2.9. Lemma. *If* $x_0^* \in \mathfrak{R}(X)$, *then*

$$\pi(x_0^*) = \inf\{M > 0: \bigwedge_{x \in X} |x_0^*(x^2)| \leqslant M|x_0^*(x)|\,\|x\|\} \leqslant \|2e_1 - x_0^*(e_1^2)e\| \cdot \|x_0^*\|$$

We shall omit the proofs of lemmas.

Proof of Theorem 2.7. Let $x_n^* \in \mathfrak{R}(X)$ and $x_n^* \underset{w}{\to} x_0^*$; then

$$x_0^*(e) = \lim_{n\to\infty} x_n^*(e) = 0 \quad \text{and} \quad x_0^*(e_1) = \lim_{n\to\infty} x_n^*(e_1).$$

From Lemmas 2.8 and 2.9 we have

$$|x_n^*(x^2)| \leqslant \pi(x_n^*)\,|x_n^*(x)|\,\|x\| \leqslant \|2e_1 - x_n^*(e_1^2)e\|\,\|x_n^*\|\,\|x\|\,|x_n^*(x)|$$
$$\leqslant \|2e_1 - x_n^*(e_1^2)e\| \cdot M \cdot \|x\|\,|x_n^*(x)|$$

Hence from the last inequality we obtain (as $n \to \infty$)

$$|x_0^*(x^2)| \leqslant \|2e_1 - x_0^*(e_1^2)e\| \cdot M\,|x_0^*(x)|\,\|x\| \leqslant M^*\,|x_0^*(x)|\,\|x\|.$$

Consequently, if $x_0^*(x) = 0$, then $x_0^*(x^2) = 0$ too, which completes the proof.

2.10. Corollary. *We have*

$$\overline{\left\{x^* \in X^*:\ x^*(x) = x^*(e_1)\frac{g_1(x) - g_2(x)}{g_1(e_1) - g_2(e_1)},\ g_1, g_2 \in \mathfrak{M}(X),\ g_1 \neq g_2\right\}} \subset \mathfrak{R}(X),$$

where the line on the left-hand side denotes weak sequential closure.

Now we can reformulate Problem 1 as follows:

Problem 1′. Does the equality

$$\mathfrak{R}(X) = \overline{\left\{x^* \in X^*:\ x^*(x) = \frac{g_1(x) - g_2(x)}{g_1(e_1) - g_2(e_1)},\ g_1, g_2 \in \mathfrak{M}(X),\quad g_1 \neq g_2\right\}}$$

hold?

Problem 1 (or the equivalent Problem 1′) can be generalized in the following manner.

Let X be an arbitrary B-algebra with unit. Let

$$U(X) = \{x^* \in R(X):\ x^* = c(g_1 - g_2),\ g_1, g_2 \in \mathfrak{M}(X),\ g_1 \neq g_2,\ c \neq 0\}$$

(evidently $U(X) = \left\{x_0^* \in \mathfrak{R}(X): \bigvee_{x_0 \in X} x_0^*(x_0) \neq 0,\ 3\left(\frac{x_0^*(x_0^2)}{2}\right)^2 \neq x_0^*(x_0)x_0^*(x_0^3)\right\}$).

Problem 2. Is it true that $\overline{U(X)} = \mathfrak{R}(X)$ (where $\overline{U(X)}$ denotes, as in the previous case, weak sequential closure $U(X)$)?

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A modern version of the E. Noether's theorems in the calculus of variations, I

by

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INTRODUCTION

A growing interest in problems connected with the Noether's theorems can be noticed in the last years. It is a result of a popularity of the Lagrange approach to physical theories.

In the period of time following the original paper of E. Noether [10] there were written many works developing its subject (Bessel-Hagen [2], Rosenfeld [11]) or treating about some mathematical foundations (de Donder [4]). Since 1950 we have had a lot of papers due as well to mathematicians as to physicists concerning those problems (Hill [9], Bergman and Schiller [1], Trautman [16], Fletcher [6], Schmutzer [12], Edelen [5], Funk [7], Steidel [13] and [14], Trautman [17] and [18], Demmig [3]).

In spite of the fact that the Lagrange formalism has a geometrical sense the authors use at every level of considerations a coordinate system for a description of geometrical objects. (One of such geometrical objects is a function, i.e. a scalar field, which being defined at the points of a space can not be considered as a function of their coordinates.) Such concept does not make easier to set off the gist of the matter and sometimes leads to misunderstandings.

As we will see, the notions of a differentiable manifold, a vector bundle and a jet-bundle are very useful in geometrical formulating of the variational problems (1).

A general variation of a functional defined on cross-sections of a finite-dimensional vector bundle M is considered, wherein the variation of the functional is induced by a variation of the cross-section, i.e. a one-parameter "smooth" family of cross-sections. It is easy to see that all such variations of a cross-section can be given by differentiable vector

(1) It is pointed out also in the recently appeared paper, *Noether equations and conservation laws* by A. Trautman in Commun. Math. Phys. 6 (1967), p. 248-261 (*added in proof*).

fields on the bundle M which preserve the fibre structure. Hence we are interested only in such vector fields. Our functional is so defined that it does not depend directly on cross-sections of the bundle M but on induced by them cross-sections of the jet-bundle $J(M)$.

We get an analogue of the Euler-Lagrange equations as a necessary and sufficient condition of invariance of the functional with respect to variations of a cross-section when these vanish on the boundary of the domain of a cross-section.

