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Some remarks on (p, q)-absolutely summing operators in l_p-spaces

by

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A linear operator $A: X \to Y$ (X, Y Banach spaces) is said to be *(p, q)-absolutely summing*, $1 \leqslant p, q < +\infty$, if there is a constant M such that for every finite set in X, $\{x_1, x_2, \ldots, x_n\}$, the inequality

$$\Big(\sum_{i=1}^{n} \|Ax_i\|^p\Big)^{1/p} \leqslant M \sup_{x^* \in K^*} \Big(\sum_{i=1}^{n} |x'(x_i)|^q\Big)^{1/q}$$

holds, where K^* is the unit ball of X^* — the dual of X.

The definition of a (p, q)-absolutely summing operator is due to Pełczyński and Mitjagin [7] and it generalizes earlier concepts of various authors. In [3] Grothendieck introduced "semi-intégrale à gauche" operators. They are exactly $(1, 1)$-absolutely summing operators. In [13] Saphar considered "Hilbert-Schmidt à gauche" operators in Banach spaces, which are $(2, 2)$-absolutely summing operators. In [10] Pietsch defined "absolut p-summierende Abbildungen", which are exactly (p, p)-absolutely summing according to our definition. In this paper we shall deal with (p, q)-absolutely summing operators in l_p-spaces. But all the results obtained here can be generalized to spaces of $\mathscr{L}_p$-type (for the definition see [8]). In the first part it is proved that every linear operator from l_1 to l_p is $\big(2p/(2p-|p-2|), 1\big)$-absolutely summing. In the second part we study (p, q)-absolutely summing operators in a Hilbert space.

0. Preliminaries. Let $(x_i)_{i \in I}$ be a finite family in a Banach space X, and K^* the unit ball of X^*. Let us put:

$$l^r(x_i, X) = \begin{cases} \Big(\sum_{i \in I} \|x_i\|^r\Big)^{1/r} & \text{if} \quad 1 \leqslant r < +\infty, \\ \sup_{i \in I} \|x_i\| & \text{if} \quad r = +\infty; \end{cases}$$

$$l^r[x_i, X] = \begin{cases} \sup_{x^* \in K^*} \Big(\sum_{i \in I} |x^*(x_i)|^r\Big)^{1/r} & \text{if} \quad 1 \leqslant r < +\infty, \\ \sup_{i \in I} \|x_i\| & \text{if} \quad r = +\infty. \end{cases}$$

By R^1 we denote the real line. If $1 < s < +\infty$, then $s^* = s(s-1)^{-1}$; if $s = 1$ (resp. $s = +\infty$), then $s^* = +\infty$ (resp. $s^* = 1$). If in a statement the braces { } are used, then this means that both assertions which are obtained by replacing in this statement the braces either by square brackets or by the round ones are true.

The following facts are simple consequences of Hölder's inequality:

(0.1) $\quad l^r\{x_i\xi_i, X\} \leqslant l^{r\cdot s}\{x_i, X\}\cdot l^{r\cdot s^*}(\xi_i, R^1), \quad 1 \leqslant r, s \leqslant +\infty;$

(0.2) $\quad l^s\{x_i, X\} = \sup\limits_{l^{s^*}(\xi_i,R^1)\leqslant 1} l^1\{x_i\xi_i, X\} = \sup\limits_{l^1(\xi_i,R^1)\leqslant 1} l^1\{x_i|\xi_i|^{1/s}, X\};$

(0.3) $\quad l^r\{x_i, X\} \leqslant l^s\{x_i, X\} \quad \textit{for} \quad r \geqslant s.$

Definition. A linear operator $A: X\to Y$ is *(p, q)-absolutely summing* if there is a constant M such that for every finite family $(x_i)_{i\in I}$ in X the inequality

$$l^p(Ax_i, Y) \leqslant Ml^q[x_i, X]$$

holds. Let $\pi_{p,q}(A)$ denote the least such constant M. By $\Pi_{p,q}(X, Y)$ we mean the set of all operators from $B(X, Y)$ for which $\pi_{p,q}(A) < +\infty$. In the same way as in [11] it can be proved that $\Pi_{p,q}(X, Y)$ is a Banach space with the norm $\pi_{p,q}(A)$.

