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A characterization of maximal ideals in commutative Banach algebras

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Let A be a commutative complex Banach algebra with unit element e. We give the following characterization of maximal ideals in A:

THEOREM 1. *A subspace $X \subset A$ of codimension 1 is a maximal ideal in A if and only if it consists of non-invertible elements.*

Clearly any maximal ideal satisfies the above condition, so it is sufficient to show that if codim $X = 1$, and if X consists of non-invertible elements, then X is a maximal ideal in A. In order to show this we shall reformulate our problem. Clearly X, as a subspace of codimension 1, is a zero set for some linear functional f. Since X contains no invertible elements, it cannot be dense in A, so f is continuous, and since $e \notin X$, we may take f in such a way that,

(1) $$f(e) = 1.$$

Such a functional f may be characterized by

(2) $$f(x) \in \sigma(x)$$

for every $x \in A$, where $\sigma(x)$ denotes the spectrum of x. In fact, if we have a functional f satisfying (2), then it also satisfies (1), and its zero set consists by (2), of non-invertible elements. On the other hand, if f satisfies (1), and its zero set consists of non-invertible elements, then for any $x \in A$ the element

$$y = x - f(x)e$$

is non-invertible in A, since $f(y) = 0$. Consequently

$$0 \in \sigma(x - f(x)e) = \sigma(x) - f(x)$$

and so (2) holds.

Thus theorem 1 is equivalent with the following

THEOREM 2 [1]. *Let A be a commutative complex Banach algebra with unit element. Then a functional $f \in A^*$ is a multiplicative linear functional if and only if* (2) *holds.*

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[1] This theorem is also true for non-commutative complex Banach algebras, cf. [4] (*added in proof*).

Proof. If f is a multiplicative and linear (non-zero) functional, then clearly (2) holds. Suppose now that (2) holds. We have then also (1). Let $x \in A$ and consider the element $\exp(\lambda x)$, where λ is a complex variable. We put

(3) $$\varphi(\lambda) = f[\exp(\lambda x)].$$

It may be easily verified that $\varphi(\lambda)$ is an entire function. Since, by (2), $\varphi(\lambda) \neq 0$, it may be written in the form

$$\varphi(\lambda) = \exp[\psi(\lambda)]$$

for some entire function $\psi(\lambda)$.

We have also the following estimation:

$$|\varphi(\lambda)| \leqslant \|f\| \exp(|\lambda|\, \|x\|);$$

this implies (cf. [2], p. 250) that $\psi(\lambda) = a\lambda + \beta$ for some complex a and β. Since, by (1), $\varphi(0) = f(\exp 0) = 1$, we have $\psi(\lambda) = a\lambda$ and so

(4) $$\varphi(\lambda) = \exp(a\lambda) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \lambda^n.$$

On the other hand, by formula (3),

(5) $$\varphi(\lambda) = f\left(\sum_{n=0}^{\infty} \frac{\lambda^n x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{f(x^n)}{n!} \lambda^n.$$

By comparising coefficients in expansion (4) and (5) we obtain

$$f(x^n) = a^n = f(x)^n$$

and, in particular,

(6) $$f(x^2) = f(x)^2$$

for any $x \in A$. We have then

$$f(xy) = f(\tfrac{1}{2}[(x+y)^2 - x^2 - y^2]) = \tfrac{1}{2}[f(x+y)^2 - f(x)^2 - f(y)^2]$$
$$= \tfrac{1}{2}([f(x)+f(y)]^2 - f(x)^2 - f(y)^2) = f(x)f(y),$$

which shows that f is a multiplicative and linear functional in A.

Remark. Theorem 2 is clearly false for real Banach algebras. E.g. for $A = C(0,1)$ the functional

$$f(x) = \int_0^1 x(t)\,dt$$

satisfies (2), but is non-multiplicative.

As a corollary we obtain a theorem on multiplicativity of measures possessing mean-value property.

THEOREM 3. *Let X be a compact Hausdorff space, let A be an algebra of complex-valued continuous functions defined on X, and let μ be a Radon measure on X such that for any function $x \in A$ we have*

(6) $$\int x\,d\mu = x(p_x),$$

where p_x is a point in X depending on the function x, and the integral is taken over the space X. Then the measure μ is multiplicative with respect to A, i.e.

(7) $$\int xy\,d\mu = \int x\,d\mu \int y\,d\mu$$

for every $x, y \in A$.

