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Orlicz spaces of finitely additive set functions*

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A serious study of finitely additive set functions appears to have started during the last part of the last century with, for instance, Jordan content. Though the Borel-Lebesgue theory of countably additive set functions has eclipsed the earlier work mostly through the first quarter of this century, the representation theory of linear functionals on spaces of bounded functions revived interest in finitely additive set functions in the early thirties, and the abstract integration relative to finitely additive measures developed extensively in the next decade. More important is the realization that finitely additive set functions provide considerable flexibility in many applications.

In probability theory, Dubins and Savage [8] have noted that countable additivity is a restrictive hypothesis and have dropped it. Even in the study of classical function spaces such as L^p Leader [13] finds it "natural to consider" the "L^p-spaces of finitely additive set functions" called the V^p-spaces. Motivated by these considerations, we shall consider spaces of set functions which are more general that the V^p-spaces but which also are endowed with an interesting structure.

Section I is devoted to defining Orlicz spaces (V^Φ) of finitely additive set functions having their values in a Banach space. In section II, it is shown that if L^Φ is the corresponding Orlicz space of point functions, there exists an isometric injection of L^Φ into V^Φ. The V^Φ-spaces generalize Leader's V^p-spaces of real finitely additive functions in much the same way the Orlicz spaces L^Φ generalize the Lebesgue spaces L^p. At the root of Leader's work is the Radon-Nikodym-Bochner theorem ([9], IV. 9. 14) which is available only in the scalar case. An extension of this theorem for vector-valued set functions is proved and then applied in sections III and IV analyzing the structure of V^Φ-spaces. In section V a representation

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of bounded linear operators on certain V^{Φ}-spaces to an arbitrary Banach space is obtained and the space of such operators is characterized. This result is specialized to L^{Φ} in representing all bounded linear operators of L^{Φ} into any Banach space whenever Φ obeys a growth condition. Some of the results contained in this paper are announced in [21] without proof.

I. The $V^{\Phi}(\mathscr{X})$-spaces

The basis for this section is finitely additive extended real-valued non-negative set function μ defined on a field Σ of subsets of a point set Ω. $\Sigma_0 \subset \Sigma$ is the ring of sets of finite μ-measure. Various Banach spaces of finitely additive set functions having their values in a real or complex Banach (or B-) space $\mathscr{X}$ will be defined and investigated.

Let φ be a left continuous non-decreasing real-valued function which does not vanish identically and satisfies $\varphi(0) = 0$. Let ψ denote the left continuous inverse of φ defined by the following convention: If φ is discontinuous at a, then $\psi(v) = a$ for $\varphi(a-) < v \leqslant \varphi(a+)$, and if $\varphi(u) = c$ for $a < u \leqslant b$ but $\varphi(u) < c$ for $u < a$, then $\varphi(c) = a$. If $\lim_{u\to\infty} \varphi(u) = l$ is finite, then $\psi(v) = \infty$ for $v > l$. By this convention, $\psi(0) = 0$ and ψ is well-defined on the positive line. φ and ψ are "generalized inverses" to each other and are mutually inverse to each other whenever both are strictly increasing and continuous.

Definition 1. If the non-decreasing functions φ and ψ are mutually inverse, and satisfy the above conditions, then the functions Φ and Ψ defined on the line by the Lebesgue integrals

$$\Phi(u) = \int_0^{|u|} \varphi(t)\,dt, \qquad \Psi(v) = \int_0^{|v|} \psi(t)\,dt$$

are called *complementary Young's functions*. A Young's function Φ *obeys the Δ_2-condition* if there exists $K < \infty$ such that $\Phi(2x) \leqslant K\Phi(x)$ for all x.

It follows that Φ and Ψ are convex, increasing, and are continuous except for at most one point, after which the function must be identically ∞. These facts and the proof of the following proposition are well-known ([24], p. 77).

Proposition 2 (Young's inequality). *If Φ and Ψ are complementary Young's functions, then*

$$|xy| \leqslant \Phi(x) + \Psi(y)$$

for all real numbers x and y. Equality holds if and only if one of the relations $y = \varphi(x)$ or $x = \psi(y)$ is satisfied.

By convention, throughout this paper, Φ and Ψ will denote (non-trivial) Young's functions.

Definition 3. If $E \in \Sigma$, a *partition* π of E is any finite disjoint collection $\{E_n\} \subset \Sigma_0$ satisfying $\bigcup_n E_n \subset E$. The partitions of a set E are partially ordered by defining $\pi_1 \leqslant \pi_2$ whenever each element of π_1 is a union of elements of π_2. In this case π_2 is said to be a *refinement* of π_1.

It may be noted that $\pi = \{E_n\}$ may be a partition of Ω, as defined above, without satisfying the relation $\Omega = \bigcup_n E_n$. This latter condition can hold only if $\mu(\Omega)$ is finite.

Definition 4. Let Φ be a Young's function, $\mathscr{X}$ a B-space, and F a finitely additive $\mathscr{X}$-valued set function defined on and vanishing on μ-null sets. If $E \in \Sigma$, $I_{\Phi}(F, E)$ is defined by

$$I_{\Phi}(F, E) = \sup_{\pi} \sum_{\pi} \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n)$$

where the supremum is taken over all partitions $\pi = \{E_n\}$ of E. (Here the convention $0/0 = 0$ is observed and will be used throughout.)

For simplicity of notation, the function $I_{\Phi}(\cdot)$ will be applied to either vector-valued set functions; the meaning will be clear from the context. Also $I_{\Phi}(F, \Omega)$ will be written as $I_{\Phi}(F)$. It is clear that $I_{\Phi}(F, E)$ is unambiguously defined as a finite number or as $+\infty$. If $I_{\Phi}(F)$ is finite, F is said to be of *Φ-bounded variation*. When $\Phi(x) = x^p$, $p \geqslant 1$, the notion of Φ-bounded variation reduces to the well-known notion of p-bounded variation. This and the Φ-bounded variation are sometimes referred to as the "Hellinger and the generalized Hellinger" integrals of F when F and μ are real-valued and countably additive [15].

Lemma 5. *For each F such that $I_{\Phi}(F)$ exists,*

$$\sum_{\pi} \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n)$$

is a non-decreasing function of the partitions π of Ω. Consequently

$$I_{\Phi}(F, E) = \lim_{\pi} \sum_{\pi} \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n),$$

where the limit is taken in the Moore-Smith sense [10] *through all partitions of $E \in \Sigma$.*

Proof. It is sufficient to show that

$$\Phi\left(\frac{\|F(A \cup B)\|}{\mu(A \cup B)}\right)\mu(A \cup B) \leqslant \Phi\left(\frac{\|F(A)\|}{\mu(A)}\right)\mu(A) + \Phi\left(\frac{\|F(B)\|}{\mu(B)}\right)\mu(B)$$

whenever $A, B \in \Sigma_0$, $A \cap B = \emptyset$. If either $\mu(A)$ or $\mu(B) = 0$, then the inequality is true and trivial. So, suppose $\mu(A), \mu(B) > 0$. The monotonicity and convexity of Φ imply

$$\Phi\left(\frac{\|F(A \cup B)\|}{\mu(A \cup B)}\right) \leqslant \Phi\left(\frac{\|F(A)\| + \|F(B)\|}{\mu(A) + \mu(B)}\right)$$
$$\leqslant \frac{\mu(A)}{\mu(A)+\mu(B)} \Phi\left(\frac{\|F(A)\|}{\mu(A)}\right) + \frac{\mu(B)}{\mu(A)+\mu(B)} \Phi\left(\frac{\|F(B)\|}{\mu(B)}\right).$$

Hence,

$$\Phi\left(\frac{\|F(A \cup B)\|}{\mu(A \cup B)}\right)\mu(A \cup B) \leqslant \Phi\left(\frac{\|F(A)\|}{\mu(A)}\right)\mu(A) + \Phi\left(\frac{\|F(B)\|}{\mu(B)}\right)\mu(B), \quad \text{q.e.d.}$$

LEMMA 6. *$I_\Phi(F, \cdot)$ is a finitely additive set function on Σ and vanishes on μ-null sets.*

Proof. The second statement is evident only the first needs a proof. Let $S_1, S_2 \in \Sigma$, $S_1 \cap S_2 = \emptyset$. If π_1 and π_2 are arbitrary partitions of S_1 and S_2 respectively, then $\pi_1 \cup \pi_2$ is a partition of $S_1 \cup S_2$. From the definition of $I_\Phi(F, \cdot)$,

$$I_\Phi(F, S_1 \cup S_2) \geqslant \sum_{\pi_1 \cup \pi_2} \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n)$$
$$= \sum_{\pi_1} \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n) + \sum_{\pi_2} \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n).$$

Taking suprema on both sides of this inequality over all partitions π_1 of S_1 and π_2 of S_2, we obtain

$$I_\Phi(F, S_1 \cup S_2) \geqslant I_\Phi(F, S_1) + I_\Phi(F, S_2).$$

To prove the reverse inequality, let $\pi = \{E_n\}$ be an arbitrary partition of $S_1 \cup S_2$. Then $\pi_1 = \{E_n \cap S_1\}$ and $\pi_2 = \{E_n \cap S_2\}$ are partitions of S_1 and S_2 respectively, and $\pi_1 \cup \pi_2$ is a refinement of π. According to lemma 5, and the definition of $I_\Phi(F, \cdot)$,

$$\sum_{\pi} \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n)$$
$$\leqslant \sum_{\pi_1} \Phi\left(\frac{\|F(E_n \cap S_1)\|}{\mu(E_n \cap S_1)}\right)\mu(E_n \cap S_1) + \sum_{\pi_2} \Phi\left(\frac{\|F(E_n \cap S_2)\|}{\mu(E_n \cap S_2)}\right)\mu(E_n \cap S_2)$$
$$\leqslant I_\Phi(F, S_1) + I_\Phi(F, S_2).$$

Consequently

$$I_\Phi(F, S_1 \cup S_2) = I_\Phi(F, S_1) + I_\Phi(F, S_2), \quad \text{q.e.d.}$$

The next theorem shows that $I_\Phi(\cdot)$ is lower semi-continuous.

THEOREM 7. *Let $\{G_\tau, \tau \in T\}$ be a net of set functions on Σ_0 such that $I_\Phi(G)$ is defined for each $\tau \in T$ and $\lim_\tau G_\tau(E) = G(E)$ in the weak topology of $\mathscr{X}$ for each $E \in \Sigma_0$. Then $I_\Phi(G, E)$ is defined and*

$$I_\Phi(G, E) \leqslant \liminf_\tau I_\Phi(G_\tau, E)$$

for each $E \in \Sigma$.

Proof. Without loss of generality, it may be assumed that $E = \Omega$. If is clear that the function G defined above on Σ_0 vanishes on μ-null sets. It is also immediate that G is finitely additive on Σ_0, so that $I_\Phi(G)$ is well defined.

Now, whether Φ is continuous or discontinuous (i.e. Φ jumps to $+\infty$ at some point),

$$\Phi\left(\frac{\|G(E)\|}{\mu(E)}\right) \leqslant \liminf_\tau \Phi\left(\frac{\|G_\tau(E)\|}{\mu(E)}\right)$$

for each $E \in \Sigma_0$ since $\|G(E)\| \leqslant \liminf_\tau \|G_\tau(E)\|$ ([9], II. 3.27). Therefore, if $\pi_0 = \{E_n\}$ is any partition of Ω, the following chain of inequalities holds:

$$\sum_{\pi_0} \Phi\left(\frac{\|G(E_n)\|}{\mu(E_n)}\right)\mu(E_n) \leqslant \sum_{\pi_0} \liminf_\tau \Phi\left(\frac{\|G_\tau(E_n)\|}{\mu(E_n)}\right)\mu(E_n)$$
$$\leqslant \liminf_\tau \sum_{\pi_0} \Phi\left(\frac{\|G_\tau(E_n)\|}{\mu(E_n)}\right)\mu(E_n)$$
$$\leqslant \liminf_\tau I_\Phi(G_\tau), \quad \text{q.e.d.}$$

In integration theory, much of the structure of spaces of integrable functions is based on the behavior of simple functions, that is, functions which assume only finitely many values. Our work, too, depends to a certain extent on the behavior of indefinite integrals of simple functions.

Definition 8. If $\pi = \{E_n\}$ is a partition of Ω, a set function of the form $\sum_\pi a_n \mu \cdot E_n$, $a_n \in \mathscr{X}$, is called a *step function*. Here $\mu \cdot E_n$ is the set function $\mu(E_n \cap \cdot)$. Of special interest are step functions of the form

$$\sum_\pi \frac{F(E_n)}{\mu(E_n)} \mu \cdot E_n$$

where F is finitely additive on Σ_0 and π is a partition of Ω. Leader [13] has termed such a function the *projection* of F on π. The projection of F on π will be denoted by F_π.

THEOREM 9. *If $I_\Phi(F)$ is defined and $\pi = \{E_n\}$ is any partition of Ω, then*

$$I_\Phi(F_\pi) = \sum_\pi \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n).$$

Consequently, $I_\Phi(F_\pi) \leqslant I_\Phi(F)$, and $I_\Phi(F, E) = \lim I_\Phi(F_\pi, E)$.

Proof. A brief computation shows that if $\pi_0 = \{E_n\}$ is any partition and $\pi = \{S_m\} \geqslant \pi_0$, then

$$\sum_\pi \Phi\left(\frac{\|F_\pi(S_m)\|}{\mu(S_m)}\right)\mu(S_m) = \sum_{\pi_0} \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n).$$

The remaining assertions of the theorem follow from lemma 5, q.e.d.

The following definitions and theorems are devoted to the introduction of a class of Banach spaces of $\mathscr{X}$-valued set functions. When μ is countably additive and has the finite subset property, Rao [18, 19] has shown that some of these spaces are equivalent to conjugate spaces of certain Orlicz spaces.

Definition 10. $A^\Phi(\Omega, \Sigma, \mu, \mathscr{X})$ $(= A^\Phi(\mathscr{X}))$ consists of all finitely additive $\mathscr{X}$-valued set functions F on Σ_0 satisfying

(i) F vanishes on μ-null sets;

(ii) $I_\Phi(F/K) \leqslant 1$ for some $K > 0$.

Using the convexity of Φ, one can easily show that whenever $I_\Phi(F) \leqslant \infty$, $F \in A^\Phi(\mathscr{X})$. By $\widetilde{A^\Phi(\mathscr{X})}$ is meant $\{F: I_\Phi(F) < \infty\}$. If Φ obeys the Δ_2-condition, then $A^\Phi(\mathscr{X}) = \widetilde{A^\Phi(\mathscr{X})}$. To see this, choose p such that $I_\Phi(F/p_0) \leqslant 1$ and a positive integer n such that $2^n/p \geqslant 1$. The monotonicity of Φ and the Δ_2-condition yield

$$I_\Phi(F) \leqslant I_\Phi\left(\frac{2^n}{p}F\right) \leqslant K^n I_\Phi\left(\frac{F}{p}\right) \leqslant K^n < \infty.$$

In the next theorem, a norm on $A^\Phi(\mathscr{X})$ is introduced relative to which $A^\Phi(\mathscr{X})$ becomes a Banach space.

THEOREM 11. *The functional $N_\Phi(F) = \inf\{k > 0: I_\Phi(F/K) \leqslant 1\}$ on $A^\Phi(\mathscr{X})$ is a norm under which $A^\Phi(\mathscr{X})$ is a Banach space.*

Proof. (i) $N_\Phi(F) = 0$ if and only if $F(E) = 0$ for all $E \in \Sigma_0$, then $I_\Phi(F/k) = 0$ for all $k > 0$. Hence $N_\Phi(F) = 0$. Conversely suppose there exists $F \in A^\Phi(\mathscr{X})$ with $N_\Phi(F) = 0$ and such that $F(E) \neq 0$ for some $E \in \Sigma_0$. Form the (trivial) partition $\pi = \{E\}$. Since $N_\Phi(F) = 0$, $I_\Phi(F/k) \leqslant 1$ for all $k > 0$. From lemma 5, it follows that $\Phi(\|F(E)\|/k\mu(E))\mu(E) \leqslant 1$ for all $k > 0$. On the other hand, because $\lim_{x\to\infty}\Phi(x) = +\infty$, $\lim_{k\to 0}\Phi(\|F(E)\|/k\mu(E))\mu(E) = \infty$, a contradiction.