A goal of this paper is to formulate The First E. Noether's Theorem, i.e. to state a necessary and sufficient condition for the functional to remain invariant with respect to a given general variation and for every restriction of a cross-section satisfying "the Euler-Lagrange condition".

At this time I would like to express my thanks to Dr. W. Tulczyjew for his help and many profitable conversations. I am also indebted to Professor K. Maurin and Professor A. Trautman for their lively interest and early encouragement.

PRELIMINARIES

Let M be a vector bundle with a base E (paracompact, orientable, n-dimensional differentiable manifold of class C^∞) and with a standard fibre $F = \boldsymbol{R}^1$. The following considerations concern also finite-dimensional vector bundles. (It is sufficient to treat some objects not like scalars but like vectors). The assumption $F = \boldsymbol{R}^1$ is made for simplification of the notation. Let π be a projection of M on E. Different coordinate charts of the manifold M will be distinguished by indices $I, J, \ldots$ If $\overset{I}{U} \subset E$, then on $\pi^{-1}(\overset{I}{U})$ we have defined the coordinate map $M \supset \pi^{-1}(\overset{I}{U}) \ni m \to (\overset{I}{\varkappa}\circ\pi(m), \overset{I}{\varrho}(m)) \in \boldsymbol{R}^{n+1}$ where $\overset{I}{\varrho}(m) = \overset{I}{r}_{\pi(m)}(m)$, $\overset{I}{r}_p\colon \pi^{-1}(p) \to F$ for $p \in \overset{I}{U}$, $\overset{I}{r}_p = A_{IJ}(p)\overset{J}{r}_p$ for $p \in \overset{I}{U} \cap \overset{J}{U}$ and A_{IJ} — a function (in the case of a finite-dimensional vector bundle — a matrix) of class C^∞ on $\overset{I}{U} \cap \overset{J}{U}$.

The coordinate charts of the manifold E will bear also indices $I, J, \ldots$ and to a domain $\overset{I}{U}$ there corresponds the coordinate chart $(\overset{I}{\varkappa}, \overset{I}{U})$ where $E \supset \overset{I}{U} \ni p \to \overset{I}{\varkappa}(p) = (\overset{I}{\xi}{}^1(p), \ldots, \overset{I}{\xi}{}^n(p)) \in \boldsymbol{R}^n$. When we have some fixed coordinate chart, we will skip the indices. By $\Gamma(M)$ we denote the space of differentiable cross-sections of the bundle M with relatively compact domains. By D_u and R_u we denote respectively the domain and the graph of a cross-section $u \in \Gamma(M)$. Let B be any vector bundle over E and $\Omega \subset E$; then by $\Gamma(\Omega, B)$ we denote the set of cross-sections of the bundle B, global over Ω. Let $(\varkappa, U)$ be a coordinate chart on E, $u \in \Gamma(M)$ and $p \in D_u \cap U$; then by $[u]_p$ we denote the set of cross-sections $u' \in \Gamma(M)$ such that $p \in D_{u'}$, $u'(p) = u(p)$ and

$$\left(\frac{\partial}{\partial\xi^i}\varrho\circ u'\circ\varkappa^{-1}\right)(\varkappa(p)) = \left(\frac{\partial}{\partial\xi^i}\varrho\circ u\circ\varkappa^{-1}\right)(\varkappa(p)), \quad i = 1, \ldots, n,$$

$$J_p^1 := \bigcup_{u\in\Gamma(M)} [u]_p, \quad J^1 := \bigcup_{p\in E} J_p^1.$$

$J^1 =: J^1(M)$ is called the *first order jet-bundle* of the vector bundle M. Taking into account, in the definition of $[u]_p$, derivatives higher than the first order we get higher order jet-bundles. The results of this paper obtained for $J^1(M)$ can be generalized for the case $J^k(M)$, where k indicates the order of the jet-bundle. In the following $J^1(M)$ will be denoted by J. The map $J \ni [u]_p \to \pi([u]_p) := p \in E$ is a projection of the bundle J on E. It will be denoted by the same symbol as the projection of the bundle M on E. In the manifold J we introduce coordinate charts $(\overset{I}{\chi}, \pi^{-1}(\overset{I}{U}))$

$$J \supset \pi^{-1}(\overset{I}{U}) \ni [u]_p \to \overset{I}{\chi}([u]_p)$$
$$:= (\overset{I}{x}{}^1([u]_p), \ldots, \overset{I}{x}{}^n([u]_p), \overset{I}{v}([u]_p), \overset{I}{v}_1([u]_p), \ldots, \overset{I}{v}_n([u]_p)) \in \boldsymbol{R}^{2n+1},$$

where

$$\overset{I}{x}{}^i([u]_p) := \overset{I}{\xi}{}^i(p), \quad \overset{I}{v}([u]_p) := \overset{I}{r}_p\circ u(p),$$
$$\overset{I}{v}_i([u]_p) := \left(\frac{\partial}{\partial\xi^i}\overset{I}{\varrho}\circ u\circ\overset{I}{\varkappa}{}^{-1}\right)(\overset{I}{\varkappa}(p)).$$

If $u \in \Gamma(M)$, then $\tilde{u} := \{[u]_p\colon p \in D_u\} \subset J$ is an n-dimensional differentiable manifold. On $\tilde{u}$ we introduce coordinate charts bearing also the indices $I, J, \ldots$ A coordinate map on $\{[u]_p\colon p \in \overset{I}{U} \cap D_u\} \subset \tilde{u}$ is given by

$$\tilde{u} \ni [u]_p \to (\overset{I}{\zeta}{}^1([u]_p), \ldots, \overset{I}{\zeta}{}^n([u]_p)) \in \boldsymbol{R}^n,$$

where $\overset{I}{\zeta}{}^i([u]_p) := \overset{I}{\xi}{}^i(p)$.

