groups on L-spaces*

by

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§ 0. Introduction. Given a semi-group S of continuous linear transformations of E into itself and an element x of E, one can ask (a) whether the closed convex hull of the orbit of x contains a common fixed point under S and (b) whether such a fixed point is unique. The answer to (a) is affirmative if the closed convex hull of the orbit of x is compact and the semi-group S is left amenable (see [2]). In order to answer (b) affirmatively, usually one assumes, among others, that S be both left and right amenable (see for example [3]). In the present paper we shall study a situation in which the left amenability of S is sufficient to conclude (b) affirmatively. More specifically, we shall postulate a certain (right) action of a semi-group S on $C(X)$, where X is a compact Hausdorff space, and we shall study the resulting (left) action of S on the dual $C(X)^*$. Throughout the paper, we prefer to speak of abstract M-spaces with units rather than $C(X)$. There are two reasons for this. First, not all M-spaces which arise naturally in this paper (such as $l_\infty(S)$, $UC_l(S)$ and $C(X)^{**}$) come neatly in the form of $C(X)$. Secondly, whenever possible, we favor order arguments over measure theoretic ones.

The basic facts on vector lattices, M-spaces and L-spaces can be found in [8].

The following is the summary of the contents. In § 1, we introduce the space $UC_l(S)$ of left uniformly continuous functions on a topological semi-group S with separately continuous multiplication, and state basic properties of $UC_l(S)$ needed in the sequel. Section § 2 is devoted to a proposition concerning certain projections in L-spaces. This proposition is crucial for the proof of the main theorem. In § 3, we define an "action" of a topological semi-group S on an M-space with unit, and the main theorem concerning this type of action is established. In § 4, we come back to $UC_l(S)$ where S is a left amenable topological semi-group. The dual $UC_l(S)^*$ is a Banach algebra, and the results of § 3 give some information about the multiplication on $UC_l(S)^*$. Next we introduce a sub-

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