(ii) $N_\Phi(aF) = |a|N_\Phi(F)$ for all scalars a and all F in $A^\Phi(\mathscr{X})$. The equality holds trivially if $a = 0$. When $a \neq 0$,

$$\begin{aligned} N_\Phi(aF) &= \inf\left\{k > 0: I_\Phi\left(\frac{aF}{k}\right) \leqslant 1\right\} \\ &= \inf\left\{k > 0: I_\Phi\left(F/\frac{k}{|a|}\right) \leqslant 1\right\} \\ &= \inf\left\{|a|t > 0: I_\Phi\left(\frac{F}{t}\right) \leqslant 1\right\} \\ &= |a|N_\Phi(F). \end{aligned}$$

(iii) $N_\Phi(F_1+F_2) \leqslant N_\Phi(F_1)+N_\Phi(F_2)$ for all $F_1, F_2 \in A^\Phi(\mathscr{X})$. From the convexity and monotonicity of Φ, it follows that, provided neither F_1 nor $F_2 = 0$,

$$\begin{aligned} I_\Phi&\left(\frac{F_1+F_2}{N_\Phi(F_1)+N_\Phi(F_2)}\right) \\ &\leqslant \frac{N_\Phi(F_1)}{N_\Phi(F_1)+N_\Phi(F_2)} I_\Phi\left(\frac{F_1}{N_\Phi(F_1)}\right) + \frac{N_\Phi(F_2)}{N_\Phi(F_1)+N_\Phi(F_2)} I_\Phi \frac{F_2}{N_\Phi(F_2)} \\ &\leqslant \frac{N_\Phi(F_1)+N_\Phi(F_2)}{N_\Phi(F_1)+N_\Phi(F_2)} = 1. \end{aligned}$$

Hence $N_\Phi(F_1+F_2) \leqslant N_\Phi(F_1)+N_\Phi(F_2)$. If either $F_1 = 0$ or $F_2 = 0$, the inequality is true and trivial.

(iv) $A^\Phi(\mathscr{X})$ is complete under $N_\Phi(\cdot)$. Suppose $\{F_n\} \subset A^\Phi(\mathscr{X})$ is a Cauchy sequence. The definition of $N_\Phi(\cdot)$ establishes the existence of a double sequence $\{N_{n,m}\}$ of positive members such that

$$\lim_{n,m} N_{n,m} = +\infty \quad \text{and} \quad I_\Phi\big(N_{n,m}(F_n - F_m)\big) \leqslant 1$$

for all positive integers m and n. First it will be shown that $\lim_n F_n(E)$ exists in the strong topology of $\mathscr{X}$ for each $E \in \Sigma_0$. If $\mu(E) = 0$, then $F_n(E) = 0$ for all n. If $\mu(E) > 0$, the definition of $I_\Phi(\cdot)$ implies

$$\Phi\left(\frac{N_{n,m}\|F_n(E) - F_m(E)\|}{\mu(E)}\right)\mu(E) \leqslant I_\Phi\big(N_{n,m}(F_n - F_m)\big) \leqslant 1$$

for all m and n. Since $\lim_{n,m} N_{n,m} = \infty$ and $\lim_{x\to\infty}\Phi(x) = \infty$, one has $\lim_{n,m}\|F_n(E) - F_m(E)\| = 0$. Since $\mathscr{X}$ is a Banach space, the set function F defined for each E in Σ_0 by

$$F(E) = \lim_n F_n(E)$$

is well-defined and vanishes on μ-null sets. It is evident that F is additive.

Next it will be shown that $F\in A^{\Phi}(\mathscr{X})$ and $\lim_n N_{\Phi}(F_n-F)=0$. The fact that $\lim_{n,m} N_{\Phi}(F_n-F_m)=0$ and the triangle inequality imply $\varrho=\lim_n N_{\Phi}(F_n)$ exists. If $\varrho=0$, then $F=0$ and $\lim_n N_{\Phi}(F_n-F)=0$, and the proof is finished. If $\varrho\neq 0$, it may be assumed that $N_{\Phi}(F_n)\neq 0$ for all n. Then

$$\lim_n \frac{F_n(E)}{N_{\Phi}(F_n)}=\frac{F(E)}{\varrho}$$

strongly in $\mathscr{X}$ for each $E\in\Sigma_0$. The lower semi-continuity of I_{Φ} (Lemma 7) guarantees

$$I_{\Phi}\left(\frac{F}{\varrho}\right)\leqslant \liminf_n I_{\Phi}\left(\frac{F_n}{N_{\Phi}(F_n)}\right)\leqslant 1.$$

Hence $F\in A^{\Phi}(\mathscr{X})$. Now let $K>0$ be arbitrary and P_K be chosen so that $N_{n,m}\geqslant K$ and for $n,m\geqslant P_K$. For each n and m the monotonicity of Φ yields $I_{\Phi}(K(F_n-F_m))\leqslant I_{\Phi}(N_{n,m}(F_n-F_m))\leqslant 1$. Another application of the lower semi-continuity (theorem 7) of $I_{\Phi}(\cdot)$ and the definition of F yield

$$I_{\Phi}(K(F_m-F))\leqslant \liminf_n I_{\Phi}(K(F_m-F_n))\leqslant 1 \quad \text{for} \quad m\geqslant P_K.$$

Thus $N_{\Phi}(F_m-F)\leqslant 1/K$ for $m\geqslant P_K$. The arbitrariness of K implies $\lim_n N_{\Phi}(F_m-F)=0$, q.e.d.

As in the Orlicz spaces of point functions, a form of the Hölder inequality can be stated for the $A^{\Phi}(\mathscr{X})$-spaces.

LEMMA 12. *If Φ and Ψ are complementary Young's functions and $\mathscr{X}^*$ is the conjugate space of $\mathscr{X}$, then*

$$\sup_{\pi}\sum_{\pi}\frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)}\leqslant 2N_{\Phi}(F)N_{\Psi}(G)$$

for all $F\in A^{\Phi}(\mathscr{X})$ and $G\in A^{\Psi}(\mathscr{X}^*)$.

Proof. If either F or G is 0, the result is immediate. Assuming neither F nor G is 0, in view of Young's inequality (Proposition 2), one obtains

$$\sum_{\pi}\frac{\frac{\|F(E_n)\|}{N_{\Phi}(F)}\,\frac{\|G(E_n)\|}{N_{\Psi}(G)}}{\mu(E_n)}=\sum_{\pi}\frac{\frac{\|F(E_n)\|}{N_{\Phi}(F)}\,\frac{\|G(E_n)\|}{N_{\Psi}(G)}}{\mu(E_n)\,\mu(E_n)}\mu(E_n)$$
$$\leqslant\sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{N_{\Phi}(F)\mu(E_n)}\right)\mu(E_n)+\sum_{\pi}\Psi\left(\frac{\|G(E_n)\|}{N_{\Psi}(G)\mu(E_n)}\right)\mu(E_n)$$
$$\leqslant I_{\Phi}\left(\frac{F}{N_{\Phi}(F)}\right)+I_{\Psi}\left(\frac{N}{N_{\Psi}(G)}\right)\leqslant 1+1=2$$

for any partition π. Hence

$$\sup_{\pi}\sum_{\pi}\frac{\frac{\|F(E_n)\|}{N_{\Phi}(F)}\,\frac{\|G(E_n)\|}{N_{\Psi}(G)}}{\mu(E_n)}\leqslant 2,$$

or,

$$\sup_{\pi}\sum_{\pi}\frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)}\leqslant 2N_{\Phi}(F)N_{\Psi}(G), \qquad \text{q. e. d.}$$

If $\Phi(x)=|x|$, the resulting $A^{\Phi}(\mathscr{X})$-space is denoted by $A^1(\mathscr{X})$. Clearly the $A^1(\mathscr{X})$-norm is precisely that of the space of bounded $\mathscr{X}$-valued set functions, $ba(\Omega,\Sigma,\mathscr{X})$ — the variation norm $\mathscr{V}(\cdot)$. The next lemma establishes an interesting and useful relationship between $A^{\Phi}(\mathscr{X})$ and $A^1(\mathscr{X})$.

LEMMA 13. *If $E\in\Sigma_0$ and $F\in A^{\Phi}(\mathscr{X})$, then $F\cdot E\in A^1(\mathscr{X})$. Moreover, there exists a constant K depending only on $\mu(E)$ such that*

$$\mathscr{V}(F\cdot E)=N_1(F\cdot E)\leqslant KN_{\Phi}(F).$$

In particular, if $\mu(\Omega)<\infty$, $A^1(\mathscr{X})\subset A^{\Phi}(\mathscr{X})$ and there exists a constant K such that $N_1(F)\leqslant KN_{\Phi}(F)$ for all $F\in A^{\Phi}(\mathscr{X})$.

Proof. From the support line property of convex functions, there exists a constant $K_0>0$ such that

$$|x|\leqslant K_0\Phi(x)+K_0.$$

Hence, if $0\neq F\in A^{\Phi}(\mathscr{X})$ and $S\in\Sigma_0$ is arbitrary

$$\frac{\|F(S)\|}{N_{\Phi}(F)}\leqslant K_0\Phi\left(\frac{\|F(S)\|}{\mu(S)N_{\Phi}(F)}\right)\mu(S)+K_0\mu(S).$$

Accordingly, if $\pi=\{E_n\}$ is a partition of E,

$$\frac{1}{N_{\Phi}(F)}\sum_{\pi}\|F(E_n)\|\leqslant K_0\sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)N_{\Phi}(F)}\right)\mu(E_n)+K_0\sum_{\pi}\mu(E_n)$$
$$\leqslant K_0 I_{\Phi}\left(\frac{F}{N_{\Phi}(F)}\right)+K_0\mu(E)<\infty.$$

Since π is arbitrary, it follows that

$$\frac{N_1(F\cdot E)}{N_{\Phi}(F)}\leqslant K_0[1+\mu(E)]=K<\infty.$$

For a more detailed study, a subclass of the $\Lambda^{\Phi}(\mathscr{X})$-spaces having a more amenable structure will be isolated; the slightly more general case, not treated in the sequel, seems to require different methods of study.

Definition 14. $V^{\Phi}(\Omega, \Sigma, \mu, \mathscr{X}) (= V^{\Phi}(\mathscr{X}))$ stands for the linear submanifold of $\Lambda^{\Phi}(\mathscr{X})$ consisting of μ-continuous functions (i.e.,

$$\lim_{\mu(E)\to 0} \|F(E)\| = 0$$

for each $F \in V^{\Phi}(\mathscr{X})$). $V^{\Phi}(\mathscr{X})$ stands for $V^{\Phi}(\mathscr{X}) \cap \Lambda^{\Phi}(\mathscr{X})$.

The following lemma due to Rao ([18], p. 85) shows that in many cases $\Lambda^{\Phi}(\mathscr{X}) = V^{\Phi}(\mathscr{X})$:

LEMMA 15. *If Φ and Ψ are complementary Young's functions and Ψ is continuous, then $\Lambda^{\Phi}(\mathscr{X}) = V^{\Phi}(\mathscr{X})$.*

The proof is given in [18], p. 85, for real-valued set functions defined on a σ-field. The proof there is applicable here *mutatis mutandis*.

In a more general setting is

THEOREM 16. *$V^{\Phi}(\mathscr{X})$ is a closed subspace of $\Lambda^{\Phi}(\mathscr{X})$. Hence $V^{\Phi}(\mathscr{X})$ is a Banach space.*

Proof. Clearly $V^{\Phi}(\mathscr{X})$ is a linear space. Therefore, all that need be shown is that $V^{\Phi}(\mathscr{X})$ is complete. Let $\{F_n\} \subset V^{\Phi}(\mathscr{X})$ be a Cauchy sequence. The completeness of $\Lambda^{\Phi}(\mathscr{X})$ establishes the existence of $F \in \Lambda^{\Phi}(\mathscr{X})$ such that $\lim_n N_{\Phi}(F_n - F) = 0$. According to lemma 15, there exists a $K > 0$ such that

$$\mathscr{V}(F_n \cdot E - F \cdot E) \leqslant K N_{\Phi}(F_n - F),$$

where K is independent of $E \in \Sigma_0$ provided $\mu(E) \leqslant 1$. Let $\varepsilon > 0$ be given and select n_0 such that

$$N_{\Phi}(F_{n_0} - F) < \varepsilon/2K.$$

Then $\mathscr{V}(F_{n_0} \cdot E - F \cdot E) < \varepsilon/2$. Since F_{n_0} is μ-continuous, there exists $\delta_1 > 0$ such that $\|F_{n_0}(E)\| < \varepsilon/2$ whenever $\mu(E) < \delta_1$. Therefore, if $\delta = \min\{\delta_1, 1\}$,

$$\big|\|F_{n_0}(E)\| - \|F(E)\|\big| \leqslant \mathscr{V}(F_{n_0} \cdot E - F \cdot E) < \varepsilon/2$$

provided $\mu(E) < \delta$. It follows that $\|F(E)\| < \varepsilon$ whenever $\mu(E) < \delta$. Therefore $F \in V^{\Phi}(\mathscr{X})$ and $V^{\Phi}(\mathscr{X})$ is a closed subspace of $\Lambda^{\Phi}(\mathscr{X})$, q. e. d.

Remark. If $\Phi(x) = |x|^p$, $1 \leqslant p \leqslant \infty$, $\mathscr{X}$ is the real line and $\mu(\Omega)$ is finite, the corresponding V^{Φ}-space is precisely the V^p-space introduced by Bochner [5, 6] and Leader [13].

As in the theory of Orlicz spaces of point functions, the structure of $V^{\Phi}(\mathscr{X})$ cannot be completely analyzed without the introduction of a second norm. Corresponding to the Orlicz norm ([24], p. 79) on $L^{\Phi}(\mathscr{X})$ is the functional $\|\cdot\|_{\Phi}$ defined for F in $V^{\Phi}(\mathscr{X})$ by

$$\|F\|_{\Phi} = \sup\left\{\sup_{\pi} \sum_{\pi} \frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)};\ G \in V^{\Psi}(\mathscr{X}^*),\ N_{\Psi}(G) \leqslant 1\right\},$$

where Ψ is complementary to Φ. Lemma 14 guarantees that

$$\|F\|_{\Phi} \leqslant \sup_{N_{\Psi}(G)\leqslant 1} 2N_{\Phi}(F) N_{\Psi}(G) = 2N_{\Phi}(F);$$

so that $\|\cdot\|_{\Phi}$ is finite on $V^{\Phi}(\mathscr{X})$.

LEMMA 17. *$\|\cdot\|_{\Phi}$ is a norm on $V^{\Phi}(\mathscr{X})$.*

Proof. $\|F\|_{\Phi} = 0$ if and only if $F = 0$. Clearly $F = 0$ implies $\|F\|_{\Phi} = 0$. If $F \in V^{\Phi}(\mathscr{X})$ and there exists $E \in \Sigma_0$ such that $F(E) \neq 0$, select $x^* \in \mathscr{X}^*$ such that $0 < \Psi(\|x^*\|) \leqslant 1/\mu(E)$ and let $H = x^* \mu \cdot E$. Then $N_{\Psi}(H) \leqslant 1$ and $\|F(E)\|\,\|H(E)\|/\mu(E) \geqslant 0$. Hence $\|F\|_{\Phi} > 0$. That $\|\cdot\|_{\Phi}$ obeys the other norm properties follows immediately from the corresponding properties of the norm in $\mathscr{X}$, q. e. d.