Let $i_{\tilde{u}}$ be the imbedding of $\tilde{u}$ in J defined by $i_{\tilde{u}}([u]_p) := [u]_p$. By $\pi_{\tilde{u}}$ we denote the 1-1 map of D_u on $\tilde{u}$ defined by $\pi_{\tilde{u}}(p) := [u]_p$. If $f \in C(J, \boldsymbol{R}^1)$, then

$$\bar{f} := f\circ i_{\tilde{u}}\circ\pi_{\tilde{u}} \in C(D_u, \boldsymbol{R}^1).$$

When on a manifold P we have a field Φ of curves (a curve $\Phi(p)$ through $p \in P$, i.e. $\Phi_0(p) = p$), then by $[\Phi]$ ($[\Phi(p)]$) we denote the vector field (the tangent vector at $p \in P$) induced by it.

Let P, Q be differentiable manifolds and $f \in C^\infty(P, Q)$; then

$$f^*\colon \overset{k}{\wedge} T^*(Q) \to \overset{k}{\wedge} T^*(P), \quad f_*\colon \overset{k}{\wedge} T(P) \to \overset{k}{\wedge} T(Q)$$

are the maps canonically induced by f.

THE FIRST NOETHER'S THEOREM

We introduce the notion of a differentiable curve in the space $\Gamma(M)$. Let $Cu(M)$ be the set of fields Ψ that

$$\bigwedge_{\Psi \in Cu(M)} \bigvee_{\varepsilon > 0} \quad [-\varepsilon, \varepsilon] \times M \ni (t, m) \to \Psi_t(m) \in M$$

differentiable and

1. $\Psi_0(m) = m$,
2. $\big(\pi(m) = \pi(m')\big) \Rightarrow \big(\pi \circ \Psi_t(m) \underset{t \in [-\varepsilon, \varepsilon]}{\equiv} \pi \circ \Psi_t(m')\big)$.

The map $[-\varepsilon, \varepsilon] \ni t \to \Psi_t(m) \in M$ is a differentiable curve in M through $m \in M$.

Every field $\Psi \in Cu(M)$ defines the field $\pi\Psi$ of curves in E

$$[-\varepsilon, \varepsilon] \times E \ni (t, p) \to (\pi\Psi)_t(p) := \pi \circ \Psi_t(m_p) \in E,$$

where $m_p \in M$ is such that $\pi(m_p) = p$.

The map $[-\varepsilon, \varepsilon] \ni t \to \mathring{\Psi}_t(u) \in \Gamma(M)$, $u \in \Gamma(M)$, where $R_{\mathring{\Psi}_t(u)} := \Psi_t(R_u)$, is called a *differentiable curve* in $\Gamma(M)$ through $u \in \Gamma(M)$ and induced by the field $\Psi \in Cu(M)$.

By an *integral functional* on $\Gamma(M)$ we mean a map

$$\Gamma(M) \ni u \to \mathscr{J}(u) := \int_{D_u} \mathscr{L}_u \in \boldsymbol{R}^1,$$

where $\mathscr{L}_u$ is defined by a differentiable map

$$J \ni [u]_p \to L([u]_p) \in \overset{n}{\wedge} T_p^*(E) \subset \overset{n}{\wedge} T^*(E)$$

as follows:

$$E \ni p \to \mathscr{L}_u(p) := L([u]_p) \in \overset{n}{\wedge} T^*(E).$$

Taking various maps L we get various integral functionals $\mathscr{J}$. An integral functional $\mathscr{J}$ is called *differentiable* at $u \in \Gamma(M)$ if for every $\Psi \in Cu(M)$ there exists

$$\frac{d}{dt} \mathscr{J} \circ \mathring{\Psi}_t(u)|_{t=0}.$$

By $\mathscr{F}$ we denote the algebra spanned by all the integral functionals differentiable on $\Gamma(M)$. The curves $\mathring{\Psi}(u)$ and $\mathring{\Phi}(u)$, where $\Psi, \Phi \in Cu(M)$, are called *equivalent* if

$$\bigwedge_{\mathscr{J} \in \mathscr{F}} \frac{d}{dt} \mathscr{J} \circ \mathring{\Psi}_t(u)|_{t=0} = \frac{d}{dt} \mathscr{J} \circ \mathring{\Phi}_t(u)|_{t=0}.$$

The equivalence class obtained in this way is denoted by $[\mathring{\Psi}(u)]$. Let us define the vector space

$$\mathfrak{W}_u\big(\Gamma(M)\big) := \{[\mathring{\Psi}(u)] \colon \Psi \in Cu(M)\}$$

of linear functionals on $\mathscr{F}$, where, if $\mathscr{J} \in \mathscr{F}$, then

$$\langle \mathscr{J}, [\mathring{\Psi}(u)] \rangle := \frac{d}{dt} \mathscr{J} \circ \mathring{\Psi}_t(u)|_{t=0}.$$