(0.4) *If $1 \leqslant q \leqslant +\infty$, then $\Pi_{\infty,q}(X, Y) = B(X, Y)$ and $\|A\| = \pi_{\infty,q}(A)$ for every A in $\Pi_{\infty,q}(X, Y)$.*

(0.5) *If $p < q$, then $\Pi_{p,q}(X, Y) = \{0\}$.*

(0.6) *If $r \leqslant p$, $s \geqslant q$, then $\Pi_{p,q}(X, Y) \supset \Pi_{r,s}(X, Y)$ and for every A in $\Pi_{p,q}(X, Y)$, $\pi_{r,s}(A) \geqslant \pi_{p,q}(A)$.*

(0.7) *If $q \leqslant s$ and $1/p-1/r = 1/q-1/s$, then $\Pi_{p,q}(X, Y) \subset \Pi_{r,s}(X, Y)$ and for every A in $\Pi_{p,q}(X, Y)$, $\pi_{p,q}(A) \geqslant \pi_{r,s}(A)$.*

The proof of (0.4) and (0.5) is easy and it is omitted; (0.6) is a simple consequence of (0.3). So it remains to prove (0.7). Let $A\in\Pi_{p,q}(X, Y)$. By (0.1) and (0.2) we have for a finite family $(x_i)_{i\in I}$ in X and $A\in\Pi_{p,q}(X, Y)$

$$\begin{aligned} l^r(Ax_i, Y) &= \sup_{l^1(\xi_i,R^1)\leqslant 1} l^1(Ax_i|\xi_i|^{1/r^*}, Y) \\ &\leqslant \sup_{l^1(\xi_i,R^1)\leqslant 1} l^p(Ax_i|\xi_i|^{1/r^*-1/p^*}, Y)\cdot l^{p^*}(|\xi_i|^{1/p^*}, R^1) \\ &\leqslant \sup_{l^1(\xi_i,R^1)\leqslant 1} l^p(Ax_i|\xi_i|^{1/q-1/s}, Y) \end{aligned}$$

(because $\sup\limits_{l^1(\xi_i,R^1)\leqslant 1} l^{p^*}(|\xi_i|^{1/p^*}, R^1) \leqslant 1$ and $1/r^*-1/p^* = 1/p-1/r = 1/q-1/s$).

But

$$\begin{aligned} \sup_{l^1(\xi_i,R^1)\leqslant 1} l^p(Ax_i|\xi_i|^{1/q-1/s}, Y) &\leqslant \sup_{l^1(\xi_i,R^1)\leqslant 1} \pi_{p,q}(A)\cdot l^q[x_i|\xi_i|^{1/q-1/s}, X] \\ &\leqslant \pi_{p,q}(A) l^s[x_i, X] \sup_{l^1(\xi_i,R^1)\leqslant 1} l^{q(s/q)^*}(|\xi_i|^{1/q-1/s}, R^1) \leqslant \pi_{p,q}(A) l^s[x_i, X] \end{aligned}$$

(because $q(s/q)^* = sq/(s-q)$ and

$$\sup_{l^1(\xi_i,R^1)\leqslant 1} l^{sq/(s-q)}(|\xi_i|^{(s-q)/sq}, R^1) \leqslant 1).$$

Thus for every finite family $(x_i)_{i\in I}$ in X

$$l^r(Ax_i, Y) \leqslant \pi_{p,q}(A) l^s[x_i, X]$$

and this implies (0.7).

(0.8) *If $A\in B(X_1, X)$, $B\in\Pi_{p,q}(X, Y)$, $C\in B(Y, Y_1)$, then $CBA \in\Pi_{p,q}(X_1, Y_1)$ and $\pi_{p,q}(CBA) \leqslant \pi_{p,q}(B)\|A\|\,\|C\|$.*

The proof is an immediate consequence of the definition. (0.8) implies that if $X = Y$, then $\Pi_{p,q}(X, Y) = \Pi_{p,q}(X, X)$ is a two-sided ideal in the algebra $B(X, X)$.

1. (p, q)-absolutely summing operators in $B(l_1, l_p)$ and $B(l_\infty, l_p)$. Grothendieck [4] (see also Lindenstrauss-Pełczyński [8] for an elementary treatment) proved that every operator from l_1 into l_2 is a "semi-intégrale à gauche". In our language this means that if $A\in B(l_1, l_2)$, then

(G) $$l^1(Ax_i, l_2) \leqslant \mathscr{G}\|A\| l^1[x_i, l_1],$$

where $\mathscr{G}$ is a universal constant independent of A and of the family $(x_i)_{i\in I}$.