The importance of $\|\cdot\|_{\Phi}$ will become apparent after it is shown that $\|\cdot\|_{\Phi}$ is actually equivalent to the N_{Φ}-norm. This will follow directly from the following lemma:

LEMMA 18. *$I_{\Phi}(F/\|F\|_{\Phi}) \leqslant 1$ for all $F \neq 0$ in $V^{\Phi}(\mathscr{X})$.*

Proof. Preliminary to the actual proof is the fact that for F in $V^{\Phi}(\mathscr{X})$ and G in $V^{\Psi}(\mathscr{X}^*)$,

$$\sup_{\pi} \sum_{\pi} \frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)} \leqslant \begin{cases} \|F\|_{\Phi} & \text{if } I_{\Psi}(G) \leqslant 1, \\ \|F\|_{\Phi} I_{\Psi}(G) & \text{if } 1 < I_{\Psi}(G) < \infty. \end{cases}$$

This follows as in the point function case ([24], p. 80) with essentially no modifications.

If Φ does not jump (i.e., $\Phi(x) < \infty$ for $x < \infty$), let $\pi = \{E_n\}$ be any partition and consider

$$G = \sum_{\pi} x^* \varphi\left(\frac{\|F(E_n)\|}{\|F\|_{\Phi}\mu(E_n)}\right) \mu \cdot E_n,$$

where x^* is an element of the unit sphere of $\mathscr{X}^*$. Since G is a step function on π, and $\|x^*\| = 1$, by theorem 9,

$$I_{\Psi}(G) = \sum_{\pi} \Psi\left(\varphi\left(\frac{\|F(E_v)\|}{\|F\|_{\Phi}\mu(E_n)}\right)\right) \mu(E_v).$$

With this choice of G,

$$\sum_{\pi}\frac{\|F(E_n)\|\,\|G(E_n)\|}{\|F\|_\Phi\,\mu(E_n)} = \sum_{\pi}\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\,\varphi\left(\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\mu(E_n)$$

$$= \sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\mu(E_n) + \sum_{\pi}\Psi\left(\varphi\left(\frac{\|(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\right)\mu(E_n),$$

since with this G there is term-by-term equality in Young's inequality,

$$= \sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\mu(E_n) + I_\Psi(G).$$

Since the left-hand side is finite, so is the right-hand side and $I_\Psi(G) < \infty$. If $I_\Psi(G) \leqslant 1$, the preliminary result yields,

$$\sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\mu(E_n) \leqslant \sum_{\pi}\frac{\|F(E_n)\|\,\|G(E_n)\|}{\|F\|_\Phi\,\mu(E_n)} \leqslant 1.$$

If $I_\Psi(G) > 1$, the same preliminary result implies

$$I_\Psi(G) \geqslant \sum_{\pi}\frac{\|F(E_n)\|\,\|G(E_n)\|}{\|F\|_\Phi\,\mu(E_n)} = \sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\mu(E_n) + I_\Psi(G);$$

so that

$$\sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\mu(E_n) = 0.$$

In either case

$$\sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\mu(E_n) \leqslant 1.$$

Since the partition π is arbitrary

$$I_\Phi\left(\frac{F}{\|F\|_\Phi}\right) = \sup_{\pi}\sum_{\Pi}\Phi\left(\frac{\|F(E_n)\|}{\|F\|_\Phi\,\mu(E_n)}\right)\mu(E_n) \leqslant 1.$$

This establishes the lemma if Φ is continuous.

Suppose Φ jumps at $M > 0$. If it can be shown that

$$\frac{\|F(E)\|}{\|F\|_\Phi\,\mu(E)} \leqslant M$$

for any $E \in \Sigma_0$ of positive μ-measure; then $\varphi\left(\|F(E)\|/\|F\|_\Phi\,\mu(E)\right) < \infty$, and the proof given above for continuous Φ applies without modification.

Since Φ jumps at M, $\Psi(x) \leqslant M$ for all x. Hence

$$\Psi(x) = \int_0^{|x|}\Psi(t)\,dt \leqslant M\,|x|$$

and $N_\Psi(H) \leqslant MN_1(H)$. For any $E \in \Sigma_0$ of positive μ-measure, consider

$$G = \frac{x^*}{M\mu(E)}\,\mu\cdot E,$$

where x^* is on the unit sphere of $\mathfrak{X}^*$. $N_\Psi(G) \leqslant MN_1(G) = 1$. Therefore

$$\frac{\|F(E)\|\,\|G(E)\|}{\mu(E)} \leqslant \|F\|_\Phi.$$

But

$$\frac{\|F(E)\|\,\|G(E)\|}{\mu(E)} = \frac{\|F(E)\|}{\mu(E)}\cdot\frac{1}{M}.$$

Hence

$$\frac{\|F(E)\|}{\|F\|_\Phi\,\mu(E)} \leqslant M, \qquad \text{q. e. d.}$$

Some important consequences of this lemma are collected below:

THEOREM 19. (a) *If Φ and Ψ are complementary Young's functions, and $F \in V^\Phi(\mathfrak{X})$, $G \in V^\Psi(\mathfrak{X}^*)$, then the following Hölder type inequalities are valid:*

$$\sup_{\pi}\sum_{\pi}\frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)} \leqslant \begin{cases} \text{(i)} & \|F\|_\Phi\,\|G\|_\Psi, \\ \text{(ii)} & N_\Phi(F)\,\|G\|_\Psi, \\ \text{(iii)} & 2N_\Phi(F)\,N_\Psi(G). \end{cases}$$

(b) $N_\Phi(F) \leqslant \|F\|_\Phi \leqslant 2N_\Phi(F)$.

Consequently $N_\Phi(\cdot)$ and $\|\cdot\|_\Phi$ are equivalent norms on $V^\Phi(\mathfrak{X})$. In particular, $V^\Phi(\mathfrak{X})$ is a Banach space under $\|\cdot\|_\Phi$ as well.

Proof. (a) If either F or G is 0, the assertions (i), (ii), (iii), and (b) hold trivially. If neither F nor G is 0, lemma 18 states $I_\Psi(G/\|G\|_\Psi) \leqslant 1$. Hence $N_\Psi(G/\|G\|_\Psi) \leqslant 1$ and

$$\sup_{\pi}\sum_{\pi}\frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)\,\|G\|_\Psi} \leqslant \|F\|_\Phi;$$

whence

$$\sup_{\pi}\sum_{\pi}\frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)} \leqslant \|F\|_\Phi\,\|G\|_\Psi.$$

This establishes (i).

To prove (ii), note that $N_\Phi(F/N_\Phi(F)) \leqslant 1$. Therefore,

$$\sup_\pi \sum_\pi \frac{\|F(E_n)\|\,\|G(E_n)\|}{N_\Phi(F)\mu(E_n)} \leqslant \|G\|_\Psi,$$

or

$$\sup_\pi \sum_\pi \frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)} \leqslant N_\Phi(F)\,\|G\|_\Psi.$$

(iii) is already proved as lemma 12.

(b) It follows from (iii) that $\|F\|_\Phi \leqslant 2N_\Phi(F)$. According to lemma 18, $I_\Phi(F/\|F_\Phi\|) \leqslant 1$. The definition of N_Φ implies $N_\Phi(F) \leqslant \|F\|_\Phi$. Combining these inequalities, one obtains

$$N_\Phi(F) \leqslant \|F\|_\Phi \leqslant 2N_\Phi(F).$$

Consequently the two norms are equivalent, q. e. d.

II. The Orlicz spaces $L^\Phi(X)$ and the injection of $L^\Phi(X)$ into $V^\Phi(X)$

The basis for this section is also a non-negative extended real-valued finitely additive set function μ defined on a field Σ of subsets of a set Ω. $\Sigma_0 \subset \Sigma$ is the ring of sets of finite μ-measure. The (point) functions to be considered will be defined on Ω, are totally μ-measurable in the sense of [9] and have their values in a real or complex B-space $\mathfrak{X}$. Orlicz spaces (possibly incomplete) of such functions, denoted by $L^\Phi(\mathfrak{X})$ will be defined, and elementary properties of these spaces will be discussed. Finally, an isometric injection of $L^\Phi(\mathfrak{X})$ into the corresponding $V^\Phi(\mathfrak{X})$-space will be constructed in a natural way. The integration procedure to be employed is that of [9], Chapter III, and coincides with the familiar Bochner integral when Σ is a σ-field and μ is a countably additive measure.

Definition 1. Let Φ be a Young's function. $L_0^\Phi(\Omega, \Sigma, \mu, \mathfrak{X})$ $(= L_0^\Phi(\mathfrak{X}))$ denotes the collection of all totally-μ-measurable ([9], III. 2.10) $\mathfrak{X}$-valued functions f such that

$$\int_\Omega \Phi\left(\frac{\|f\|}{k}\right)d\mu \leqslant 1 \quad \text{for some } k > 0.$$

By the *norm* $N_\Phi(f)$ *of an element* $f \in L_0^\Phi(\mathfrak{X})$ is meant the quantity

$$N_\Phi(f) = \inf\left\{k > 0:\ \int_\Omega \Phi\left(\frac{\|f\|}{k}\right)d\mu \leqslant 1\right\}.$$

LEMMA 2. *$L_0^\Phi(\mathfrak{X})$ is a semi-normed linear space under the semi-norm $N_\Phi(\cdot)$. Moreover $N_\Phi(f) = 0$ if and only if f is μ-null.*

Proof. The second statement follows from standard properties of the integral. The proof of the fact that $N_\Phi(\cdot)$ is a semi-norm is essentially the same as the proof of theorem I. 11 and need not be repeated here.

In view of lemma 2, it is natural to consider classes of functions in $L_0^\Phi(\mathfrak{X})$ equivalent under the relation: f is equivalent to g if and only if $f-g$ is a μ-null function. With this identification, we denote the resulting space by $L^\Phi(\Omega, \Sigma, \mu, \mathfrak{X})$ $(= L^\Phi(\mathfrak{X}))$. The following theorem is a direct consequence of lemma 2:

THEOREM 3. *$L^\Phi(\mathfrak{X})$ is a normed linear space.*

Unfortunately $L^\Phi(\mathfrak{X})$ is not, in general, a complete space. If it is insisted that Σ be a σ-field and μ be countably additive, then the $L^\Phi(\mathfrak{X})$ spaces defined above are automatically complete, and in fact, coincide with the familiar Orlicz spaces [11, 22, 24] when $\mathfrak{X} = R$, the real line. For this reason, it is natural to call $L^\Phi(\mathfrak{X})$ an Orlicz space. Hereafter a function $f \in L^\Phi(\mathfrak{X})$ means that f is a representative of its equivalence class. Paralleling ([22], Th. 6) is the following

THEOREM 4. *If f is in $L^\Phi(\mathfrak{X})$ and g is in $L^\Psi(\mathfrak{X}^*)$, where Φ and Ψ are complementary in the sense of Young, then $\|f\|\,\|g\|$ is integrable and*

$$\int_\Omega \|f\|\,\|g\|\,d\mu \leqslant 2N_\Phi(f)N_\Psi(g).$$

The proof is essentially the same as that of [22], Th. 6, and will be omitted.

The next result links the $L^\Phi(\mathfrak{X})$ and the $V^\Phi(\mathfrak{X})$-spaces and allows many properties of $L^\Phi(\mathfrak{X})$ to be deduced from the corresponding properties of $V^\Phi(\mathfrak{X})$.

THEOREM 5. *Let Φ be continuous.*

(a) *Each $f \in L^\Phi(\mathfrak{X})$ is integrable on all sets of finite μ-measure. Consequently, the set function λf defined on Σ_0 by*

$$\lambda f(E) = \int_E f\,d\mu, \qquad E \in \Sigma_0,$$

is finitely additive and μ-continuous.

(b) *If $f \in L^\Phi(\Omega, \Sigma, \mu, \mathfrak{X})$, then $\lambda f \in V^\Phi(\Omega, \Sigma, \mu, \mathfrak{X})$. The mapping λ: $L^\Phi(\mathfrak{X}) \to V^\Phi(\mathfrak{X})$ is linear and $N_\Phi(f) = N_\Phi(\lambda f)$. Hence λ is an isometric injection of $L^\Phi(\mathfrak{X})$ into $V^\Phi(\mathfrak{X})$.*

Proof. The proof of the fact that $f \in L^\Phi(\mathfrak{X})$ is integrable on all sets of finite measure is based on the support line property of convex functions. Its proof is the same as the proof of lemma I. 13 with an obvious notational modification. The finite additivity of the set function λf follows from standard properties of the integral. Again from the support line property of Φ, for $0 \neq f \in L^{\Phi}(\mathscr{X})$, one has

$$\frac{\|\lambda f(E)\|}{N_{\Phi}(f)} \leqslant \int_E \frac{\|f\| d\mu}{N_{\Phi}(f)} \leqslant K \int_E \Phi\left(\frac{\|f\|}{N_{\Phi}(f)}\right) d\mu + K\mu(E),$$

for some $K > 0$, and all $E \in \Sigma_0$. The μ-continuity of the right-hand side therefore implies the μ-continuity of the left-hand side.

(b) To prove $\lambda f \in V^{\Phi}(\mathscr{X})$, it will be shown that $\|\lambda f\|_{\Phi} < \infty$. Let $G \in V^{\Psi}(\mathscr{X}^*)$, $N_{\Psi}(G) \leqslant 1$ and the partition π be arbitrary. Then

$$\sum_{\pi} \frac{\|\lambda f(E_n)\| \, \|G(E_n)\|}{\mu(E_n)} = \sum_{\pi} \frac{\left\|\int_{E_n} f d\mu\right\| \, \|G(E_n)\|}{\mu(E_n)}$$

$$\leqslant \sum_{\pi} \int_{E_n} \|f\| d\mu \frac{\|G(E_n)\|}{\mu(E_n)} = \int_{\Omega} \|f\| \, \|g_{\pi}\| d\mu, \qquad g_{\pi} = \sum_{\pi} \frac{G(E_n)}{\mu(E_n)} \chi_{E_n},$$

but

$$\int_{\Omega} \Psi(\|g_{\pi}\|) d\mu = I_{\Psi}(G_{\pi}) \leqslant I_{\Psi}(G)$$

by theorem I. 9, where Ψ is complementary to Φ. Since $N_{\Psi}(G) \leqslant 1$, $I_{\Psi}(G) \leqslant 1$ and $N_{\Psi}(g_{\pi}) \leqslant 1$. An application of theorem 4 yields

$$\sum_{\pi} \frac{\|\lambda f(E_n)\| \, \|G(E_n)\|}{\mu(E_n)} \leqslant \int_{\Omega} \|f\| \, \|g_{\pi}\| d\mu \leqslant 2N_{\Phi}(f) N_{\Psi}(g_{\pi}) \leqslant 2N_{\Phi}(f).$$

Since π and G were arbitrary,

$$\|\lambda f\|_{\Phi} = \sup_{N_{\Psi}(G) \leqslant 1} \sup_{\pi} \sum_{\pi} \frac{\|\lambda f(E_n)\| \, \|G(E_v)\|}{\mu(E_n)} \leqslant 2N_{\Phi}(f).$$

Therefore $\|\lambda f\|_{\Phi} < \infty$. According to lemma I. 18, $I_{\Phi}(\lambda f / \|\lambda f\|_{\Phi}) \leqslant 1$. Hence $f \in V^{\Phi}(\mathscr{X})$.

To prove $N_{\Phi}(f) = N_{\Phi}(\lambda f)$, it suffices to show

$$\Phi\left(\frac{\|f\|}{k}\right) d\mu = I_{\Phi}\left(\frac{\lambda f}{k}\right)$$

for any $k > 0$ such that $I_{\Phi}(\lambda f/k) \leqslant 1$. (To complete the proof a few results from section IV concerning the behavior of $I_{\Phi}(\cdot)$ are needed. Since these results are independent of the $L^{\Phi}(\mathscr{X})$ theory, they will be used here.)