The vectors of the space $\mathfrak{W}_u\big(\Gamma(M)\big)$ will be denoted by $[\mathring{\Psi}(u)]$ or $\mathfrak{X}_u$. It is easy to see that $\mathfrak{X}_u \in \mathfrak{W}_u\big(\Gamma(M)\big)$ satisfies the Leibniz formula. Hence it can be considered as a kind of a vector tangent to $\Gamma(M)$ at $u \in \Gamma(M)$ provided that an analogue of a differentiable structure of a manifold is so chosen for $\Gamma(M)$ that it coincides with $\mathscr{F}$. Any pair $(u, \Psi) \in \Gamma(M) \times Cu(M)$ induces the field $\overset{*}{\Psi}$ of curves in J through points of submanifold $\tilde{u} \subset J$, as follows:

$$[-\varepsilon, \varepsilon] \times \tilde{u} \ni (t, [u]_p) \to \overset{*}{\Psi}_t([u]_p) := [\mathring{\Psi}_t(u)]_{(\pi\Psi)_t(p)} \in J.$$

The map $\overset{*}{\Psi}([u]_p) \colon [-\varepsilon, \varepsilon] \to J$ is a differentiable curve in J through $[u]_p \in \tilde{u}$. The curve $\overset{*}{\Psi}([u]_p)$ defines the vector $X_{[u]_p} \in T_{[u]_p}(J)$. The field on $\tilde{u}$ of vectors tangent to J defined by the field $\overset{*}{\Psi}$ of curves in J will be denoted by $[\overset{*}{\Psi}]$. For interrelations between components of this field see Appendix.

Now we define a map which will be needed in the following. By j we denote an element of the bundle J. Let us recall that

$$J \ni j \to L(j) \in \overset{n}{\wedge} T_{\pi(j)}^*(E) \subset \overset{n}{\wedge} T^*(E).$$

We know that $\pi^* \colon \overset{n}{\wedge} T^*(E) \to \overset{n}{\wedge} T^*(J)$. Then we define

$$\pi_j^* \colon \overset{n}{\wedge} T_{\pi(j)}^*(E) \to \overset{n}{\wedge} T^*(J)$$

as follows: if $\omega \in \overset{n}{\wedge} T_{\pi(j)}^*(E)$, then $\pi_j^*(\omega) := (\pi^*\omega)(j) \in \overset{n}{\wedge} T_j^*(J)$.

Now we define $\mathfrak{L} \in \Gamma\big(J, \overset{n}{\wedge} T^*(J)\big)$ as follows:

(1) $$J \ni j \to \mathfrak{L}(j) := \pi_j^*\big(L(j)\big) \in \overset{n}{\wedge} T^*(J).$$

If X is a field on $\tilde{u}$ of vectors tangent to J, then by $\hat{X}$ we will denote any differentiable extension of X on the manifold J such that $\pi_* \hat{X} = \pi_* X$.

LEMMA 1. *If* $\mathscr{J} \in \mathscr{F}$, $\Psi \in Cu(M)$ *and* $u \in \Gamma(M)$, *then*

$$\langle \mathscr{J}, [\mathring{\Psi}(u)] \rangle = \int_{D_u} \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* (\pounds_{\hat{X}} \mathfrak{L}), \quad \textit{where } X = [\overset{*}{\Psi}].$$

Proof. We introduce the following notation:

$$u_t := \overset{\circ}{\Psi}_t(u),$$

$$\varphi_t(p) := (\pi\Psi)_t(p), \qquad X := [\overset{*}{\Psi}], \qquad \mathsf{w}_p := \bigwedge_{i=1}^{n}\left(\frac{\partial}{\partial\xi^i}\right)_p,$$

$$\mathsf{v}_p := \bigwedge_{i=1}^{n} d\xi^i(p), \qquad L'([u]_p) := \langle \mathsf{w}_p, L([u]_p)\rangle,$$

$$\mathsf{V}_{[u]_p} := \bigwedge_{i=1}^{n} dx^i([u]_p),$$

where coordinate charts are fixed.

Let $\{I_\nu\}_1^k$ be such that $D_u \subset \bigcup_{\nu=1}^{k} \overset{I_\nu}{U}$. By $\{\eta_\nu\}_1^\infty$ we denote a partition of unity subordinate to the covering $\{\overset{I_\nu}{U} \cap D_u\}_1^k$. Then

$$\mathscr{J}(u_t) = \int_{D_{u_t}} \mathscr{L}_{u_t} = \sum_\nu \int_{D_{u_t}} \eta_\nu \mathscr{L}_{u_t} = \sum_\nu \int_{D_u} (\eta_\nu \mathscr{L}_{u_t})\circ\varphi_t.$$

It should be noted that in view of the relative compactness of D_u the above sum has only a finite number of non-zero terms.

$$\begin{aligned}\frac{d}{dt}\mathscr{J}(u_t)\Big|_{t=0} &= \sum_\nu \int_{D_u} \frac{d}{dt}(\eta_\nu \mathscr{L}_{u_t})\circ\varphi_t\Big|_{t=0}\\ &= \sum_\nu \int_{D_u}\left[\mathscr{L}_u \frac{d}{dt}\eta_\nu\circ\varphi_t\Big|_{t=0} + \eta_\nu \frac{d}{dt}\mathscr{L}_{u_t}\circ\varphi_t\Big|_{t=0}\right]\\ &= \int_{D_u}\sum_\nu \frac{d}{dt}\eta_\nu\circ\varphi_t\Big|_{t=0}\mathscr{L}_u + \int_{D_u}\frac{d}{dt}\mathscr{L}_{u_t}\circ\varphi_t\Big|_{t=0}\\ &= \int_{D_u}\frac{d}{dt}\mathscr{L}_{u_t}\circ\varphi_t\Big|_{t=0}.\end{aligned}$$