Orlicz theorem [9] states that if $\sum\limits_{n=1}^{\infty} x_n$ is an unconditionally convergent series in l_1, then

$$\sum_{n=1}^{\infty}\|x_n\|^2 < +\infty.$$

This implies that the identity operator in l_1 is (2.1)-absolutely summing. Combining this with (0.8) we conclude that for every $A\in B(l_1, l_2)$ (resp. $A\in B(l_1, l_\infty)$)

(O) $$l^2(Ax_i, l_1) \leqslant \mathscr{O}\|A\| l^1[x_i, l_1], \quad (\text{resp. } l^2(Ax_i, l_\infty) \leqslant \mathscr{O}\|A\| l^1[x_i, l_1]),$$

where $\mathscr{O}$ is a constant independent of A and of the family $(x_i)_{i\in I}$.

The following theorem is a generalization of these results to the case of l_p-spaces, $1 \leqslant p \leqslant +\infty$:

THEOREM. *Let* $1 \leqslant p \leqslant +\infty$. *Then*

(1.1) $\quad B(l_1, l_p) = \Pi_{r(p),1}(l_1, l_p)$, *where* $\dfrac{1}{r(p)} = 1 - \left|\dfrac{1}{p} - \dfrac{1}{2}\right|$.

(1.2) $\quad$ *Moreover, if* $r < r(p)$, *then* $\Pi_{r,1}(l_1, l_p) \neq B(l_1, l_p)$.

Proof of (1.1). We restrict our attention to the case where $1 \leqslant p \leqslant 2$. The proof for $2 \leqslant p \leqslant +\infty$ is analogous.

Let $A \in B(l_1, l_p)$ and let $(x_i)_{i\in I}$, $x_i = (\omega_n^i)$, be a finite family in l_1. We need to prove the inequality

$$\boldsymbol{l}^{r(p)}(Ax_i, l_p) \leqslant M\boldsymbol{l}^1[x_i, l_1],$$

where M is a constant which does not depend on $(x_i)_{i\in I}$. Let, for $k = 1, 2, \ldots, A(e_k) = a_k$; $a_k = (a_n^k) \in l_p$ (e_k denotes throughout this paper the sequence (δ_n^k); $\delta_n^k = 0$ if $n \neq k$ and $\delta_n^k = 1$ otherwise).

Since $1/r^*(p) = 1/p - \frac{1}{2}$ (by (0.2)), we can write

$$\boldsymbol{l}^{r(p)}(Ax_i, l_p) = \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \boldsymbol{l}^1(Ax_i|\xi_i|^{1/p-1/2}, l_p)$$
$$= \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \sum_{i\in I} |\xi_i|^{1/p-1/2}\|Ax_i\|_p$$

($\|\ \|_p$ denotes the norm in l_p). Choose for $i \in I$, b_i such that

$$b_i = (\beta_n^i) \in l_{p^*} = l_p^*, \qquad \|b_i\|_{p^*} \leqslant 1,$$

$$\|Ax_i\|_p = \langle b_i, Ax_i\rangle = \sum_{k,n=1}^{\infty} \beta_n^i a_n^k \omega_n^i.$$

Hence

$$\boldsymbol{l}^{r(p)}(Ax_i, l_p) = \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \sum_{i\in I} \sum_{k,n=1}^{\infty} |\xi_i|^{1/p-1/2} \beta_n^i a_n^k \omega_n^i.$$

Let $(\xi_i)_{i\in I}$ be a family of real numbers with $\boldsymbol{l}^1(\xi_i, R^1) \leqslant 1$, and let N be a natural number. Consider the function

$$f(z) = \sum_{i\in I} \sum_{k,n=1}^{N} |\xi_i|^{z-1/2} \operatorname{sgn}\beta_n^i |\beta_n^i|^{p(1-z)/(p-1)} \operatorname{sgn} a_n^k |a_n^k|^{pz} \omega_k^i$$

in the strip $\frac{1}{2} \leqslant \operatorname{Re} z \leqslant 1$.