Let k be such that $I_{\Phi}(\lambda f/k) \leqslant 1$ and $E \in \Sigma_0$ be arbitrary. For each partition π of E, consider the μ-simple function

$$\frac{f_{\pi}}{k} = \sum_{\pi} \frac{\int_{E_n} f d\mu}{\mu(E_n)} \chi_{E_n}.$$

From (a), $f\chi_E/k \in L^1(\mathscr{X})$. The proof of [9], IV. 8. 17, shows that

$$\lim_{\pi} \int_E \frac{\|f - f_{\pi}\|}{k} d\mu = 0,$$

where the limit is taken through all partitions π of E. It follows that the net $\{f_{\pi}\}$ converges in μ-measure to $f\chi_E$. By virtue of the continuity of Φ and [9], III. 2.12, the net $\Phi(\|f_{\pi}\|/k)$ converges to $\Phi(\|f\|/k)\chi_E$ in μ-measure. Now, for each $E_0 \in \Sigma_0$ contained in E,

$$\text{(A)} \qquad \int_{E_0} \Phi\left(\frac{\|f_{\pi}\|}{k}\right) d\mu = \sum_{\pi} \Phi\left(\frac{\left\|\int_{E_n} f d\mu\right\|}{k\mu(E_n)}\right) \mu(E_v \cap E_0) = I_{\Phi}\left(\left(\frac{\lambda f}{k}\right)_{\pi}, E_0\right)$$

for each partition π of E. Since $\lambda(f\chi_E)$ is an indefinite integral, the hypotheses of lemma IV. 5 (b) are satisfied and

$$\text{(B)} \qquad I_{\Phi}(\lambda f, E \cap \cdot) = \lim_{\pi} I_{\Phi}(\lambda f_{\pi}, E \cap \cdot)$$

strongly in $V^1(R)$. This fact and [7], III. 2.15, imply

$$\lim_{\pi, \Delta} \int_E \left| \Phi\left(\frac{\|f_{\pi}\|}{k}\right) - \Phi\left(\frac{\|f_{\Delta}\|}{k}\right) \right| d\mu = 0,$$

where the limit is taken through all partitions π and Δ of E. Hence the net $\{\Phi(\|f_{\pi}\|/k), \pi \text{ a partition of } E\}$ determines $\Phi(\|f\|/k)\chi_E$ in the sense of [9], Chapter III; i.e.,

$$\int_E \Phi\left(\frac{\|f\|}{k}\right) d\mu = \lim_{\pi} \int_E \Phi\left(\frac{\|f_{\pi}\|}{k}\right) d\mu.$$

On the other hand, (B) implies

$$I_{\Phi}\left(\frac{\lambda f}{k}, E\right) = \lim_{\pi} I_{\Phi}\left(\frac{f_{\pi}}{k}, E\right) = \lim_{\pi} \int_E \Phi\left(\frac{\|f_{\pi}\|}{k}\right) d\mu,$$

by (A),

$$I_{\Phi}\left(\frac{\lambda f}{k}, E\right) = \int_E \Phi\left(\frac{\|f\|}{k}\right) d\mu,$$

from above. Hence

$$I_\Phi\left(\frac{\lambda f}{k}, E\right) = \int_E \Phi\left(\frac{\|f\|}{k}\right) d\mu$$

for each $E \in \Sigma_0$. From the definitions of $I_\Phi(\lambda f/k)$ and $\int_\Omega \Phi(\|f\|/k)\,d\mu$, it follows that

$$\int_\Omega \Phi\left(\frac{\|f\|}{k}\right) d\mu = \sup_{E \in \Sigma_0} \int_E \Phi\left(\frac{\|f\|}{k}\right) d\mu$$

$$= \sup_{E \in \Sigma_0} I_\Phi\left(\frac{\lambda f}{k}, E\right), \quad \text{from above,}$$

$$= I_\Phi\left(\frac{\lambda f}{k}, \Omega\right) = I_\Phi\left(\frac{\lambda f}{k}\right), \quad \text{q. e. d.}$$

Remark. In general the mapping λ is not a surjection. Indeed, if λ were always onto $V^\Phi(\mathscr{X})$, this would imply that each member of $V^\Phi(\mathscr{X})$ had a Radon-Nikodym derivative. This is not true in general — even when Σ is a σ-field and μ is a countably additive finite measure on Σ. In fact, a counterexample may be found, for instance, in [23], p. 135. In addition, the image of $L^\Phi(\mathscr{X})$ under λ may fail to be a closed subspace of $V^\Phi(\mathscr{X})$. Since λ is an isometry, this will be true if and only if $L^\Phi(\mathscr{X})$ is complete.

Recall that μ has the finite subset property (FSP) if and only if for every set $E \in \Sigma$ of infinite measure, there exists $E_0 \in \Sigma_0$, $E_0 \subset E$ such that $0 < \mu(E_0) < \mu(E)$. When μ has FSP, the classical Orlicz norm may be introduced on the $L^\Phi(\mathscr{X})$-spaces.

THEOREM 6. *Let Φ be continuous and μ prove the finite subset property. Then*

(a) *the functional $\|\cdot\|_\Phi$ defined for $f \in L^\Phi(\mathscr{X})$ by*

$$\|f\|_\Phi = \sup\left\{\int_\Omega \|f\|\,\|g\|\,d\mu \colon g \in L^\Psi(\mathscr{X}^*),\ N_\Psi(g) \leqslant 1\right\},$$

where Ψ is complementary to Φ, is a norm.

(b) *If $f \in L^\Phi(\mathscr{X})$, then $N_\Phi(f) \leqslant \|f\|_\Phi \leqslant 2N_\Phi(f)$. Consequently, $N_\Phi(\cdot)$ and $\|\cdot\|_\Phi$ are equivalent norms on $L^\Phi(\mathscr{X})$.*

(c) *$\|\lambda f\|_\Phi = \|f\|_\Phi$ for all $f \in M^\Phi(\mathscr{X})$, where $M^\Phi(\mathscr{X})$ is the closed subspace of $L^\Phi(\mathscr{X})$ determined by μ-simple functions, and λ is the injection of theorem 5.*

Proof. (a) From theorem 4, it immediately follows that $\|\cdot\|_\Phi$ is well defined and finite on $L^\Phi(\mathscr{X})$. Arguments paralleling those of [24], p. 78-83, show that $\|\cdot\|_\Phi$ is positive homogeneous and obeys the triangle inequality. Furthermore, it is clear that $\|f\|_\Phi = 0$ if f is μ-null. The FSP will be used

to prove the converse of this statement. If f is not μ-null, there exists by the proof of [7], III. 2.22, a sequence of μ-simple functions converging to f in μ-measure and satisfying $\|f_n\| \leqslant 2\|f\|$. It follows that there exists a set $E \in \Sigma$ and a positive number a such that $\|f\chi_E\| \geqslant a\chi_E$. By FSP, there exists $E_0 \in \Sigma_0$, $E_0 \subset E$ and $0 < \mu(E_0)$. For this E_0, one has $\|f\chi_{E_0}\| \geqslant a\chi_{E_0}$. If $0 \neq x^* \in \mathscr{X}^*$ is chosen such that $\Psi(\|x^*\|)\mu(E_0) \leqslant 1$ and $g = x^*\chi_E$, then $N_\Psi(g) \leqslant 1$ and

$$\int_\Omega \|f\|\,\|g\|\,d\mu > 0.$$

Hence $\|f\|_\Phi > 0$. This proves (a).

(b) Let $f \in L^\Phi(\mathscr{X})$ be arbitrary. It will be shown first that $\|\lambda f\|_\Phi \leqslant \|f\|_\Phi$. Let $\varepsilon > 0$ be given; choose $G \in V^\Psi(\mathscr{X}^*)$, $N_\Psi(G) \leqslant 1$ and the partition π such that

$$\sum_\pi \frac{\|\lambda f(E_n)\|\,\|G(E_n)\|}{\mu(E_n)} \geqslant \|\lambda f\|_\Phi - \varepsilon.$$

Let

$$g = \sum_\pi \frac{G(E_n)}{\mu(E_n)} \chi_{E_n}.$$

Then $G_\pi = \lambda g$ and $N_\Psi(g) = N_\Psi(G_\pi) \leqslant N_\Psi(G) \leqslant 1$ by theorem 5 (b) and the choice of G. With this g, one has

$$\|f\|_\Phi \geqslant \int_\Omega \|f\|\,\|g\|\,d\mu = \sum_\pi \frac{\int_{E_n} \|f\|\,d\mu\,\|G(E_n)\|}{\mu(E_n)}$$

$$\geqslant \sum_\pi \frac{\left\|\int_{E_n} f\,d\mu\right\|\,\|G(E_n)\|}{\mu(E_n)} = \sum_\pi \frac{\|\lambda f(E_n)\|\,\|G(E_n)\|}{\mu(E_n)} \geqslant \|\lambda f\|_\Phi - \varepsilon$$

by the choice of G.

Since $\varepsilon > 0$ is arbitrary, it follows that $\|f\|_\Phi \geqslant \|\lambda f\|_\Phi$. But

$$\|\lambda f\|_\Phi \geqslant N_\Phi(\lambda f) = N_\Phi(f)$$

by theorem I. 19 (b) and theorem 5. On the other hand, from the definition of $\|f\|_\Phi$ and theorem 4, one has $\|f\|_\Phi \leqslant 2N^\Phi(f)$. Combining these inequalities yields

$$N_\Phi(f) \leqslant \|f\|_\Phi \leqslant 2N_\Phi(f),$$

and (b) follows.

(c) It is sufficient to prove $\|f\|_\Phi = \|\lambda f\|_\Phi$ for all μ-simple functions $f \in L^\Phi(\mathscr{X})$. From the proof of (b), $\|\lambda f\|_\Phi \leqslant \|f\|_\Phi$; thus it remains only to prove the opposite inequality. Let

$$f = \sum_{i=1}^n x_i \chi_{E_i}, \qquad x_i \in \mathscr{X}, \qquad \{E_i\}_{i=1}^n \subset \Sigma_0$$

be disjoint. If $g \in L^{\Psi}(\mathscr{X}^*)$, $N_{\Psi}(g) \leqslant 1$, then

$$\int_{\Omega} \|f\| \, \|g\| d\mu = \sum_{i=1}^{n} \|x_i\| \int_{E_i} \|g\| d\mu = \sum_{\pi} \frac{\|\lambda f(E_i)\| \int_{E_i} \|g\| d\mu}{\mu(E_i)},$$

where π is the partition $\{E_i\}_{i=1}^{n}$. Let $x^* \in \mathscr{X}^*$ be on the unit sphere and define G on Σ_0 by

$$G(E) = \int_{E} x^* \|g\| d\mu.$$

By the choice of x^* and theorem 5 (b),

$$N_{\Psi}(G) = N_{\Psi}(x^* \|g\|) = N_{\Psi}(g) \leqslant 1.$$

Moreover

$$\sum_{\pi} \frac{\|\lambda f(E_i)\| \int_{E_i} \|g\| d\mu}{\mu(E_i)} = \sum_{\pi} \frac{\|\lambda f(E_i)\| \, \|G(E_i)\|}{\mu(E_i)}.$$

It follows from this that $\|f\|_{\Phi} \leqslant \|\lambda f\|_{\Phi}$, q. e. d.

III. A generalization of the Radon-Nikodym-Bochner theorem

In this section, we shall investigate some special properties of vector-valued set functions. A generalization of the Radon-Nikodym-Bochner theorem ([9], IV. 9. 14) for vector-valued finitely additive set functions is proved. This result is employed in this paper crucially in the study of the structure of $V^{\Phi}(\mathscr{X})$. By ba $(\Omega, \Sigma, \mathscr{X})$ is meant, as usual, the collection of all finitely additive $\mathscr{X}$-valued set functions of bounded variation defined on a field Σ of subsets of Ω, where $\mathscr{X}$ is a real or complex B-space. ca $(\Omega, \Sigma, \mathscr{X})$ is the collection of all countably additive members of ba $(\Omega, \Sigma, \mathscr{X})$. Ω and Σ will be fixed throughout this section. The following lemmas are extracted from [9], IV. 9, for ready reference.

LEMMA 1. *There exists a totally disconnected Hausdorff space S_1 and an isomorphism $\tau: \Sigma \to \Sigma_1$, the field of all open and closed subsets of S_1 in the sense that $\tau(E \cup F) = \tau(E) \cup \tau(F)$, $\tau(E \cap F) = \tau(E) \cap \tau(F)$, and $\tau(E') = \tau(E)'$ $(E' = \Omega - E)$ for all $E, F \in \Sigma$.*

LEMMA 2. (a) *If C is the complex plane, there is an isometric isomorphism T of* ba (Ω, Σ, C) *onto* ba (S_1, Σ_1, C) *determined by the correspondence $T\mu(E_1) = \mu(\tau^{-1}(E_1))$ for $\mu \in$* ba (S_1, Σ_1, C) *and $E_1 \in \Sigma_1$.*

(b) *Each μ_1 in* ba (S_1, Σ_1, C) *has a unique extension to a regular countably additive measure μ_2 in* ca (S_1, Σ_2, C) *where Σ_2 is the σ-field generated by Σ_1. The correspondence $U: \mu_1 \to \mu_2$ is an isometric isomorphism of* ba (S_1, Σ_1, C) *onto* ca (S_1, Σ_2, C).

The following theorem generalizes the Radon-Nikodym-Bochner theorem ([9], IV. 9. 14) to vector-valued set functions and constitutes the main result of this section:

THEOREM 3. *Let $\mathscr{X}$ be a Banach space, μ be a non-negative element of* ba (Ω, Σ, C) *and $F \in$* ba $(\Omega, \Sigma, \mathscr{X})$ *be μ-continuous. If*

$$\left\{ \frac{F(E)}{\mu(E)} : \left\| \frac{F(E)}{\mu(E)} \right\| \leqslant n, \quad E \in \Sigma \right\}$$

is weakly sequentially compact in $\mathscr{X}$ for each positive integer n, then for each $\varepsilon > 0$ there exists a μ-simple function f_ε such that $\mathscr{V}\big(F(\cdot) - \int_{(\cdot)} f_\varepsilon d\mu\big) < \varepsilon$, where $\mathscr{V}$ is the variation norm on ba $(\Omega, \Sigma, \mathscr{X})$.

Proof. The notation of lemmas 1 and 2 will be used throughout the proof. In the notation of lemma 2, let $M = UT$; by lemma 2, M is an isometric isomorphism of ba (Ω, Σ, C) onto ca (S_1, Σ_2, C). Let $M(\mu) = \mu_2$, and define F_1 on Σ_1 by $F_1(E_1) = F(\tau^{-1}(E_1))$ for $E_1 \in \Sigma_1$. Since F is μ-continuous, it follows that F_1 is μ_2-continuous on Σ_1. Next F_1 will be extended to Σ_2, the σ-field generated by the field Σ_1. According to [9], III. 7.1, for each E in Σ_2, there exists a sequence $\{E_n\}$ in Σ_1 such that $\mu_2\text{-}\lim_n E_n = E$ in the sense that $\lim_n \mu_2(E \Delta E_n) = 0$, where Δ is the symmetric difference operation. Now, if $E \in \Sigma_2$ is arbitrary and $\{E_n\} \subset \Sigma_1$ satisfies $\mu_2\text{-}\lim_n E_n = E$, then from the μ_2-continuity of F_1 on Σ_1 and the simple relations $\mu_2(E_m \Delta E_n) = \mu_2(E_m - E_m \cap E_n) + \mu_2(E_n - E_m \cap E_n)$ and $\|F_1(E_m) - F_1(E_n)\| = \|F_1(E_m - E_m \cap E_n) - F_2(E_n - E_m \cap E_n)\| \leqslant \|F_1(E_m - E_m \cap E_n)\| + \|F(E_n - E_n \cap E_m)\|$, it follows that $\lim_{n,m} \|F_1(E_n) - F_1(E_m)\| = 0$. The completeness of $\mathscr{X}$ insures the existence of an element x_E in $\mathscr{X}$ such that $\lim_n F_1(E_n) = x_E$ strongly in $\mathscr{X}$. It is clear that x_E is independent of the sequence $\{E_n\}$ converging to E since any two sequences converging to E can be combined into a single sequence converging to E. Define the function F_2 on Σ_2 by $F_2(E_1) = F_1(E_1)$ for all E_1 in Σ_1 and by $F_2(E) = x_E$ if E is not in Σ_1 but is in Σ_2.