Let $p \in \overset{I}{U}$; then

$$\begin{aligned}\frac{d}{dt}\mathscr{L}_{u_t}\circ\varphi_t(p)|\Big|_{t=0} &= \frac{d}{dt}(\langle\mathsf{w}, \mathscr{L}_{u_t}\rangle\mathsf{v})\circ\varphi_t(p)\Big|_{t=0} = \frac{d}{dt}L'([u_t]_{\varphi_t(p)})\mathsf{v}_{\varphi_t(p)}\Big|_{t=0}\\ &= \frac{d}{dt}L'([u_t]_{\varphi_t(p)})\Big|_{t=0}\mathsf{v}_p + L'([u]_p)\frac{d}{dt}\mathsf{v}_{\varphi_t(p)}\Big|_{t=0}\\ &= \overline{\mathop{\pounds}_X L'}(p)\mathsf{v}_p + \overline{L'}(p)(\mathop{\pounds}_{X_0}\mathsf{v})(p),\end{aligned}$$

where $X_0 = \pi_* X$.

Now we will bring the right-hand side of the thesis to the form above. We must find the form of $i_{\tilde{u}}^*(\mathop{\pounds}_{\hat{X}}\mathfrak{L})$ and therefore we are interested in $\mathop{\pounds}_{\hat{X}}\mathfrak{L}$ only on $\tilde{u}$. Thus

$$\begin{aligned}(\mathop{\pounds}_{\hat{X}}\mathfrak{L})([u]_p) &= (\mathop{\pounds}_{\hat{X}} L'\mathsf{V})([u]_p)\\ &= (\mathop{\pounds}_{X} L')([u]_p)\mathsf{V}_{[u]_p} + L'([u]_p)(\mathop{\pounds}_{\hat{X}}\mathsf{V})([u]_p).\end{aligned}$$

Since $\mathsf{V} = \pi^*\mathsf{v}$, we have $\mathop{\pounds}_{\hat{X}}\mathsf{V} = \mathop{\pounds}_{\hat{X}}\pi^*\mathsf{v} = \pi^*\mathop{\pounds}_{\pi_*\hat{X}}\mathsf{v} = \pi^*\mathop{\pounds}_{X_0}\mathsf{v}$.

Above we have made use of the identity

$$f^*\mathop{\pounds}_{f_*X}\omega = \mathop{\pounds}_X f^*\omega, \tag{2}$$

where ω is an exterior differential form.

It is easy to see that the thesis does not depend on the choice of an extension $\hat{X}$ of the field X defined on $\tilde{u}$ only, q. e. d.

In $\Gamma(M)\times Cu(M)$ we can introduce three equivalence relations:

$$\big((u,\Psi)\underset{r_1}{\sim}(v,\Phi)\big) \overset{\mathrm{df}}{\Leftrightarrow} (u = v,\ [\Psi] \underset{R_u}{\equiv} [\Phi]),$$

$$\big((u,\Psi)\underset{r_2}{\sim}(v,\Phi)\big) \overset{\mathrm{df}}{\Leftrightarrow} (u = v,\ [\overset{*}{\Psi}] \underset{\tilde{u}}{\equiv} [\overset{*}{\Phi}]),$$

$$\big((u,\Psi)\underset{r_3}{\sim}(v,\Phi)\big) \overset{\mathrm{df}}{\Leftrightarrow} (u = v,\ [\overset{\circ}{\Psi}(u)] = [\overset{\circ}{\Phi}(u)]).$$

LEMMA 2. $r_1 \Leftrightarrow r_2 \Rightarrow r_3$.

Proof. Let $\Psi \in Cu(M)$. If the field $[\overset{*}{\Psi}]$ has the form

$$[\overset{*}{\Psi}] = \overset{I}{X}\frac{\partial}{\partial \overset{I}{x}{}^i} + \overset{I}{X}{}^0\frac{\partial}{\partial \overset{I}{v}} + \overset{I}{Y}{}^i\frac{\partial}{\partial \overset{I}{v}{}^i}$$

at points of the set $\pi^{-1}(\overset{I}{U}) \cap \tilde{u}$, then the field $[\Psi]$ has the form

$$[\Psi] = \overset{I}{X}{}^i\circ f\frac{\partial}{\partial(\overset{I}{\xi}{}^i\circ\pi)} + \overset{I}{X}{}^0\circ f\frac{\partial}{\partial\overset{I}{\varrho}}$$

at points of the set $\pi^{-1}(\overset{I}{U}) \cap R_u$, where f is the canonical 1-1 map of R_u on $\tilde{u}$ given by

$$R_u \ni u(p) \to f(u(p)) := [u]_p \in \tilde{u}.$$

Hence $r_1 \Rightarrow r_2$. For the proof of the implication $r_2 \Rightarrow r_1$ see Appendix 1.

It follows from Lemma 1 that $r_2 \Rightarrow r_3$, q. e. d.