Proof. Let $\bar{A}$ denote the completion of A in the Sup-norm $\|\cdot\|$ on X. By formula (6) the functional

$$f(x) = \int x\,d\mu$$

is continuous with respect to the Sup-norm in A, and so it may be extended onto $\bar{A}$. The set $\bar{A}$ is clearly a Sup-norm algebra. We show that f satisfies on $\bar{A}$ condition (2). For any $x \in A$ formula (2) is satisfied by condition (6). If $x_n \to x$ in $\bar{A}$, $x_n \in A$, then $f(x_n) \to f(x)$, and $f(x_n) = x_n(p_n)$, so we have

$$|x(p_n) - f(x)| \leqslant |x(p_n) - x_n(p_n)| + |f(x_n) - f(x)| \to 0$$

which means, by the compactness of X, that there is a point $p \in X$ such that

$$x(p) = f(x).$$

So formula (2) holds. The conclusion is now a consequence of Theorem 2.

Remark. The above result is not obvious even in the case when X is the unit disc on the complex plane and A is the algebra of all functions continuous on the disc and holomorphic in its interior.

We prove now a generalization of a part of Theorem 2 replacing there the complex plane by an arbitrary commutative semi-simple Banach algebra with unit element.

THEOREM 4. *Let A_1 and A_2 be two commutative Banach algebras, with unit elements, and suppose that A_2 is semi-simple. If T is a linear mapping of A_1 into A_2 such that*

(8) $$\sigma(Tx) \subset \sigma(x)$$

for any $x \in A_1$, then it is a multiplicative mapping, i.e.

(9) $$Txy = TxTy$$

for every $x, y \in A_1$.

Proof. Let f be a multiplicative and linear functional on A_2 and put

$$F(x) = f(Tx)$$

for any $x \in A_1$. So F is a linear functional on A_1. We have also

$$F(x) = f(Tx) \in \sigma(Tx) \subset \sigma(x),$$

and so, by Theorem 2, F is a multiplicative and linear functional in the algebra A_1. It follows that

$$F(xy) = F(x)F(y),$$

or

(10) $$f(Txy) = f(Tx)f(Ty) = f(TxTy).$$

Formula (9) is now a consequence of the fact that (10) holds for any multiplicative and linear functional f on A_2, and that A_2 is semi-simple.

Remark. Theorem 4 does not remain true if A_2 fails to be semi-simple. In fact, if $A_1 = A_2 = A$ and if A is direct sum of a radical algebra and the complex plane, then any mapping T of A into A, satisfying

$$Te = e,$$

satisfies formula (8), but not necessarily formula (9).

The reciproque theorem of Theorem 4 is true under an additional condition that T sends the unit element of A_1 on the unit of A_2. We may omit, however, the assumption that A_2 is a semi-simple algebra.

THEOREM 5. *Let A_1, A_2 be commutative Banach algebras with unit elements denoted respectively by e_1 and e_2. Let T be a multiplicative linear mapping of A_1 into A_2, i.e. a linear mapping satisfying condition* (9). *Then, for any $x \in A_1$, relation* (8) *is satisfied provided that*

(10) $$Te_1 = e_2.$$

Proof. We have

$$e_2 = Te_1 = Txx^{-1} = TxTx^{-1}$$

for any element x which is invertible in A_1. This implies that for any such x the element Tx is invertible in A_2 and

(11) $$T(x^{-1}) = (Tx)^{-1}.$$

If $\lambda \notin \sigma(x)$, then $x - \lambda e_1$ is invertible in A_1 and so is the element $T(x-\lambda e_1) = Tx - \lambda e_2$ in A_2. Therefore $\lambda \notin \sigma(Tx)$ and formula (8) holds.

Remark. The assumption that $Te_1 = e_2$ is essential here. For, if we take the natural imbedding

$$T: A_1 \to A_1 \oplus A_2,$$

then it clearly satisfies (9), but neither (8) nor (10). In this case, however, we have

(12) $$\sigma(x) \subset \sigma(Tx).$$

But we can produce also an example non satisfying this condition. If A is the algebra of all continuous functions defined on the unit disc, holomorphic in its interior, then the mapping $T: A \to A \oplus A$, given by the formula

$$x(t) \to [x(2t), 0],$$

is clearly a multiplicative mapping satisfying neither (8) nor (12).

Concerning generalizations, it is obvious that the results given here are also true for complete locally bounded, or p-normed algebras (for the definition cf. e.g. [3]).

Theorem 2 is also true for complete multiplicatively convex locally convex algebras, provided the functional f is continuous. It is, however, false without an assumption of m-convexity. In paper [1], p. 301 (cf. remark 3.6), there is given an example of a commutative, locally convex, complete metric algebra A, consisting of entire functions of one complex variable, such that only invertible elements of A are scalar multiples of the unit element. Any functional in this algebra which satisfies condition (1) satisfies also condition (2), but it must not be a multiplicative functional.

References

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