The next step is to show that the thus defined F_2 is strongly countably additive, μ_2-continuous, and of bounded variation. Let $x^* \in \mathscr{X}^*$ be arbitrary. By lemma 2, $x^* F_1$ has a unique countably additive extension, say G_x^* to Σ_2. On account of the fact that F_1 is μ_2-continuous on Σ_1, $x^* F_1$ is μ_2-continuous on Σ_1. An application of [9], IV. 9. 13, implies G_x^* is μ_2-continuous. Replacing G_x^* in the argument used to define F_2, one finds $\lim_n x^*(F_1(E_n)) = G_x^*(E)$ for any sequence $\{E_n\}$ in Σ_1 such that $\mu_2 - \lim_n E_n = E$ in $\Sigma_2 - \Sigma_1$. Since strong convergence implies weak convergence, it follows that $G_x^*(E) = x^*(F_2(E))$ for all $E \in \Sigma_2$. But G_x^* is countably additive and $x^*\in\mathscr{X}^*$ is arbitrary. Therefore F_2 is weakly countably additive. An application of a result of Pettis ([9], IV. 10.1) shows that F_2 is strongly countably additive on Σ_2. To show F_2 is μ_2-continuous, note that $G_x^*(=x^*F_2)$ is μ_2-continuous for each $x^*\in\mathscr{X}^*$. Therefore $x^*F_2(N)=0$ for all $x^*\in\mathscr{X}^*$ whenever $N\in\Sigma_2$ and $\mu_2(N)=0$. Hence F_2 vanishes on μ_2-null sets. Another theorem of Pettis ([9], IV. 10.1) then implies F_2 is μ_2-continuous.

Next, it will be shown that $\mathscr{V}(F_2)=\mathscr{V}(F)$. For, $\mathscr{V}(F)\leqslant\mathscr{V}(F_2)$ since F is "defined" only on a subclass (under identification) of Σ_2. From [13], Th. 9,

$$0=\lim_\pi \mathscr{V}(x^*F_\pi-x^*F)=\lim_\pi \mathscr{V}\big(M(x^*F_\pi-x^*F)\big)$$
$$=\lim_\pi \mathscr{V}\big(M(x^*F_\pi)'M(x^*F)\big),$$

since M is an isometry. Therefore

$$\lim_\pi M(x^*F_\pi)(E_2)=M(x^*F)(E_2)$$

for all $E_2\in\Sigma_2$ and $x^*\in\mathscr{X}^*$. But $M(x^*F)(E_1)=x^*F_1(E_1)$ for all $E_1\in\Sigma_1$. Lemma 2 (b) implies, by the uniqueness of the extension of x^*F_1 to Σ_2, that $M(x^*F)(E_2)=x^*F_2(E_2)$ for all $E_2\in\Sigma_2$. Now, if π is any partition of Ω, and $x^*\in\mathscr{X}^*$

$$M(x^*F_\pi)=M\Big(\sum_\pi\frac{x^*F(E_n)}{\mu(E_n)}\mu\cdot E_n\Big)=\sum_\pi\frac{x^*F(E_n)}{\mu(E_n)}M(\mu\cdot E_n)$$
$$=\sum_\pi\frac{x^*F(E_n)}{\mu(E_n)}\mu_2\cdot\tau(E_n)$$
$$=\sum_\pi\frac{x^*F_2\big(\tau(E_n)\big)}{\mu_2\big(\tau(E_n)\big)}\mu_2\cdot\tau(E_n)=x^*F_{2_{\tau(\pi)}}$$

where $\tau(\pi)=\{\tau(E_n):\ E_n\in\pi\}$. Combining these results, one obtains

$$\lim_\pi x^*F_{2_{\tau(\pi)}}(E)=x^*F_2(E)$$

for all $E\in\Sigma_2$ and $x^*\in\mathscr{X}^*$. It follows that

$$\|F_2(E)\|\leqslant\liminf_\pi\|F_{2_{\tau(\pi)}}(E)\|$$

for all $E\in\Sigma_2$. In view of this, if $\{S_i\}_{i=1}^n$ is a partition of Σ_2, one has

$$\text{(B)}\qquad \sum_{i=1}^n\|F_2(S_1)\|\leqslant\sum_{i=1}^n\liminf_\pi\|F_{2_{\tau(\pi)}}(S_i)\|$$
$$\leqslant\liminf_\pi\sum_{i=1}^n\|F_{2_{\tau(\pi)}}(S_i)\|\leqslant\liminf_\pi\mathscr{V}(F_{2_{\tau(\pi)}}).$$

But

$$\mathscr{V}(F_{2_{\tau(\pi)}})=\sum_\pi\|F_2(\tau E_i)\|=\sum_\pi\|F(E_i)\|\leqslant\mathscr{V}(F)$$

for $\pi=\{E_i\}$. Hence

$$\liminf_\pi\mathscr{V}(F_{2_{\tau(\pi)}})\leqslant\mathscr{V}(F).$$

Since the partition $\{S_i\}$ is arbitrary, it follows, from (B) and the preceding inequality, that $\mathscr{V}(F_2)\leqslant\mathscr{V}(F)$. This and (A) imply that $\mathscr{V}(F_2)=\mathscr{V}(F)$.

Finally, using the second part of the hypothesis, it will be shown that

$$\Big\{\frac{F_2(E)}{\mu_2(E)}:\ \frac{\|F_2(E)\|}{\mu_2(E)}\leqslant n\Big\}=A_n$$

is weakly sequentially compact for each positive integer n. In order to prove this, it is sufficient to show that A_n is contained in the strong closure of

$$\Big\{\frac{F(E)}{\mu(E)}:\ \frac{\|F(E)\|}{\mu(E)}\leqslant n+1\Big\}=B_{n+1}$$

for each positive integer n. Suppose $E\in\Sigma_2$ is such that $F_2(E)/\mu_2(E)\in A_n$. If $E\in\Sigma_1$, then

$$\frac{F_2(E)}{\mu_2(E)}=\frac{F_2\big(\tau(E_0)\big)}{\mu_2\big(\tau(E_0)\big)}$$

for some $E_0\in\Sigma,=F(E_0)/\mu(E_0)\in B_{n+1}$.

If $E\notin\Sigma_1$, the definition of F_2 establishes the existence of a sequence $\{E_m\}\subset\Sigma_1$ such that

$$\lim_m F_2(E_m)=F_2(E)\ \text{(strongly)}\quad\text{and}\quad\lim_m\mu_2(E_m)=\mu_2(E).$$

Since $\|F_2(E)/\mu_2(E)\|\leqslant n$, it follows that there exists m_0 such that

$$\Big\|\frac{F_2(E_m)}{\mu_2(E_m)}\Big\|\leqslant n+1\quad\text{for}\quad m\geqslant m_0.$$

The above argument shows that $F_2(E_m)/\mu_2(E_m)\in B_{n+1}$. Moreover

$$\lim_m\Big\|\frac{F_2(E_m)}{\mu_2(E_m)}-\frac{F_2(E)}{\mu_2(E)}\Big\|=0.$$

Hence $F_2(E)/\mu_2(E)$ belongs to the strong closure of B_{n+1}. It follows that A_n is contained in the strong closure of B_{n+1}. Since B_{n+1} is weakly sequentially compact by hypothesis, the Eberlein-Smulian theorem ([9], V. 6.1) implies A_n is weakly sequentially compact.

In view of the established properties of F_2, F_2 satisfies the hypothesis of Phillips' generalization of the classical Radon-Nikodym theorem to vector-valued functions ([16], p. 134). This theorem guarantees the existence of a unique $F \in L^1(S_1, \Sigma_2, \mu_2, \mathscr{X})$ such that

$$F_2(E_2) = \int_{E_2} f d\mu_2$$

for all $E_2 \in \Sigma_2$. By virtue of the fact that Σ_2 is the σ-field generated by Σ_1, the Σ_1-simple functions are dense in $L^1(S_1, \Sigma_2, \mu_2, \mathscr{X})$ ([9], III. 8.3). Therefore, if $\varepsilon > 0$ is given, there exists a Σ_1-simple function

$$f_\varepsilon = \sum_{i=1}^{n} a_i \chi_{E_i}, \quad a_i \in X,\ E_i \in \Sigma_1,\ E_i \cap E_j = \emptyset,\ i \neq j,$$

satisfying

$$\int_{S_1} \|f - f_\varepsilon\| d\mu_2 < \varepsilon.$$

If F_ε is the indefinite integral of f_ε, then $\mathscr{V}(F_2 - F_\varepsilon) < \varepsilon$ by [9], III. 2.20. Hence

$$\mathscr{V}\big((F_2 - F_\varepsilon)|\Sigma_1\big) < \mathscr{V}(F_2 - F_\varepsilon) < \varepsilon$$

where $(F_2 - F_\varepsilon)|\Sigma_1$ is the restriction of $F_2 - F_\varepsilon$ to Σ_1. But $F_2(E_1) = F\big(\tau^{-1}(E_1)\big)$ for all $E_1 \in \Sigma_1$. Therefore if H_ε is the indefinite integral of

$$h_\varepsilon = \sum_{i=1}^{n} a_i \chi \tau^{-1}(E_i),$$

then $H_\varepsilon(E) = F_\varepsilon\big(\tau(E)\big)$ for all $E \in \Sigma$. Hence

$$\mathscr{V}(F - \int_{(\cdot)} h_\varepsilon d\mu) = \mathscr{V}\big((F_2 - F_\varepsilon)|\Sigma_1\big) < \varepsilon, \qquad \text{q. e. d.}$$

The hypothesis of theorem 3 cannot be materially weakened. If the condition on the set function were deleted, the resulting statement would be false. Indeed, if Σ were a σ-field and μ were countably additive and finite on Σ, the truth of this statement would imply that each $\mathscr{X}$-valued μ-continuous countably additive set function had a representation as an indefinite Bochner integral. A counter example cited before also applies to this situation and can be found in [23], p. 135. However, if the conditions on F are relaxed, but the conditions on $\mathscr{X}$ are strengthened, then the following result may be stated.

The following corollary is basic to the main result of the second part of this section.

COROLLARY 4. *Let $\mathscr{X}$ be a reflexive B-space. Let μ be a non-negative element of* $\mathrm{ba}(\Omega, \Sigma, C)$ *and* $F \in \mathrm{ba}(\Omega, \Sigma, \mathscr{X})$ *be μ-continuous. Then for each $\varepsilon > 0$, there is a μ-simple function f_ε such that*

$$\mathscr{V}(F(\cdot) - \int_{(\cdot)} f_\varepsilon d\mu) < \varepsilon.$$

Consequently, if $\mu(\Omega) < \infty$ and $\mathscr{X}$ is reflexive, step functions are dense in $V^1(\Omega, \Sigma, \mu, \mathscr{X})$.

Proof. The first assertion immediately follows from the fact that boundedness and weak sequential compactness are equivalent in a reflexive B-space. The second assertion follows from the facts that step functions in $V^1(\mathscr{X})$ are merely indefinite integrals of μ-simple functions and that the norm of $V^1(\mathscr{X})$ is the variation norm $\mathscr{V}(\cdot)$, q. e. d.

IV. The structure of $V^\Phi(\mathscr{X})$

The principal results of this section deal with the closed subspace of $V^\Phi(\mathscr{X})$ determined by step functions. The main result here shows that step functions are dense in $V^\Phi(\mathscr{X})$ if Φ obeys the Δ_2-condition and $\mathscr{X}$ is a reflexive Banach space.

Throughout this section $\mathscr{X}$ is a Banach space, Φ a continuous Young's function, Ω a point set, Σ a field of subsets of Ω, μ an finitely additive extended real-valued non-negative set function defined for $E \in \Sigma \cdot \Sigma_0 \subset \Sigma$ is the ring of sets of finite μ-measure.

Definition 1. $S^\Phi(\Omega, \Sigma, \mu, \mathscr{X}) (= S^\Phi(\mathscr{X}))$ is the closed subspace of $V^\Phi(\Omega, \Sigma, \mu, \mathscr{X}) (= V^\Phi(\mathscr{X}))$ determined by step functions. (If $\Phi(x) = |x|$, $S^\Phi(\mathscr{X})$ will be denoted by $S^1(\mathscr{X})$).

THEOREM 2. *If F belongs to $S^\Phi(\mathscr{X})$, then*

$$\lim_\pi N_\Phi(F - F_\pi) = 0.$$

Proof. Clearly $\lim_\pi F_\pi = F$ in $V^\Phi(\mathscr{X})$ for all step functions F. Moreover, $N_\Phi(F_\pi) \leqslant N_\Phi(F)$ for all $F \in S^\Phi(\mathscr{X})$ and all partitions π. An application of [9], II. 1.18, yields the result, q. e. d.

LEMMA 3. (a) *If F belongs to $S^1(\mathscr{X})$, then $\mathscr{V}(F, \cdot)$, the variation of F, belongs to $V^1(R)$ where R is the real line.*

(b) *If F belongs to $V^\Phi(\mathscr{X})$, then $I_\Phi\big(\mathscr{V}(F, \cdot)\big) = I_\Phi(F)$.*

Proof. (a) $\mathscr{V}(F, \cdot)$ is clearly additive and of bounded variation. It remains to show that $\mathscr{V}(F, \cdot)$ is μ-continuous. Since $F \in S^1(\mathscr{X})$, for each $\varepsilon > 0$, there exists a partition π_0 such that

$$\mathscr{V}(F - F_\pi) = N_1(F - F_\pi) < \varepsilon \quad \text{for} \quad \pi \geqslant \pi_0.$$

Thus for any partition $\Delta = \{E_n\}$

$$\sum_{\Delta} |\mathscr{V}(F, E_n) - \mathscr{V}(F_\pi, E_n)| \leqslant \sum_{\Delta} \mathscr{V}(F\cdot E_n - F_\pi \cdot E_n)$$

$$= \sum_{\Delta} \mathscr{V}\big((F-F_\pi)\cdot E_n\big) \leqslant \mathscr{V}(F-F_\pi) = N_1(F-F_\pi) \quad \text{for} \quad \pi \geqslant \pi_0.$$

Hence $\lim_\pi \mathscr{V}(F_\pi, \cdot) = \mathscr{V}(F, \cdot)$ strongly in $A^1(R)$. But each $\mathscr{V}(F_\pi, \cdot)$ is evidently μ-continuous and consequently is in $V^1(R)$. The conclusion follows directly from theorem I. 16.