Remark. Let us notice that in general the implication $r_3 \Rightarrow r_1$ does not occur. For example, a curve $\overset{\circ}{\Psi}(u)$ in $\Gamma(M)$ such that

$$\partial R_{\overset{\circ}{\Psi}_t(u)} \underset{t}{\equiv} \partial R_u$$

can be given by a field $\Psi \in Cu(M)$ that curves $\Psi(m)$, $m \in M$, lie inside of fibres as well as by a field $\Phi \in Cu(M)$ that for some $m \in M$ curves $\Phi(m)$ are transversal to fibres.

Now we can pass from dealing with fields of curves to dealing with vector fields. We define the following vector spaces:

$$T_M := \{X \in \Gamma(M, T(M)) \colon \pi(m) = \pi(m') \Rightarrow \pi_* X_m = \pi_* X_{m'}\},$$

$$W_M(u) := \{X \in \Gamma(R_u, T(M)) \colon \bigwedge_X \bigvee_{\Psi \in Cu(M)} X = [\Psi]|_{R_u}\},$$

$$W_J(u) := \{\boldsymbol{X} \in \Gamma(\tilde{u}, T(J)) \colon \bigwedge_{\boldsymbol{X}} \bigwedge_{\Psi \in Cu(M)} \boldsymbol{X} = [\overset{*}{\Psi}]\}.$$

Now we can formulate Lemma 2 in a new form:

LEMMA 2'. *There are canonical homomorphisms onto* $-h_u$, H *and the canonical isomorphism* I:

$$T_M \xrightarrow{h_u} W_M(u) \xrightarrow{I} W_J(u) \xrightarrow{H} \mathfrak{W}_u(\Gamma(M)).$$

Proof. It is sufficient to give the explicite form of these maps:

$$T_M \ni X \to h_u(X) := X|_{R_u} \in W_M(u),$$

$$W_M(u) \ni [\Psi]|_{R_u} \to I([\Psi]|_{R_u}) := [\overset{*}{\Psi}] \in W_J(u),$$

$$W_J(u) \ni [\overset{*}{\Psi}] \to H([\overset{*}{\Psi}]) := [\overset{\circ}{\Psi}(u)] \in \mathfrak{W}_u(\Gamma(M)).$$

The correctness of above definitions follows from Lemma 1 and Lemma 2, q. e. d.

Now we formulate Lemma 1 in terms of vector fields.

LEMMA 1'. *Let* $\mathcal{J} \in \mathcal{F}$, $\mathfrak{X}_u \in \mathfrak{W}_u(\Gamma(M))$; *then*

$$\langle \mathcal{J}, \mathfrak{X}_u \rangle = \int_{D_u} \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* (\underset{\hat{X}}{\pounds} \mathfrak{L}), \quad \textit{where} \quad \boldsymbol{X} \in H^{-1}(\mathfrak{X}_u).$$

There is a unique decomposition $\boldsymbol{X} = \boldsymbol{X}_\perp + (i_{\tilde{u}} \circ \pi_{\tilde{u}})_* \boldsymbol{X}_0$ for $\boldsymbol{X} \in W_J(u)$, where $\boldsymbol{X}_0 \in \Gamma(D_u, T(E))$ and $(\pi)_* \boldsymbol{X}_\perp = 0$. Using (2) we have

$$\pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* (\underset{\hat{X}}{\pounds} \mathfrak{L}) = \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* (\underset{\boldsymbol{X}_\perp}{\pounds} \mathfrak{L}) + \underset{\boldsymbol{X}_0}{\pounds} \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* \mathfrak{L}$$

$$= \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* (\underset{\boldsymbol{X}_\perp}{\pounds} \mathfrak{L}) + d\boldsymbol{X}_0 \lrcorner\, \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* \mathfrak{L}$$

The last expression follows from the identity $\underset{X}{\pounds}\, \omega = X \lrcorner\, d\omega + dX \lrcorner\, \omega$ and from the fact that $d\pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* \mathfrak{L} = 0$.

Let $\{\eta_\nu\}_1^\infty$ be as before a partition of unity subordinate to a covering $\{\overset{I_\nu}{U} \cap D_u\}_1^k$ of D_u. We define the following $(n-1)$-form on E:

(3) $$N_{\boldsymbol{X}_\perp}(\mathcal{L}_u) := \sum_{\nu,k} (-1)^k \eta_\nu \overset{I_\nu}{\boldsymbol{X}_\perp^0} \frac{\overline{\partial \overset{I_\nu}{L}}}{\partial \overset{I_\nu}{v_k}} d\overset{I_\nu}{\xi^1} \wedge \ldots \overset{k}{\wedge} \ldots \wedge d\overset{I_\nu}{\xi^n},$$

where $\overset{I_\nu}{L}$, $\nu = 1, 2, \ldots$, are the coefficients in the expression of the form L in the coordinate system connected with the domain $\overset{I_\nu}{U}$.

For the proof of correctness of this definition see Appendix.

By an Euler-Lagrange's derivative of the n-form $\mathcal{L}_u$ with respect to $X \in W_M(u)$ we mean

$$[\mathcal{L}_u]_X := \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* (\underset{\boldsymbol{X}_\perp}{\pounds} \mathfrak{L}) - dN_{\boldsymbol{X}_\perp}(\mathcal{L}_u), \quad \text{where} \quad \boldsymbol{X} = I(X).$$

We define Noether's expression for the n-form $\mathcal{L}_u$ and $X \in W_M(u)$ as follows:

$$\mathcal{N}_X(\mathcal{L}_u) := N_{\boldsymbol{X}_\perp}(\mathcal{L}_u) + \boldsymbol{X}_0 \lrcorner\, \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* \mathfrak{L},$$

where $\boldsymbol{X} = I(X)$.