Putting $z = 1/p$ we have

$$f\left(\frac{1}{p}\right) = \sum_{i\in I} \sum_{k,n=1}^{N} |\xi_i|^{1/p-1/2} \beta_n^i a_n^k \omega_k^i.$$

Hence

$$\boldsymbol{l}^{r(p)}(Ax_i, l_p) \leqslant \sup_N \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} f(1/p).$$

The function $f(z)$ is analytic and bounded in the strip $\frac{1}{2} \leqslant \operatorname{Re} z \leqslant 1$. If $\operatorname{Re} z = \frac{1}{2}$ and $\operatorname{Im} z = t$, then

$$|f(z)| =$$
$$\left|\sum_{i\in I} \sum_{k,n=1}^{N} |\xi_i|^{t\sqrt{-1}} \operatorname{sgn}\beta_n^i |\beta_n^i|^{-\frac{p}{p-1} t\sqrt{-1}} |\beta_n^i|^{\frac{p}{(p-1)}\frac{1}{2}} \cdot \operatorname{sgn} a_n^k |a_n^k|^{tp\sqrt{-1}} |a_n^k|^{p/2} \cdot \omega_k^i\right|$$
$$\leqslant 2\left|\sum_{i\in I} \sum_{k,n=1}^{N} \varepsilon_n^i |\beta_n^i|^{p/2(p-1)} \eta_n^k |a_n^k|^{p/2} \omega_k^i\right|,$$

where ε_n^i and η_n^k are real numbers with $|\varepsilon_n^i| \leqslant 1$ and $|\eta_n^k| \leqslant 1$. Hence, by Schwartz inequality,

$$|f(z)| \leqslant 2\sum_{i\in I} \left|\sum_{n=1}^{N} \left(\sum_{k=1}^{N} \eta_n^k |a_n^k|^{p/2} \omega_k^i\right) (\varepsilon_n^i |\beta_n^i|^{p/2(p-1)})\right|$$
$$\leqslant 2\sum_{i\in I} \left[\sum_{n=1}^{N} \left(\sum_{k=1}^{N} \eta_n^k |a_n^k|^{p/2} \omega_k^i\right)^2\right]^{1/2} \cdot \left[\sum_{n=1}^{N} |\varepsilon_n^i|^2 |\beta_n^i|^{p/(p-1)}\right]^{1/2}$$
$$\leqslant 2\sum_{i\in I} \left[\sum_{n=1}^{N} \left(\sum_{k=1}^{N} \eta_n^k |a_n^k|^{p/2} \omega_k^i\right)^2\right]^{1/2}$$

(because $\|b_i\|_{p^*} \leqslant 1$, i.e. $\sum_{n=1}^{\infty} |\beta_n^i|^{p/(p-1)} \leqslant 1$).

Define an operator $C\colon l_1 \to l_2$ by putting $C(e_k) = c_k$, where $c_k = (\gamma_n^k)$, $\gamma_n^k = \eta_n^k |a_n^k|^{p/2}$ if $k, n \leqslant N$, and otherwise $\gamma_n^k = 0$. C is a well defined bounded linear operator, because

$$\|C\| = \sup_k \|c_k\|_2 \leqslant \sup_k \left(\sum_{n=1}^{\infty} |\eta_n^k|^2 |a_n^k|^p\right)^{1/2} \leqslant \sup_k \|a_k\|_p^{p/2} \leqslant \|A\|^{p/2}.$$

Now returning to the estimation of $f(z)$ for $\operatorname{Re} z = \frac{1}{2}$ we obtain

$$|f(z)| \leqslant 2\sum_{i\in I} \left[\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \gamma_n^k \omega_k^i\right)^2\right]^{1/2} = 2\sum_{i\in I} \|Cx_i\|_2 = 2\boldsymbol{l}^1(Cx_i, l_2).$$

By (G)

$$\boldsymbol{l}^1(Cx_i, l_2) \leqslant \mathscr{G}\|C\|\boldsymbol{l}^1[x_i, l_1] \leqslant \mathscr{G}\cdot\|A\|^{p/2}\boldsymbol{l}^1[x_i, l_1].$$

Thus

$$|f(z)| \leqslant 2\mathscr{G}\|A\|^{p/2}\boldsymbol{l}^1[x_i, l_1].$$

Using inequality (O) we can prove analogously that if $\operatorname{Re} z = 1$, then

$$|f(z)| \leqslant 2\mathscr{O}\|A\|^{p}\boldsymbol{l}^1[x_i, l_1].$$

Now, let us make use of the following lemma ([1], Chap. VI, § 10, Theorem 3).