(b) Let $F \in V^\Phi(\mathscr{X})$. From the definition of $\mathscr{V}(F, \cdot)$, it is seen that $\lim_\pi \mathscr{V}(F_\pi, E) = \mathscr{V}(F, E)$ for each $E \in \Sigma_0$. In addition,

$$\mathscr{V}(F_\pi, \cdot) = \sum_\pi \frac{\|F(E_n)\|}{\mu(E_n)} \mu \cdot E_n;$$

from this and theorem I. 9, it follows that

$$I_\Phi\big(\mathscr{V}(F_\pi, \cdot)\big) = I_\Phi(F_\pi) \leqslant I_\Phi(F).$$

These facts combined with the lower semi-continuity of $I_\Phi(\cdot)$ (theorem I. 7) imply

$$I_\Phi\big(\mathscr{V}(F, \cdot)\big) \leqslant \liminf_\pi I_\Phi\big(\mathscr{V}(F_\pi, \cdot)\big) \leqslant I_\Phi(F).$$

To prove the reverse inequality, note that $\mathscr{V}(F, E) \geqslant \|F(E)\|$ for each $E \in \Sigma_0$. The monotonicity of Φ yields the inequality $I_\Phi\big(\mathscr{V}(F, \cdot)\big) \geqslant I_\Phi(F)$ and the result, q. e. d.

Lemma 4. *If $F \in V^\Phi(\mathscr{X})$ and $F\cdot E \in S^1(\mathscr{X})$ for each $E \in \Sigma_0$, then $I_\Phi\big(\mathscr{V}(F, \cdot), \cdot\big) \in V^1(R)$ (where $I_\Phi\big(\mathscr{V}(F, \cdot), \cdot\big)$ has the value $I_\Phi\big(\mathscr{V}(F, \cdot), E\big)$ for each $E \in \Sigma_0$). Consequently, $I_\Phi(F, \cdot) \in V^1(R)$.*

Proof. For notational convenience the set function $\mathscr{V}(F, \cdot)$ will be denoted by G. From lemma 3 (b) it follows that $I_\Phi(G, E)$ is finite for each $E \in \Sigma_0$. That $I_\Phi(G, \cdot)$ is additive on Σ_0 is assured by lemma I. 6. Clearly $I_\Phi(G, \cdot)$ is of bounded variation and vanishes on η-null sets. Hence $I_\Phi(G, \cdot) \in A^1(R)$. To show that $I_\Phi(G, \cdot) \in V^1(R)$, it must be shown that $I_\Phi(G, \cdot)$ is μ-continuous. By hypothesis, $F\cdot E \in S^1(\mathscr{X})$ for all $E \in \Sigma_0$. From lemma 3 (a), $G\cdot E \in V^1(R)$ for all $E \in \Sigma_0$. Hence by a theorem of Bochner and Phillips [6],

$$\lim_{n\to\infty} (G \wedge n\mu)\cdot E = \lim_n G\cdot E \wedge n\mu = G\cdot E$$

strongly in $V^1(R)$ for $E \in \Sigma_0$ where $G \wedge n\mu$ is the set function defined by

$$G \wedge n\mu(E) = \inf_{A \subset E}\{G(A) + \mu(E-A):\ A \in \Sigma\}.$$

The lower semicontinuity of $I_\Phi(\cdot)$ implies

$$I_\Phi(G, E) \leqslant \liminf_n I_\Phi(G \wedge n\Phi, E);$$

on the other hand, the monotonicity of Φ insures $I_\Phi(G, E) \geqslant I_\Phi(G \wedge n\mu, E)$ for each $E \in \Sigma_0$. Hence

$$I_\Phi(G) = \lim_n I_\Phi(G \wedge n\mu)$$

and if $\pi = \{E_n\}$ is any partition of Ω and $A = \bigcup_n E_n$, then

$$\sum_\pi |I_\Phi(G, E_n) - I_\Phi(G \wedge n\mu, E_n)|$$

$$= \sum_\pi \big(I_\Phi(G, E_n) - I_\Phi(G \wedge n\mu, E_n)\big)$$

$$= \sum_\pi I_\Phi(G, E_n) - \sum_\pi I_\Phi(G \wedge n\mu, E_n)$$

$$\leqslant I_\Phi(G, A) - I_\Phi(G \wedge n\mu, A) + I_\Phi(G, A') - I_\Phi(G \wedge n\mu, A')$$

$$= I_\Phi(G) - I_\Phi(G \wedge n\mu).$$

Since the partition π is arbitrary,

$$N_1\big(I_\Phi(G, \cdot) - I_\Phi(G \wedge n\mu, \cdot)\big) \leqslant I_\Phi(G) - I_\Phi(G \wedge n\mu),$$

and

$$\lim_n I_\Phi(G \wedge n\mu, \cdot) = I_\Phi(G, \cdot)$$

strongly in $A^1(R)$. But $I_\Phi(G \wedge n\mu, E) \leqslant \Phi(n)\mu(E)$ for each $E \in \Sigma_0$. Hence $I_\Phi(G \wedge n\mu, \cdot)$ is μ-continuous. Therefore $I_\Phi(G, \cdot) \in V^1(R)$ because $V^1(R)$ is a closed subspace of $A^1(R)$. The last statement follows immediately from the above and lemma 3 (b), q. e. d.

Lemma 5. (a) *If $F \in V^\Phi(\mathscr{X})$, then for each choice of $\varepsilon > 0$, there exists a set $A \in \Sigma_0$ such that $I_\Phi(F, A') < \varepsilon$.*

(b) *If $F \in V^\Phi(\mathscr{X})$ and $F\cdot E \in S^1(\mathscr{X})$ for each E in Σ_0, then*

$$I_\Phi(F, \cdot) = \lim_\pi I_\Phi(F_\pi, \cdot)$$

strongly in $V^1(R)$.

Proof. (a) Let $\varepsilon > 0$ be given. According to lemma I. 5, there exists a partition $\pi = \{E_n\}$ such that

$$0 \leqslant I_\Phi(F) - I_\Phi(F_\pi) < \varepsilon.$$

Let $A = \bigcup_n E_n$. Then $A \in \Sigma_0$ and $I_\Phi(F_\pi) \leqslant I_\Phi(F, A) \leqslant I_\Phi(F)$. Whence

$$I_\Phi(F, A') = I_\Phi(F) - I_\Phi(F, A) \leqslant I_\Phi(F) - I_\Phi(F_\pi) < \varepsilon.$$

This proves (a).

(b) First note that if $E \in \Sigma_0$ and $\mu(E) \neq 0$, then

$$\Phi\left(\frac{\|F(E)\|}{\mu(E)}\right)\mu(E) \leqslant I_\Phi(F, E);$$

hence

$$\Phi\left(\frac{\|F(E)\|}{\mu(E)}\right) \leqslant \frac{I_\Phi(F, E)}{\mu(E)}.$$

In view of this inequality, if $\pi = \{E_n\}$ is any partition, one has

$$I_\Phi(F_\pi, \cdot) = \sum_\pi \Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu\cdot E_n \leqslant \sum_\pi \frac{I_\Phi(F, E_n)}{\mu(E_n)}\mu\cdot E_n = I_\Phi(F, \cdot)_\pi;$$

so that

$$I_\Phi(F_\pi, \cdot) \leqslant I_\Phi(F, \cdot)_\pi$$

for all partitions π. Now for any partition π,

$$\begin{aligned} N_1\big(I_\Phi(F, \cdot) - I_\Phi(F_\pi, \cdot)\big) &\leqslant N_1\big(I_\Phi(F, \cdot) - I_\Phi(F, \cdot)_\pi\big) + N_1\big(I_\Phi(F, \cdot)_\pi - I_\Phi(F_\pi, \cdot)\big) \\ &= N_1\big(I_\Phi(F, \cdot) - I_\Phi(F, \cdot)_\pi\big) + I_\Phi(F, \cdot)_\pi(\Omega) - I_\Phi(F_\pi, \cdot)(\Omega), \end{aligned}$$

since $I_\Phi(F_\pi, E) \leqslant I_\Phi(F, E)_\pi$ for all $E \in \Sigma_0$ and the variation of a non-negative set function is its value on the whole set Ω. But

$$I_\Phi(F, \cdot)_\pi(\Omega) = I_\Phi(F, \Omega) = I_\Phi(F),$$

and

$$I_\Phi(F_\pi, \cdot)(\Omega) = I_\Phi(F_\pi, \Omega) = I_\Phi(F_\pi).$$

Hence

$$\lim_\pi\big(I_\Phi(F) - I_\Phi(F_\pi)\big) \leqslant \lim_\pi N_1\big(I_\Phi(F, \cdot) - I_\Phi(F, \cdot)_\pi\big) + 0$$

by theorem I. 9. Now let $\varepsilon > 0$ be given. By part (a), there exists on $A \in \Sigma_0$ such that $I_\Phi(F, A') < \varepsilon$. For any partition π refining the trivial partition $\{A\}$, one has

$$\begin{aligned} N_1\big(I_\Phi(F, \cdot) - I_\Phi(F, \cdot)_\pi\big) &\leqslant N_1\big(I_\Phi(F, \cdot)\cdot A - I_\Phi(F, \cdot)_\pi\cdot A\big) + N_1\big(I_\Phi(F, \cdot)\cdot A'\big) + N_1\big(I_\Phi(F, \cdot)_\pi\cdot A\big) \\ &\leqslant N_1\big(I_\Phi(F, \cdot)\cdot A - I_\Phi(F, \cdot)_\pi\cdot A\big) + 2N_1\big(I_\Phi(F, \cdot)\cdot A'\big), \\ &= N_1\big(I_\Phi(F, \cdot)\cdot A - I_\Phi(F, \cdot)_\pi\cdot A\big) + 2I_\Phi(F, A') \\ &\leqslant N_1\big(I_\Phi(F, \cdot)\cdot A - I_\Phi(F, \cdot)_\pi\cdot A\big) + \varepsilon \end{aligned}$$

by the choice of A.

Now $I_\Phi(F, \cdot)\cdot A \in V^1(R)$ by lemma 4. Since $\mu(A) < \infty$, corollary III. 5 implies

$$\lim_\pi N_1\big(I_\Phi(F, \cdot)\cdot A - I_\Phi(F, \cdot)_\pi\cdot A\big) = 0.$$

Since $\varepsilon > 0$ is arbitrary, this implies

$$\lim_\pi N_1\big(I_\Phi(F, \cdot) - I_\Phi(F, \cdot)_\pi\big) = 0,$$

which, in view of the above inequalities, implies

$$\lim_\pi N_1\big(I_\Phi(F, \cdot) - I_\Phi(F_\pi, \cdot)\big) = 0, \quad \text{q. e. d.}$$

The principal results of this section are the following theorem and its corollaries.

THEOREM 6. *If $F \in V^\Phi(\mathfrak{X})$ satisfies* (i) $I_\Phi(aF) < \infty$ *for all scalars a, and* (ii) $F\cdot E \in S^1(\mathfrak{X})$ *for all $E \in \Sigma_0$, then*

$$\lim_\pi N_\Phi(F - F_\pi) = 0.$$

Proof. First it will be shown $\lim_\pi I_\Phi(F - F_\pi) = 0$. Let $\varepsilon > 0$ be given. According to lemma 5(a), there exists a set $A \in \Sigma_0$ such that $I_\Phi(2F, A') < \varepsilon$. Now if $\{A\} \leqslant \pi_1 \leqslant \pi_2$, then

$$I_\Phi(F_{\pi_2} - F_{\pi_1}, A') = I_\Phi\big(\mathscr{V}(F_{\pi_2} - F_{\pi_1}\cdot), A'\big),$$

by lemma 4,

$$\begin{aligned} &\leqslant I_\Phi\big(\mathscr{V}(F_{\pi_2}, \cdot) + \mathscr{V}(F_{\pi_1}\cdot \cdot), A'\big) \\ &\leqslant I_\Phi\big(2\,\mathscr{V}(F, \cdot), A'\big) \quad \text{by monotonicity of } \Phi, \\ &= I_\Phi(2F, A'), \quad \text{by lemma 4}, \\ &< \varepsilon \quad \text{by the choice of } A. \end{aligned}$$

Hence

$$I_\Phi(F - F_{\pi_1}, A') \leqslant \liminf_{\pi_2} I_\Phi(F_{\pi_2} - F_{\pi_1}, A') < \varepsilon.$$

Therefore, it can and will be assumed that $\mu(\Omega) < \infty$.

Let $\delta > 0$ be given and fix $\gamma > 0$ such that $\Phi(\gamma)\mu(\Omega) < \delta/2$. For each partition π, form the μ-simple function

$$f_\pi = \sum_\pi \frac{F(E_n)}{\mu(E_n)}\chi_{E_n}.$$

Consider

$$\int_\Omega \|f_{\pi_1} - f_{\pi_2}\|\,d\mu = N_1(F_{\pi_1} - F_{\pi_2}).$$

The facts $F\cdot E\in S^1(\mathscr{X})$ for every $E\in\Sigma_0$ and $\mu(\Omega)<\infty$ imply $F\in S^1(\mathscr{X})$. Consequently, from theorem 2, it follows that

$$0=\lim_{\pi_1,\pi_2} N_1(F_{\pi_1}-F_{\pi_2})=\lim_{\pi_1,\pi_2}\int_\Omega \|f_{\pi_1}-f_{\pi_2}\|\,d\mu.$$

If $E_{\pi_1,\pi_2}=\{\omega: \|f_{\pi_1}(\omega)-f_{\pi_2}(\omega)\|\geqslant\gamma\}$, then $E_{\pi_1,\pi_2}\in\Sigma$ and

$$0=\lim_{\pi_1,\pi_2}\int_\Omega\|f_{\pi_1}-f_{\pi_2}\|\,d\mu\geqslant\lim_{\pi_1,\pi_2}\int_{E_{\pi_1,\pi_2}}\|f_{\pi_1}-f_{\pi_2}\|\,d\mu\geqslant\lim_{\pi_1,\pi_2}\gamma\mu(E_{\pi_1,\pi_2})\geqslant 0.$$

Whence $\lim_{\pi_1,\pi_2}\mu(E_{\pi_1,\pi_2})=0$.

Now consider for $\pi_1\leqslant\pi_2$

$$\begin{aligned}I_\Phi(F_{\pi_2}-F_{\pi_1})&=I_\Phi(F_{\pi_2}-F_{\pi_1},E'_{\pi_1,\pi_2})+I_\Phi(F_{\pi_2}-F_{\pi_1},E_{\pi_1,\pi_2})\\&\leqslant\Phi(\gamma)\mu(E'_{\pi_1,\pi_2})+I_\Phi(F_{\pi_2}-F_{\pi_2},E_{\pi_1,\pi_2})\\&\leqslant\Phi(\gamma)\mu(\Omega)+I_\Phi\big(\mathscr{V}(F_{\pi_2}-F_{\pi_1}),E_{\pi_1,\pi_2}\big)\end{aligned}$$

by lemma 3(b),

$$\begin{aligned}&\leqslant\delta/2+I_\Phi\big(\mathscr{V}(F_{\pi_2},\cdot)+\mathscr{V}(F_{\pi_1},\cdot),E_{\pi_1,\pi_2}\big)\\&\leqslant\delta/2+I_\Phi\big(2\mathscr{V}(F),E_{\pi_1,\pi_2}\big),\quad\text{by monotonicity of }\Phi,\\&=\delta/2+I_\Phi(2F,E_{\pi_1,\pi_2}),\quad\text{lemma 3(b)}.\end{aligned}$$

But $\lim_{\pi_1,\pi_2}\mu(E_{\pi_1,\pi_2})=0$. By the μ-continuity of $I_\Phi(2F,\cdot)$ (lemma 4) there exists a partition π_0 such that

$$I_\Phi(2F,E_{\pi_1,\pi_2})<\delta/2\quad\text{for}\quad\pi_2\geqslant\pi_1\geqslant\pi_0.$$

Whence

$$I_\Phi(F_{\pi_2}-F_{\pi_1})<\delta/2+\delta/2=\delta$$

for $\pi_2\geqslant\pi_1\geqslant\pi_0$. This combined with the lower semi-continuity (theorem I. 7) of $I_\Phi(\cdot)$ yields

$$I_\Phi(F-F_{\pi_1})\leqslant\liminf_{\pi_2}I_\Phi(F_{\pi_2}-F_{\pi_1})\leqslant\delta\quad\text{for}\quad\pi_1\geqslant\pi_0.$$

Since $\delta>0$ is arbitrary, it follows that $\lim_\pi I_\Phi(F-F_\pi)=0$. To complete the proof of the theorem, note that if F satisfies the hypothesis of the theorem, so does F/k for any $k>0$. From the above analysis for each $k>0$, there exists π_k such that

$$I_\Phi\left(\frac{F-F_\pi}{k}\right)=I_\Phi\left(\frac{F}{k}-\frac{F_\pi}{k}\right)\leqslant 1$$

for all $\pi\geqslant\pi_k$. From the definition of $N_\Phi(\cdot)$, one has

$$N_\Phi(F-F_\pi)\leqslant k$$

for $\pi\geqslant\pi_k$. Since $k>0$ is arbitrary, it follows that

$$\lim_\pi N_\Phi(F-F_\pi)=0,\qquad\text{q. e. d.}$$

Combining the results of section 1 with those of this section given the following result which will play a major role in the presentation theorems of section V.