Hence if $X \in W_M(u)$ and $\mathfrak{X}_u = H \circ I(X)$, then

(4) $$\langle \mathcal{J}, \mathfrak{X}_u \rangle = \int_{D_u} [\mathcal{L}_u]_X + d\mathcal{N}_X(\mathcal{L}_u).$$

We define vector subspaces:

$$D_M(u) := \{X \in W_M(u) \colon X \underset{\partial R_u}{\equiv} 0\},$$

$$D_J(u) := I(D_M(u)), \quad \mathfrak{D}_u(\Gamma(M)) := H(D_J(u)).$$

A cross-section $u \in \Gamma(M)$ is called *extremal* for $\mathcal{J} \in \mathcal{F}$ if and only if, for every $\mathfrak{X}_u \in \mathfrak{D}_u(\Gamma(M))$, $\langle \mathcal{J}, \mathfrak{X}_u \rangle = 0$. By $\mathcal{S}(\mathcal{J})$ we denote the set of all $u \in \Gamma(M)$ extremal for a given $\mathcal{J} \in \mathcal{F}$.

LEMMA 3. *We have*

$$(u \in \mathcal{S}(\mathcal{J})) \Leftrightarrow \Big(\bigwedge_{X \in W_M(u)} [\mathcal{L}_u]_X \underset{D_u}{\equiv} 0\Big).$$

Proof. If $X \in D_M(u)$, then $\mathcal{N}_X(\mathcal{L}_u) \underset{\partial\tilde{u}}{\equiv} 0$. Hence for $X \in D_M(u)$ we have $\langle \mathcal{J}, H \circ I(X) \rangle = \int_{D_u} [\mathcal{L}_u]_X$ and so the implication $\Leftarrow$ is proved.

Let $X \in W_M(u)$ and $\boldsymbol{X} = I(X)$. It follows directly from the definitions that for $p \in \overset{I}{U} \cap D_u$

$$(5)\qquad \begin{aligned} [\mathscr{L}_u]_X|_p &= \overline{\overset{I}{\boldsymbol{X}}{}^0_\perp}\left(\overline{\frac{\partial \overset{I}{L}}{\partial \overset{I}{v}}} - \frac{\partial}{\partial \overset{I}{\xi}{}^i}\,\overline{\frac{\partial \overset{I}{L}}{\partial \overset{I}{v}_i}}\right)\overset{I}{\mathsf{v}}|_p;\\ [\mathscr{L}_u]_X &= \sum_\nu \eta_\nu \overline{\overset{I_\nu}{\boldsymbol{X}}{}^0_\perp}\left(\overline{\frac{\partial \overset{I_\nu}{L}}{\partial \overset{I_\nu}{v}}} - \frac{\partial}{\partial \overset{I_\nu}{\xi}{}^i}\,\overline{\frac{\partial \overset{I_\nu}{L}}{\partial \overset{I_\nu}{v}_i}}\right)\overset{I_\nu}{\mathsf{v}}\end{aligned}$$

and, which is of the highest importance, that $[\mathscr{L}_u]_X(p)$ depends on X_p only. We will prove that if $u \in \mathscr{S}(\mathscr{J})$, then for every $\overset{I}{U}$

$$(6)\qquad A := \overline{\frac{\partial \overset{I}{L}}{\partial \overset{I}{v}}} - \frac{\partial}{\partial \overset{I}{\xi}{}^i}\,\overline{\frac{\partial \overset{I}{L}}{\partial \overset{I}{v}_i}} \equiv 0 \quad \text{on } \overset{I}{U} \cap D_u.$$

Let us suppose that there exists a point $p \in \overset{I}{U} \cap D_u$ such that $A(p) \neq 0$; then there exists a neighborhood Ω of p that $\Omega \subset \overset{I}{U} \cap D_u$ and $A(q) \neq 0$ for $q \in \Omega$. Now let $X \in D_M(u)$ be such that $\boldsymbol{X}_\perp = 0$ on $\complement \pi_{\tilde{u}}(\Omega)$ and $\overline{\overset{I}{\boldsymbol{X}}{}^0_\perp} \geqslant 0$. Then $\overline{\overset{I}{\boldsymbol{X}}{}^0_\perp} A$ has a constant sign on Ω. Hence $\int_{D_u} [\mathscr{L}_u]_X \neq 0$ (a contradiction). Now from (4) and (5) the implication $\Rightarrow$ follows, q. e. d.