LEMMA. *Let $f(z)$ be an analytic and bounded function in a strip $\alpha \leqslant \operatorname{Re} z \leqslant \beta$. If $|f(z)| \leqslant C$ on the straight lines $\operatorname{Re} z = \alpha$ and $\operatorname{Re} z = \beta$, then $|f(z)| \leqslant C$ on the whole strip.*

By this lemma we obtain:

$$\left|f\left(\frac{1}{p}\right)\right| \leqslant 2\max(\mathscr{G}\|A\|^{p/2}, \mathscr{C}\|A\|^{p})\boldsymbol{l}^1[x_i, l_1].$$

This together with

$$\boldsymbol{l}^{r(p)}(Ax_i, l_p) \leqslant \sup_N \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \left|f\left(\frac{1}{p}\right)\right|$$

implies the inequality

$$\boldsymbol{l}^{r(p)}(Ax_i, l_p) \leqslant 2\max(\mathscr{G}\|A\|^{p/2}, \mathscr{C}\|A\|^{p})\boldsymbol{l}^1[x_i, l_1],$$

i.e.

$$\pi_{r(p),1}(A) \leqslant 2\max(\mathscr{G}\|A\|^{p/2}, \mathscr{C}\|A\|^{p}).$$

This completes the proof of (1.1).

Proof of 1.2. At first we are going to prove the case $1 \leqslant p \leqslant 2$. The proof of this case is due to Pełczyński. Let $f_1, f_2, \ldots, f_{2^N}$ be the first 2^N functions of the orthonormal Walsh system on the interval $[0, 1]$ (see [6]). For $n = 1, 2, \ldots, 2^N$, $i = 1, 2, \ldots, 2^N$, $f_n(\cdot)$ is a constant ω_n^i on the interval $\big((i-1)/2^N, i/2^N\big)$ where $\omega_n^i = \pm 1$. Let us put $\omega_n^i = 0$ for $n > 2^N$ or $i > 2^N$. Let $x_i = (\omega_n^i)$ and let $I = \{1, 2, \ldots, 2^N\}$. If the injection operator of l_1 into l_p is $(r, 1)$-absolutely summing, then there is a constant M which does not depend on N such that

$$\boldsymbol{l}^r(x_i, l_p) \leqslant M\boldsymbol{l}^1[x_i, l_1].$$

But

$$\boldsymbol{l}^1[x_i, l_1] = \sup_{\|(\xi_n)\|_\infty\leqslant 1} \sum_{i=1}^{2^N}\Big|\sum_{n=1}^{2^N}\xi_n\omega_n^i\Big|$$
$$= \sup_{\|(\xi_n)\|_\infty\leqslant 1} 2^N\int_0^1\Big|\sum_{n=1}^{2^N}\xi_n f_n(t)\Big|\,dt$$
$$\leqslant \sup_{\|(\xi_n)\|_\infty\leqslant 1} 2^N\Big[\int_0^1\Big|\sum_{n=1}^{2^N}\xi_n f_n(t)\Big|^2 dt\Big]^{1/2}$$
$$= \sup_{\|(\xi_n)\|_\infty\leqslant 1} 2^N\Big[\sum_{n=1}^{2^N}\xi_n^2\Big]^{1/2} = 2^N\cdot 2^{N/2}.$$

On the other hand,

$$\boldsymbol{l}^r(x_i, l_p) = \Big[\sum_{i=1}^{2^N}\Big(\sum_{n=1}^{2^N}|\omega_n^i|^p\Big)^{r/p}\Big]^{1/r} = 2^{N/p}\cdot 2^{N/r}.$$

So we have $2^{N/p}\cdot 2^{N/r} \leqslant M2^N\cdot 2^{N/2}$ for every N, but this is impossible unless

$$\frac{1}{r}+\frac{1}{p}\leqslant 1+\frac{1}{2}, \quad \text{i.e.} \quad r \geqslant \frac{2p}{3p-2} = r(p).$$