Corollary 7. *If Φ obeys the Δ_2-condition and $\mathscr{X}$ is reflexive, then $S^\Phi(\mathscr{X})=V^\Phi(\mathscr{X})$.*

Proof. Let $F\in V^\Phi(\mathscr{X})$. From lemma I. 13 and corollary III. 4 it follows that $F\cdot E\in S^1(\mathscr{X})$ for all $E\in\Sigma_0$. Since Φ obeys the Δ_2-condition $I_\Phi(aF)<\infty$ for all scalars a. Hence $F\in S^\Phi(\mathscr{X})$, q. e. d.

Applying the preceding results to the $L^\Phi(\mathscr{X})$-spaces, one has the following theorem:

Theorem 8. *If Φ obeys the Δ_2-condition, then μ-simple functions are dense in $L^\Phi(\mathscr{X})$.*

Proof. Let $f\in L^\Phi(\mathscr{X})$ and λ be the isometric injection of $L^\Phi(\mathscr{X})$ into $V^\Phi(\mathscr{X})$ of theorem II. 5. Since μ-simple functions are dense in $L^1(\mathscr{X})$ by [9], III. 3.8, it follows that $\lambda f\cdot E\in S^1(\mathscr{X})$ for each $E\in\Sigma_0$. The Δ_2-condition ensures that $I_\Phi(a\lambda f)$ is finite for all scalars a. Hence, by theorem 6,

$$\lim_\pi N_\Phi\big(\lambda f-(\lambda f)_\pi\big)=0.$$

Since $\lambda^{-1}(\lambda f)_\pi$ is a μ-simple function and λ is an isometry, the conclusion immediately follows, q. e. d.

Remark. If $\Phi(x)=|x|^p$, $1\leqslant p<\infty$, this result was proved in [9], III. 3.8. Generalizing that method, one can also obtain an independent proof for this theorem.

V. Linear operators on $V^\Phi(\mathscr{X})$

In this section, the general bounded linear operator from $S^\Phi(\mathscr{X})$ to an arbitrary B-space $\mathscr{Y}$ is characterized in terms of a certain "integral" representation involving operator-valued set functions. Then it is shown that the space of such operators is isometrically isomorphic to a space of operator-valued set functions $W^\Psi\big(B(\mathscr{X},\mathscr{Y})\big)$ which is related to a $V^\Psi(\mathscr{X}^*)$-space in a natural way. As corollaries specializations of these results are given for operators on $L^\Phi(\mathscr{X})$.

The problem of representing bounded linear operators on $L^p(\mathscr{X})$, $1\leqslant p<\infty$, to an arbitrary B-space has been a recurring one. Although some special cases have been resolved [4, 7, 16, 17], only partial results have been obtained in the case of the general bounded linear operator on $L^\Phi(\mathscr{X})$. One of the papers cited above, the paper of Bochner and Taylor

[7] allows itself to be abstracted to the $V^{\Phi}(\mathscr{X})$ setting. It is this paper and [13] which motivate the considerations of this section.

Let Ω, Σ, Σ_0, and μ be as in section I. $\mathscr{X}$ and $\mathscr{Y}$ are B-spaces with conjugate spaces $\mathscr{X}^*$ and $\mathscr{Y}^*$ respectively. $B(\mathscr{X},\mathscr{Y})$ is the space of bounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$ with uniform norm. Φ is a continuous Young's function. The first lemma is concerned with associating an operator-valued set function with a bounded linear operator on $S^{\Phi}(\mathscr{X})$.

LEMMA 1. *Let $h\in B\big(S^{\Phi}(\mathscr{X}),\mathscr{Y}\big)$. Then there exists a unique finitely additive $B(\mathscr{X},\mathscr{Y})$-valued μ-continuous set function H, defined on Σ_0 and satisfying*

$$\text{(A)}\qquad h(F)=\lim_{\pi}\sum_{\pi}\frac{H(E_n)[F(E_n)]}{\mu(E_n)}$$

for all $F\in S^{\Phi}(\mathscr{X})$, where the limit is taken in the Moore-Smith sense through all partitions π of Ω.

Proof. If $x_1, x_2\in\mathscr{X}$ are arbitrary and a, b are any scalars, then $h\big((ax_1+bx_2)\mu\cdot E\big)=h(ax_1\mu\cdot E)+h(bx_2\mu\cdot E)=ah(x_1\mu\cdot E)+bh(x_2\mu\cdot E)$ for any $E\in\Sigma_0$. Hence the set function H defined for each E in Σ_0 by

$$H(E)=h(\cdot\mu\cdot E)$$

is a linear operator from $\mathscr{X}$ to $\mathscr{Y}$ satisfying $H(E)[x]=h(x\mu\cdot E)$ for all $x\in\mathscr{X}$ and each $E\in\Sigma_0$.

Now consider

$$\begin{aligned}\|h(E)[x]\|&=\|h(x\mu\cdot E\|\leqslant\|h\|N_{\Phi}(x\mu\cdot E)\\&=\|h\|\,\|x\|N_{\Phi}(\mu\cdot E)=\|h\|\,\|x\|\inf\Big\{K>0\colon\ I_{\Phi}\Big(\frac{\mu\cdot E}{K}\Big)\leqslant 1\Big\}\\&=\|h\|\,\|x\|\inf\Big\{K>0\colon\ \Phi\Big(\frac{1}{K}\Big)\mu(E)\leqslant 1\Big\}\end{aligned}$$

for any $x\in\mathscr{X}$, $E\in\Sigma_0$. This shows first that

$$H(E)\in B(\mathscr{X},\mathscr{Y})\quad\text{for each}\quad E\in\Sigma_0,$$

and second that

$$\begin{aligned}&\lim_{\mu(E)\to 0}\|H(E)\|_{B(\mathscr{X},\mathscr{Y})}\\&\leqslant\|h\|\lim_{\mu(E)\to 0}\inf\Big\{K>0\colon\ \Phi\Big(\frac{1}{K}\Big)\mu(E)\leqslant 1\Big\}=0,\end{aligned}$$

since $\Phi(x)<\infty$ for $x<\infty$. To show H is finitely additive, let $E_1, E_2\in\Sigma_0$, $E_1\cap E_2=\emptyset$. Then

$$\begin{aligned}H(E_1\cup E_2)[x]&=h\big(x\mu\cdot(E_1\cup E_2)\big)\\&=h\big(x(\mu\cdot E_1+\mu\cdot E_2)\big)=h(x\mu\cdot E_1)+h(x\mu\cdot E_2)\\&=H(E_1)[x]+H(E_2)[x]=H(E_1)+H(E_2)[x],\end{aligned}$$

for all $x\in\mathscr{X}$. Therefore H is finitely additive.

To establish representation (A), let $F\in S^{\Phi}(\mathscr{X})$. Then $\lim_{\pi}N_{\Phi}(F-F_{\pi})=0$ by theorem IV.2. The continuity of h implies

$$\lim_{\pi}\|h(F)-h(F_{\pi})\|\leqslant\|h\|\lim_{\pi}N_{\Phi}(F-F_{\pi})=0.$$

Thus $h(F)=\lim_{\pi}h(F_{\pi})$. Now, the linearity of h and the definition of H yield

$$\begin{aligned}h(F_{\pi})&=h\Big(\sum_{\pi}\frac{F(E_n)}{\mu(E_n)}\mu\cdot E_n\Big)\\&=\sum_{\pi}\frac{h\big(F(E_n)\mu\cdot E_n\big)}{\mu(E_n)}=\sum_{\pi}\frac{H(E_n)[F(E_n)]}{\mu(E_n)}\end{aligned}$$

for any partition π of Ω. Therefore

$$h(F)=\lim_{\pi}h(F_{\pi})=\lim_{\pi}\sum_{\pi}\frac{H(E_n)[F(E_n)]}{\mu(E_n)}.$$

The uniqueness is simple. For, if

$$h(F)=\lim_{\pi}\sum_{\pi}\frac{H'(E_n)[F(E_n)]}{\mu(E_n)}$$

for all $F\in S^{\Phi}(\mathscr{X})$, then by the definition of H,

$$H(E)[x]=h(x\mu\cdot E)=\lim_{\pi}\sum_{\pi}\frac{H'(E_n)[x\mu(E_n\cap RE)]}{\mu(E_n)}=H'(E)[x],$$

for each $x\in\mathscr{X}$ and $E\in\Sigma_0$. Hence $H=H'$, q. e. d.

Now that a representation of members of $B\big(S^{\Phi}(\mathscr{X}),\mathscr{Y}\big)$ in terms of operator-valued set functions has been established, one immediate question is this: what space of operator-valued set functions has a topology compatible with the uniform operator topology of $B\big(S^{\Phi}(\mathscr{X}),\mathscr{Y}\big)$? The following is devoted to defining a space of set functions $W^{\Psi}\big(B(\mathscr{X},\mathscr{Y})\big)$ which is then shown to be isometrically isomorphic to $B\big(S^{\Phi}(\mathscr{X}),\mathscr{Y}\big)$.

Definition 2. The space $W^{\Phi}(\Omega, \Sigma, \mu, B(\mathscr{X}, \mathscr{Y})) = W^{\Phi}(B(\mathscr{X}, \mathscr{Y}))$ consists of all finitely additive μ-continuous $B(\mathscr{X}, \mathscr{Y})$-valued set functions H defined on Σ_0 which satisfy

(i) $y^*H \in V^{\Phi}(\mathscr{X}^*)$ for all $y^* \in Y^*$.

(ii) $\sup_{\|Y^*\| \leqslant 1} N_{\Phi}(y^*H) < \infty$ where y^*H is the set function defined for $E \in \Sigma_0$, by

$$y^*H(E)[x] = y^*\big(H(E)[x]\big).$$

(Here Φ is an arbitrary Young's function).

The additivity of $H(\cdot)$ and the fact that $H(E) \in B(\mathscr{X}, \mathscr{Y})$ for $E \in \Sigma_0$ imply $y^*H(\cdot)$ is a finitely additive $\mathscr{X}^*$-valued set function on Σ_0; so (i) and (ii) are meaningful.

Theorem 3. *The functional $\|\cdot\|_{W^{\Phi}}$ defined for $H \in W^{\Phi(B(\mathscr{X},\mathscr{Y}))}$ by $\|H\|_{W^{\Phi}} = \sup\{N_{\Phi}(y^*H) : y^* \in Y^*, \|y^*\| \leqslant 1\}$ is a norm, and $W^{\Phi}(B(\mathscr{X}, \mathscr{Y}))$ is a normed linear space.*

Proof. If $H = 0$, evidently $\|H\|_{W^{\Phi}} = 0$. On the other hand, if $H \neq 0$, then there exists $E \in \Sigma_0$ and $x \in \mathscr{X}$ such that $H(E)[x] \neq 0$. By a consequence of the Hahn-Banach theorem, there exists y_0^* in the unit ball of Y^* such that $y_0^*H(E)[x] \neq 0$. Consequently $y_0^*H \neq 0$ and $N_{\Phi}(y_0^*H) > 0$. Therefore $\|H\|_{W^{\Phi}} > 0$.

The other two properties of a norm follow directly from the corresponding properties of N_{Φ}, q. e. d.

A proof of the completeness of $W^{\Phi}(B(\mathscr{X}, \mathscr{Y}))$ can be given in a way which is only a slight modification of the completeness proof for $A^{\Phi}(\mathscr{X})$, but the completeness of the $W^{\Phi}(B(\mathscr{X}, \mathscr{Y}))$ spaces of interest here is a consequence of the main theorem of this section. The next two lemmas are preliminary to this result.

Lemma 4. *If $F \in V^{\Phi}(\mathscr{X})$, then $\|F\|_{\Phi} \leqslant I_{\Phi}(F)+1$.*

Proof. If Ψ is complementary to Φ, by definition,

$$\|F\|_{\Phi} = \sup_{N_{\Psi}(G)\leqslant 1} \sup_{\pi} \sum_{\pi} \frac{\|F(E_n)\|\,\|G(E_n)\|}{\mu(E_n)}$$

$$\leqslant \sup_{N_{\Psi}(G)\leqslant 1}\left(\sup_{\pi}\sum_{\pi}\Phi\left(\frac{\|F(E_n)\|}{\mu(E_n)}\right)\mu(E_n) + \sup_{\pi}\sum_{\pi}\Psi\left(\frac{\|G(E_n)\|}{\mu(E_n)}\right)\mu(E_n)\right)$$

$$= \sup_{N_{\Psi}(G)\leqslant 1}\big(I_{\Phi}(F)+I_{\Psi}(G)\big) \leqslant I_{\Phi}(F)+1, \quad \text{q. e. d.}$$

Lemma 5. *If $x^* \in \mathscr{X}^*$ and $\varepsilon > 0$ is arbitrary, there exists $x \in \mathscr{X}$ such that $x^*(x) = \|x^*\|$ and $\big|\|x\|-1\big| < \varepsilon$.*

Proof. This immediately follows from the definition of $\|x^*\|$ and the linearity of x^*, q. e. d.

The principal result of this section is

Theorem 7. *Let Φ be a continuous Young's function and Ψ be its complementary function. Then*

(a) *To each $h \in B(S^{\Phi}(\mathscr{X}), \mathscr{Y})$ there corresponds a unique H in $W^{\Psi}(B(\mathscr{X}, \mathscr{Y}))$ such that*

$$h(F) = \lim_{\pi} \sum_{\pi} \frac{H(E_n)[F(E_n)]}{\mu(E_n)} \tag{B}$$

for all $F \in S^{\Phi}(\mathscr{X})$.

(b) *The correspondence $h \to H$ maps $B(S^{\Phi}(\mathscr{X}), \mathscr{Y})$ linearly onto $W^{\Psi}(B(\mathscr{X}, \mathscr{Y}))$, and if $S^{\Phi}(\mathscr{X})$ is normed with $\|\cdot\|_{\Phi}$, then*

$$\|h\|_{B(S^{\Phi}(\mathscr{X}),\mathscr{Y})} = \|H\|_{W^{\Psi}(B(\mathscr{X}),\mathscr{Y})}.$$

Consequently, $B(S^{\Phi}(\mathscr{X}), \mathscr{Y})$ and $W^{\Psi}(B(\mathscr{X}, \mathscr{Y}))$ are isometrically isomorphic.

Proof. (a) Let the norm of S^{Φ} be the Orlicz norm $\|\cdot\|_{\Phi}$ and $h \in B(S^{\Phi}(\mathscr{X}), \mathscr{Y})$. If $h = 0$, then $H = 0 \in W^{\Psi}(B(\mathscr{X}, \mathscr{Y}))$ satisfies (a). So let $h \neq 0$. Lemma 1 establishes the existence of a unique μ-continuous finitely additive $B(\mathscr{X}, \mathscr{Y})$-valued set function H which satisfies (B). To complete the proof of (a), it remains to show that $H \in W^{\Psi}(B(\mathscr{X}, \mathscr{Y}))$.