Now let $2 \leqslant p \leqslant +\infty$. Define an operator $A_N: l_1 \to l_p$ by putting $A_N(e_k) = a_k$, where $a_k = (\omega_k^n)$ and ω_k^n are as before. It is not difficult to prove that if $B(l_1, l_p) = \Pi_{r,1}(l_1, l_p)$, then there is a constant M such that for every $A \in B(l_1, l_p)$

$$M\|A\| \geqslant \pi_{p,q}(A).$$

Thus for every N,

$$\boldsymbol{l}^r(A_N x_i, l_p) \leqslant M\|A_N\|\boldsymbol{l}^1[x_i, l_1].$$

But

$$\boldsymbol{l}^r(A_N x_i, l_p) = \Big(\sum_{i=1}^{2^N}\|A_N x_i\|_p^r\Big)^{1/r}$$
$$= \Big[\sum_{i=1}^{2^N}\Big(\sum_{n=1}^{2^N}\Big|\sum_{k=1}^{2^N}\omega_k^n\omega_k^i\Big|^p\Big)^{r/p}\Big]^{1/r}$$
$$\geqslant \Big[\sum_{i=1}^{2^N}\Big(\sum_{k=1}^{2^N}|\omega_k^i|^2\Big)^r\Big]^{1/r} = 2^{N/r}\cdot 2^N.$$

On the other hand, $\boldsymbol{l}^1[x_i, l_1] \leqslant 2^N\cdot 2^{N/2}$ and

$$\|A_N\| = \sup_k\|a_k\|_p = \sup_k\Big(\sum_{n=1}^{2^N}|\omega_k^n|^p\Big)^{1/p} = 2^{N/p}.$$

Hence $2^{N/r}\cdot 2^N \leqslant M\cdot 2^{N/p}\cdot 2^N\cdot 2^{N/2}$ for every N, but this is impossible unless $1/r \leqslant 1/p + \frac{1}{2}$, i.e.

$$r \geqslant \frac{2p}{p+2} = r(p).$$

Thus the proof of the theorem is completed.

Remarks. 1. Slightly improving the proof of (1.1), one can show that

$$\pi_{r(p),1}(A) \leqslant \mathscr{G}^{2/p}\cdot\mathscr{C}^{|2/p-1|}\|A\|.$$

2. For the special case of $p = \frac{4}{3}$ and the injection operator, (1.1) was proved by Littlewood [5].

3. The following result is proved in [8], p. 289 and p. 320:

THEOREM. *If $1 \leqslant p \leqslant +\infty$, then $B(l_\infty, l_p) = \Pi_{r(p),2}(l_\infty, l_p)$, where $r(p) = \max(2, p)$.*

2. (p, q)**-absolutely summing operators in a Hilbert space.** Let $1 \leqslant r \leqslant +\infty$. By $\mathfrak{S}_r$ we mean the class of all operators $A \in B(l_2, l_2)$ which can be decomposed into a product $A = UBV$, where V is a unitary operator, U is an isometric operator on the image of B (i.e. $\|UBx\| = \|Bx\|$ for $x \in l_2$) and B is an operator for which $B\big((\xi_n)\big) = (|\lambda_n|^{1/r}\xi_n)$ and (λ_n) is a fixed sequence from l_1.

The next result is due to Mitjagin (unpublished) except 2.4, which is due to Pietsch [11] and Pełczyński [10].

THEOREM. *Let* $1 \leqslant q \leqslant p \leqslant +\infty$.

2.1. *If* $1/q - 1/p \geqslant \frac{1}{2}$ *or* $p = +\infty$, *then* $\Pi_{p,q}(l_2, l_2) = B(l_2, l_2)$.

2.2. *If* $q \leqslant 2$ *and* $1/q - 1/p < \frac{1}{2}$, *then* $\Pi_{p,q}(l_2, l_2) = \mathfrak{S}_r$ *where* $1/r = 1/p - 1/q + \frac{1}{2}$.

2.3. *If* $q \geqslant 2$, *then* $\Pi_{p,q}(l_2, l_2) \supset \mathfrak{S}_r$, *where* $1/r = q/2p$.

2.4. *If* $q = p$, *then* $\Pi_{p,q}(l_2, l_2) = \mathfrak{S}_2$.

Proof. By Orlicz theorem the identity operator in l_2 is $(2, 1)$-absolutely summing. Hence by (0.8) all operators from $B(l_2, l_2)$ are $(2, 1)$-absolutely summing. Thus to obtain (2.1), it is enough to apply (0.4), (0.6) and (0.7). Now we shall prove the inclusion $\Pi_{p,q}(l_2, l_2) \supset \mathfrak{S}_r$ in item 2.2 and simultaneously in item 2.3. Let $A \in \mathfrak{S}_r$, $1/r = 1/p - 1/q + \frac{1}{2}$ (resp. $1/r = q/2p$ in item 2.3) and let $A = UBV$ be a decomposition of A as in the definition of $\mathfrak{S}_r$. By (0.8) it is enough to prove that $B \in \Pi_{p,q}(l_2, l_2)$. Consider operators $B_s\colon l_2 \to l_2$ $(0 \leqslant s \leqslant \frac{1}{2})$ defined in the following way: $B_s\big((\xi_n)\big) = (|\lambda_n|^{1/s}\xi_n)$; $B_{1/2}$ is an operator of the Hilbert-Schmidt type. But every Hilbert-Schmidt operator is (1,1)-absolutely summing (see [12], p. 42); this, by (0.7), implies that $B_{1/2}$ is (q, q)-absolutely summing. By 2.1, already proved, B_0 is $\big(2q/(2-q), q\big)$-absolutely summing (resp. (∞, q)-absolutely summing). Thus there is a constant M such that for every finite family $(x_i)_{i\in I}$ in l_2 the inequalities

$$\boldsymbol{l}^q(B_{1/2}x_i, l_2) \leqslant M\boldsymbol{l}^q[x_i, l_2],$$
$$\boldsymbol{l}^p(B_0 x_i, l_2) \leqslant M\boldsymbol{l}^q[x_i, l_2]$$

hold, where $p = 2q/(2-q)$ (resp. $p = +\infty$).

Let $(x_i)_{i\in I}$ be a finite family in l_2, $x_i = (\alpha_n^i)$. Let us choose a family $(y_i)_{i\in I}$ such that $y_i = (\beta_n^i)$, $\|y_i\| \leqslant 1$,

$$\|Bx_i\| = \|B_{1/r}x_i\| = \langle B_{1/r}x_i, y_i\rangle = \sum_{n=1}^{\infty} |\alpha_n^i|\,|\beta_n^i|\,|\lambda_n|^{1/r}$$

If $1/p = 1/q + 1/r - \frac{1}{2}$ (resp. $1/p = 2/qr$), then

$$\boldsymbol{l}^p(Bx_i, l_2) = \sup_{\boldsymbol{l}^1(\xi_i, R^1) \leqslant 1} \boldsymbol{l}^1(B_{1/r}x_i|\xi_i|^{1/p^*}, l_2)$$
$$= \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \sum_{i\in I} \|B_{1/r}x_i\|\,|\xi_i|^{1/p^*} = \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \sum_{i\in I}\sum_{n=1}^{\infty} |\xi_i|^{1/p^*}|\lambda_n|^{1/r}|\alpha_n^i|\,|\beta_n^i|.$$

Consider the function

$$f(s) = \sum_{i\in I}\sum_{n=1}^{N} |\xi_i|^{3/2-1/q-s}|\lambda_n|^s|\alpha_n^i|\,|\beta_n^i|$$

for $0 \leqslant s \leqslant \frac{1}{2}$ (resp. $f(s) = \sum_{i\in I}\sum_{n=1}^{N}|\xi_i|^{1-2/q\cdot s}|\lambda_n|^s|\alpha_n^i|\,|\beta_n^i|$) where $(\xi_i)_{i\in I}$ is a family of real numbers with $\boldsymbol{l}^1(\xi_i, R^1) \leqslant 1$, N is a natural number and (λ_n), (α_n^i), (β_n^i) are as before. For $s = 0$, and $s = \frac{1}{2}$ (by Hölder inequality)

$$f(0) \leqslant \sum_{i\in I}|\xi_i|^{3/2-1/q}\|B_0x_i\| \leqslant \boldsymbol{l}^{2q/(2-q)}(B_0x_i, l_2) \leqslant M\boldsymbol{l}^q[x_i, l_2]$$

(resp. $f(0) \leqslant \sum_{i\in I}|\xi_i|\,\|B_0x_i\| \leqslant \boldsymbol{l}^\infty(B_0x_i, l_2) \leqslant M\boldsymbol{l}^q[x_i, l_2]$),

$$f(\tfrac{1}{2}) \leqslant \sum_{i\in I}|\xi_i|^{1-1/q}\|B_{1/2}x_i\| \leqslant \boldsymbol{l}_q(B_{1/2}x_i, l_2) \leqslant M\boldsymbol{l}^q[x_i, l_2].$$

Now the following lemma will be useful:

LEMMA. *The function*

$$f(s) = \sum_{k=1}^{k}|\varepsilon_i|\,|\eta_i|^s, \qquad -\infty < s < \infty$$