First assume that Ψ is continuous (i.e., $\Psi(x) < \infty$ for $x < \infty$). Let $\varepsilon > 0$ be given and y^* in the unit ball of $\mathscr{Y}^*$ be fixed but arbitrary. For an arbitrary partition $\pi = \{E_n\}$ of Ω, form the $\mathscr{X}$-valued step function $G = \sum_{\pi} x_n \mu \cdot E_n$ where the $x_n \in \mathscr{X}$ are subject to:

(i) $x_n = z_n \psi\left(\dfrac{\|y^*H(E_n)\|}{\|h\|\,\mu(E_n)}\right)$, $\psi = \Psi'$,

(ii) The $z_n \in \mathscr{X}$ satisfy $y^*H(E_n)(z_n) = \|y^*H(E_n)\|$,

(iii) $\left|\Phi\left(\|z_n\|\,\psi\left(\dfrac{\|y^*H(E_n)\|}{\|h\|\,\mu(E_n)}\right)\right) - \Phi\left(\psi\left(\dfrac{\|y^*H(E_n)\|}{\|h\|\,\mu(E_n)}\right)\right)\right| < \dfrac{\varepsilon}{m\mu(E_n)}$,

where m is the number of sets in π. The existence of such z_n's is ensured by lemma 5 and the continuity of Φ. According to lemma 4,

$$1+I_{\Phi}(G) \geqslant \|G\|_{\Phi} \geqslant \frac{y^*h(G)}{\|h\|}, \quad \text{since } \|y^*\| \leqslant 1,$$

$$= \frac{1}{\|h\|} y^*h\Big(\sum_{\pi} x_n \mu \cdot E_n\Big) = \frac{1}{\|h\|}\sum_{\pi} y^*h(x_n\mu \cdot E_n)$$

$$= \frac{1}{\|h\|}\sum_{\pi} y^*H(E_n)[x_n], \quad \text{by the definition of } H \text{ in lemma 1,}$$

$$= \frac{1}{\|h\|}\sum_{\pi} y^*H(E_n)[z_n]\,\psi\left(\frac{\|y^*H(E_n)\|}{\|h\|\,\mu(E_n)}\right)$$

by the choice of the x_n's,

$$= \sum_\pi \left(\frac{\|y^* H(E_n)\|}{\|h\|} \psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \right)$$

by the choice of the z_n's,

$$= \sum_\pi \frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \mu(E_n).$$

Hence by equality in Young's inequality, we have $I_\Phi(G)+1$

$$\text{(C)} \qquad \sum_\pi \Phi \left(\psi \frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \mu(E_n) + \sum_\pi \Psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \mu(E_n).$$

But G is a step function on π. According to theorem I.1.9,

$$I_\Phi(G) = \sum_\pi \Phi \left(\frac{\|G(E_n)\|}{\mu(E_n)} \right) \mu(E_n) = \sum_\pi \Phi \left(\|z_n\| \psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \right) \mu(E_n)$$

by the definition of G. Substituting this into (C) yields

$$1 + \sum_\pi \Phi \left(\|z_n\| \psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \right) \mu(E_n)$$

$$\geqslant \sum_\pi \Phi \left(\psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_v)} \right) \right) \mu(E_n) + \sum_\pi \Psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \mu(E_n).$$

Hence

$$1 + \sum_\pi \left| \Phi \left(\|z_n\| \psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \right) - \Phi \left(\psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \right) \right| \mu(E_n)$$

$$\geqslant \sum_\pi \Psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \mu(E_n).$$

From the choice of the z_n's in (iii), it follows that

$$\sum_\pi \Psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \mu(E_n) \leqslant 1 + \varepsilon.$$

Since the partition π is arbitrary,

$$I_\Psi \left(\frac{y^* H}{\|h\|} \right) = \sup_\pi \sum_\pi \Psi \left(\frac{\|y^* H(E_n)\|}{\|h\| \mu(E_n)} \right) \mu(E_n) \leqslant 1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $I_\Psi(y^* H/\|h\|) \leqslant 1$. Therefore $y^* H \in V^\Psi(\mathscr{X}^*)$ and

$$N_\Psi(y^* H) \leqslant \|h\|.$$

Since y^* was arbitrary in the unit ball of y^*, it follows that $H \in W^\Psi(B(\mathscr{X}, \mathscr{Y}))$ and

$$\text{(D)} \qquad \|H\|_{W^\Psi} = \sup_{\|y^*\| \leqslant 1} N_\Psi(y^* H) \leqslant \|h\|.$$

This proves (a) in the case that Ψ is continuous. If Ψ jumps at $M > 0$, the same proof will be applicable provided it is shown that

$$\Psi \frac{\|y^* H(E)\|}{\|h\| \mu(E)} \leqslant M$$

for any $E \in \Sigma_0$ and y^* in the unit ball of y^*. Let E and y^* be fixed but arbitrary subject to these conditions. If $\mu(E) = 0$, then $H(E) = 0$, and the quotient above becomes $0/0 = 0$ by convention. If $\mu(E) > 0$ and $x \in \mathscr{X}$ is arbitrary, then

$$|y^* H(E)[x]| = |y^* h[x\mu \cdot E]| \leqslant \|y^* h\| \, \|x\mu \cdot E\|_\Phi.$$

Now

$$\|x\mu \cdot E\|_\Phi = \sup_{N_\Psi(G) \leqslant 1} \sup_\pi \sum_\pi \frac{\|x\mu(E \cap E_n)\|}{\mu(E_n)} \|G(E_n)\|.$$

In order that $N_\Psi(G) \leqslant 1$, it must be true that $\|G(E)\|/\mu(E) \leqslant M$ for all $E \in \Sigma_0$ (otherwise $I_\Psi(G/K) = \infty$ for $K \leqslant 1$). Hence

$$\|x\mu \cdot E\|_\Phi \leqslant \sup_\pi \sum_\pi \|x\mu(E \cap E_n)\| \, M$$

$$= \sup_\pi \sum_\pi \|x\| \mu(E \cap E_n) \, M = M \|x\| \mu(E).$$

Therefore

$$|y^* H(E)[x]| \leqslant \|y^* h\| \, M \, \|x\| \mu(E)$$

$$\leqslant \|y^*\| \, \|h\| \, M \, \|x\| \mu(E) \leqslant \|h\| \, M \, \|x\| \mu(E),$$

since $\|y^*\| \leqslant 1$. Whence

$$\|y^* H(E)\| \leqslant \sup_{\|x\| \leqslant 1} \|h\| \, M \, \|x\| \mu(E) = \|h\| \, M \mu(E).$$

Accordingly, $\|y^* H(E)\|/\|h\| \mu(E) \leqslant M$, which completes the proof of (a).

(b) To show the correspondence $h \to H$ is onto $W^\Psi(B(\mathscr{X}, \mathscr{Y}))$, let $H \in W^\Psi(B(\mathscr{X}, \mathscr{Y}))$ be arbitrary. An $h \in B(S^\Phi(\mathscr{X}), \mathscr{Y})$ will be defined such that it corresponds to H. Define h on all step functions $F_\pi \in S^\Phi(\mathscr{X})$ by

$$h(F_\pi) = \sum_\pi \frac{H(E_n)[F(E_n)]}{\mu(E_n)}.$$

It is easy to see that h is linear on the step functions. Moreover, if $y^* \epsilon y^*$ and $\pi = \{E_n\}$ is a partition of Ω, then

$$|y^* h(F_\pi)| = \left| y^* \sum_\pi \frac{H(E_n)[F(E_n)]}{\mu(E_n)} \right| = \left| \sum_\pi \frac{y^* H(E_n)[F(E_n)]}{\mu(E_n)} \right|$$
$$\leqslant \sum_\pi \frac{\|y^* H(E_n)\| \, \|F(E_n)\|}{\mu(E_n)} \leqslant N_\Psi(y^* H) \|F\|_\Phi$$

by theorem I. 1.21. (a). Hence

$$\|h(F_\pi)\|_y = \sup_{\|y^*\| \leqslant 1} |y^* h(F_\pi)| \leqslant \sup_{\|y^*\| \leqslant 1} N(y^* H) \|F\|_\Phi = \|H\|_{w^\Psi} \|F\|_\Phi,$$

and h is bounded. Now,

$$\|h(F_{\pi_1}) - h(F_{\pi_2})\| = \|h(F_{\pi_1} - F_{\pi_2})\| \leqslant \|H\|_W \|F_{\pi_1} - F_{\pi_2}\|_\Phi$$
$$\leqslant \|H\|_{w^\Phi} (\|F_{\pi_1} - F\|_\Phi + \|F - F_{\pi_2}\|_\Phi).$$

Therefore by IV. 3 and I. 19 (b)

$$\lim_{\pi_1, \pi_2} \|h(F_{\pi_1}) - h(F_{\pi_2})\| = 0.$$

Thus $\lim_\pi h(F_\pi)$ exists strongly in $\mathscr{Y}$, and we *define* $h(F)$ for $F \in S^\Phi(\mathscr{X})$ by

$$h(F) = \lim_\pi \sum_\pi \frac{H(E_n)[F(E_n)]}{\mu(E_n)} \quad \left(= \lim_\pi h(F_\pi)\right).$$

Computations which are the same as those above show

$$\text{(E)} \qquad \|h(F)\| \leqslant \|H\|_{W^\Psi} \|F\|_\Phi.$$

Hence $h \in B(S^\Phi(\mathscr{X}), \mathscr{Y})$. Clearly $H(E) = h(\cdot \mu \cdot E)$. Therefore h is mapped into H by the correspondence of (a).

Now, the correspondence $h \to H$ from $B(S^\Phi(\mathscr{X}), \mathscr{Y})$ onto $W^\Psi(B(\mathscr{X}, \mathscr{Y}))$ is obviously linear. Moreover, if $h \to H$, then (D) yields $\|H\|_{W^\Psi} \leqslant \|h\|$ and (E) yields $\|h\| \leqslant \|H\|_{W^\Psi}$. Thus $\|h\| = \|H_{W^\Psi}\|$, q. e. d.

An immediate consequence of theorem 6 is that the conjugate space to $S^\Phi(\mathscr{X})$ is $V^\Psi(\mathscr{X}^*)$ since $W^\Psi(B(\mathscr{X}, \mathscr{Y}))$ collapses to $V^\Psi(B(\mathscr{X}, \mathscr{Y}))$ in the case where $\mathscr{Y} = \mathscr{C}$, the scalars.

Corollary 7. *Let Φ obey the Δ_2-condition and $\mathscr{X}$ be reflexive. Then theorem 6 remains true if $S^\Phi(\mathscr{X})$ is replaced by $V^\Phi(\mathscr{X})$.*

Proof. Under these hypotheses $S^\Phi(\mathscr{X}) = V^\Phi(\mathscr{Y})$, q. e. d.

Corollary 8. *If Φ and its complementary functions Ψ each obey the Δ_2-condition, and $\mathscr{X}$ is reflexive, $V^\Phi(\mathscr{X})$ is reflexive.*

Proof. Two applications of corollary 7 yield the result, q. e. d.

Applying these results to the $L^\Phi(\mathscr{X})$ spaces, one has the following corollary:

Corollary 9. *Let Φ be a continuous Young's function and Ψ be its complementary function, and μ have FSP, then*

(a) *to each $h \in B(M^\Phi(\mathscr{X}), \mathscr{Y})$, where $M^\Phi(\mathscr{X})$ is the closed subspace of $L^\Phi(\mathscr{X})$ determined by μ-simple functions, there corresponds a unique $H \in W^\Psi(B(\mathscr{X}, \mathscr{Y}))$ such that*

$$h(f) = \lim_\pi \sum_\pi \frac{H(E_n)\left[\int_{E_n} f d\mu\right]}{\mu(E_n)}$$

for all $f \in M^\Phi(\mathscr{X})$.

(b) *The correspondence $h \to H$ maps $B(M^\Phi(\mathscr{X}), \mathscr{Y})$ linearly onto $W^\Psi(B(\mathscr{X}, \mathscr{Y}))$ and if M^Φ is normed with $\|\cdot\|_\Phi$, then*

$$\|h\|_{B(M^\Phi(\mathscr{X}), \mathscr{Y})} = \|H\|_{W^\Psi(B(\mathscr{X}, \mathscr{Y}))}.$$

Consequently $B(M^\Phi(\mathscr{X}), \mathscr{Y})$ and $W^\Psi(B(\mathscr{X}, \mathscr{Y}))$ are isometrically isomorphic.

(c) *If Φ obeys the Δ_2-condition, $M^\Phi(\mathscr{X})$ may be replaced by $L^\Phi(\mathscr{X})$ in* (a) *and* (b).

Proof. From the definitions of $M^\Phi(\mathscr{X})$ and $S^\Phi(\mathscr{X})$, λ, the isometric isomorphism of theorem II. 5, maps $M^\Phi(\mathscr{X})$ onto a dense subset of $S^\Phi(\mathscr{X})$ isometrically by theorem I. 6. The result immediately follows from theorem 6, corollary 7, and theorem III.9.

Remark 1. A specialization of this result shows that $(M^\Phi(\mathscr{X}))^*$ is isometrically isomorphic to $V^\Psi(\mathscr{X}^*)$ and when Φ obeys the Δ_2-condition $(L^\Phi(\mathscr{X}))^*$ is equivalent to $V^\Psi(\mathscr{X}^*)$.

Remark 2. Employing the general bilinear vector integral of Bartle [3], we see that the representation

$$h(f) = \lim_\pi \sum_\pi \frac{H(E_n)\left[\int_{E_n} f d\mu\right]}{\mu(E_n)}$$

takes the form $h(f) = \int_\Omega f dH$.

The referee has kindly brought the author's attention to the interesting papers of Albrycht and Orlicz [1, 2] which contain certain results related to the earlier part of this work. Unfortunately, the author was not aware of these papers at the time of the preparation of this work. However the work of [1, 2] and the present paper largely complement each other.

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The spectrum of an infinite product measure

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The infinite product of certain probability measures on compact abelian groups was discussed by Varopoulus [4]; the measures he considered are included in the present description. We are mainly interested in showing how the orthogonality criterion of Kakutani [1] may be used in place of the almost-everywhere-convergence calculations of [4]. Besides this we give some elementary facts which aid in constructing examples in the harmonic analysis of measure algebras.

0. Let $G_1, G_2, G_3, \ldots$ be compact abelian groups $\neq 0$, e_n the unit measure at 0 in G_n, m_n the normalized Haar measure of G_n. Let $0 < a_n < 1$ for $1 \leqslant n$ and $\mu_n = a_n e_n + (1-a_n) m_n$. Finally,

$$G = \prod_1^\infty G_n, \quad \mu = \prod_1^\infty \mu_n.$$

THEOREM 1. *The Fourier transform $\hat{\mu}$ vanishes at infinity in $\Gamma = \hat{G}$, if and only if $a_n \to 0$* (Varopoulos [4], p. 3806).

THEOREM 2. *The measure μ, as an element of the complex Banach algebra $M(G)$, has purely real spectrum if and only if*

(i) $\sum_{a_n > 1/2} (1-a_n) < \infty$.

(ii) *For some integer* $k \geqslant 1$, $\sum_{a_n \leqslant 1/2} a_n^k < \infty$.

1. The proofs are divided into one paragraph for the first theorem, and two for the second.

Proof of Theorem 1. It is well-known that a continuous character of G is composed in an obvious way from a *finite* number of characters $\gamma_1, \gamma_2, \ldots, \gamma_s$ on $G_1, G_2, \ldots, G_s$ respectively, and that $\hat{\mu}$ takes the value $\prod_1^s \hat{\mu}(\gamma_s)$ on the composite character. This degenerates to $\Pi' a_n$, Π' being the product over non-trivial components γ_n. It is clear that if

$$a < \limsup_{n\to\infty} |a_n| < b,$$