$(\varepsilon_i, \eta_i$ *are real numbers) is convex; thus if* $a \leqslant b$, *then*

$$\sup_{a\leqslant s\leqslant b} f(s) = \max\big(f(a), f(b)\big).$$

By this lemma we easily conclude that

$$f\left(\frac{1}{r}\right) \leqslant \max\big(f(0), f(\tfrac{1}{2})\big) \leqslant M\boldsymbol{l}^q[x_i, l_2].$$

But

$$\sup_N \sup_{\boldsymbol{l}^1(\xi_i R^1)\leqslant 1} f\left(\frac{1}{r}\right) \geqslant \boldsymbol{l}^p(Bx_i, l_2).$$

Thus $\boldsymbol{l}^p(Bx_i, l_2) \leqslant M\boldsymbol{l}^q[x_i, l_2]$, i.e. $B \in \Pi_{p,q}(l_2, l_2)$. This proves 2.3 and one part of 2.2. To complete the proof of 2.2, it remains to show the inclusion $\mathfrak{S}_r \supset \Pi_{p,q}(l_2, l_2)$ for $q \leqslant 2$ and $1/r = 1/p - 1/q + \frac{1}{2} > 0$. It is easy to verify that the identity operator is not (p, q)-absolutely summing in this case. Since $\Pi_{p,q}(l_2, l_2)$ is a two-sided ideal, Calkin's theorem (see [2], p. 89) implies that every (p, q)-absolutely summing operator is compact. But if A is compact, then $A = UBV$, where V is a unitary operator, U is an isometric operator on the image of B and B is an operator for which $B\big((\xi_n)\big) = (\lambda_n\xi_n)$ and $(\lambda_n) \in c_0$ (see [2]). Since A is (p, q)-absolutely summing (resp. $A \in \mathfrak{S}_r$) is and only if B is (p, q)-absolutely summing (resp. $B \in \mathfrak{S}_r$), it is enough to prove that if $B \in \Pi_{p,q}(l_2, l_2)$, $1/r = 1/p - 1/q + \frac{1}{2}$ and $q \leqslant 2$, then

$$\Big(\sum_{i=1}^{\infty}|\lambda_i|^r\Big)^{1/r} < +\infty.$$

So let $B \in \Pi_{p,q}(l_2, l_2)$, $I = \{1, 2, \ldots, N\}$. Then (e_i denotes as before the i-th unit vector)

$$\Big(\sum_{i=1}^{N} |\lambda_i|^r\Big)^{1/r} = \boldsymbol{l}^r(\lambda_i, R^1)$$

$$= \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \boldsymbol{l}^1(\lambda_i|\xi_i|^{1/r^*}, R^1) \leqslant \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \boldsymbol{l}^p(\lambda_i|\xi_i|^{1/r^*-1/p^*}, R^1)\boldsymbol{l}^{p^*}(|\xi_i|^{1/p^*}, R^1)$$

$$\leqslant \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \boldsymbol{l}^p(Be_i|\xi_i|^{1/q-1/2}, l_2) \leqslant \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \pi_{p,q}(B)\boldsymbol{l}^q[e_i|\xi_i|^{1/q-1/2}, l_2]$$

$$\leqslant \pi_{p,q}(B) \sup_{\boldsymbol{l}^1(\xi_i, R^1)\leqslant 1} \boldsymbol{l}^2[e_i, l_2]\boldsymbol{l}^{q\cdot(2/q)^*}(|\xi_i|^{1/q-1/2}, R^1) \leqslant \pi_{p,q}(B).$$

We have used the equalities

$$\frac{1}{r^*} - \frac{1}{p^*} = \frac{1}{q} - \frac{1}{2}, \quad q\left(\frac{2}{q}\right)^* = \frac{2q}{q-2}, \quad \boldsymbol{l}^2[e_i, l_2] = 1.$$

Thus

$$\Big(\sum_{i=1}^{N} |\lambda_i|^r\Big)^{1/r} \leqslant \pi_{p,q}(B);$$

his implies that $(\lambda_n) \in l_r$, whence $B \in \mathfrak{S}_r$.

2.4 was proved by Pełczyński [10] and Pietsch [11]. This completes the proof of the theorem.

CONJECTURE. *If* $2 < q < p < +\infty$, *then* $\Pi_{p,q}(l_2, l_2) = \mathfrak{S}_{2p/q}$.

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