For every $\mathscr{J} \in \mathscr{F}$ and for every open $\Omega \subset E$ we define the functional $\mathscr{J}_\Omega \in \mathscr{F}$ in the following way:

$$\Gamma(M) \ni u \to \mathscr{J}_\Omega(u) := \mathscr{J}(u|_{\Omega \cap D_u}) \in \boldsymbol{R}^1.$$

$\mathscr{J} \in \mathscr{F}$ is called *invariant* at a point $u \in \Gamma(M)$ with respect to $X \in T_M$ if and only if

$$\langle \mathscr{J}, H \circ I \circ h_u(X)\rangle = 0.$$

THE FIRST NOETHER'S THEOREM. *Let $\mathscr{J} \in \mathscr{F}$, $u \in \mathscr{S}(\mathscr{J})$, $X \in T_M$; then*

$$\left(\begin{matrix}\textit{For every } \Omega \subset D_u, \mathscr{J}_\Omega \textit{ is invariant}\\ \textit{at the point } u \textit{ with respect to } X\end{matrix}\right) \Leftrightarrow \left(d\mathscr{N}_{h_u(X)}(\mathscr{L}_u) \underset{D_u}{\equiv} 0\right).$$

The proof is an immediate consequence of (4) and Lemma 3.

It is easy to see that for $X, Y \in W_M(u)$, $a, b \in \boldsymbol{R}^1$

$$\mathscr{N}_{aX+bY}(\mathscr{L}_u) = a\mathscr{N}_X(\mathscr{L}_u) + b\mathscr{N}_Y(\mathscr{L}_u).$$

Now we can state the second variant of

THE FIRST NOETHER'S THEOREM. *Let $\mathscr{J} \in \mathscr{F}$, $u \in \mathscr{S}(\mathscr{J})$, $\{X_i\}_1^r$ be a base of a subspace $T \subset T_M$; then*

$$\left(\begin{matrix}\textit{For every } \Omega \subset D_u \textit{ and every}\\ X \in T, \mathscr{J}_\Omega \textit{ is invariant at}\\ \textit{the point } u \textit{ with respect to } X\end{matrix}\right) \Leftrightarrow \left(d\mathscr{N}_{h_u(X_i)}(\mathscr{L}_u) \underset{D_u}{\equiv} 0,\ i = 1, \ldots, r\right).$$

Remark. The assumption D_u as relatively compact is not weighty. If we introduce a new definition that a cross-section of M is called *extremal* for $\mathscr{J} \in \mathscr{F}$ if and only if

$$\bigwedge_{K \subset E} (K\text{-compact}) \Rightarrow \left(u|_K \in \mathscr{S}(\mathscr{J})\right),$$

hen it is clear that for u extremal in this sense The First Noether's Theotem is valid.

APPENDIX

1. Let $\boldsymbol{X} \in T(J)$; then in the base induced by the coordinate chart $(\overset{I}{\chi}, \pi^{-1}(\overset{I}{U}))$ the vector $\boldsymbol{X}_\perp$ has the form

$$\boldsymbol{X}_\perp = \overset{I}{\boldsymbol{X}}{}^0_\perp \frac{\partial}{\partial \overset{I}{v}} + \overset{I}{\boldsymbol{Y}}{}^i_\perp \frac{\partial}{\partial \overset{I}{v}_i}.$$

But if $\boldsymbol{X} \in W_J(u)$ and $\boldsymbol{X} = [\overset{*}{\Psi}]$, then

$$\overset{I}{\boldsymbol{Y}}{}^i_\perp([u]_p) = \frac{d}{dt}\overset{I}{v}_i \circ \overset{*}{\Psi}_t([u]_p)|_{t=0} = \frac{d}{dt}\left(\frac{\partial \overset{I}{\varrho} \circ \overset{*}{\Psi}_t(u)}{\partial \overset{I}{\xi}{}^i}(p)\right)\Bigg|_{t=0}$$

$$= \frac{\partial}{\partial \overset{I}{\xi}{}^i} \frac{d\overset{I}{\varrho} \circ \overset{\circ}{\Psi}_t(u)(p)}{dt}\Bigg|_{t=0} = \frac{\partial \overset{I}{\boldsymbol{X}}{}^0_\perp}{\partial \overset{I}{\xi}{}^i}([u]_p).$$

2. The following identities are valid on $\pi^{-1}(\overset{I}{U} \cap \overset{J}{U}) \subset J$:

$$\overset{J}{v} = A_{JI} \circ \pi\, \overset{I}{v},$$

$$\overset{J}{v}_k = \frac{\partial A_{JI}}{\partial \overset{J}{\xi}{}^k} \circ \pi\, \overset{I}{v} + A_{JI} \circ \pi\, \frac{\partial \overset{I}{\xi}{}^i}{\partial \overset{J}{\xi}{}^k} \circ \pi\, \overset{I}{v}_i,$$

$$\frac{\partial}{\partial \overset{I}{v}} = A_{JI} \circ \pi \frac{\partial}{\partial \overset{J}{v}} + \frac{\partial A_{JI}}{\partial \overset{J}{\xi}{}^k} \circ \pi \frac{\partial}{\partial \overset{J}{v}_k},$$

$$\frac{\partial}{\partial \overset{I}{v}_i} = A_{JI}\circ\pi \, \frac{\partial \overset{I}{\xi}{}^i}{\partial \overset{J}{\xi}{}^k}\circ\pi\, \frac{\partial}{\partial \overset{J}{v}_k},$$

$$\overset{I}{X}{}^i = \overset{J}{X}{}^k \frac{\partial \overset{I}{\xi}{}^i}{\partial\overset{J}{\xi}{}^k}\circ\pi,$$

$$\overset{I}{X}{}^0 = \overset{J}{X}{}^0 A_{IJ}\circ\pi + \overset{J}{X}{}^i \frac{\partial A_{IJ}}{\partial \overset{J}{\xi}{}^i}\circ\pi\,\overset{J}{v},$$

$$\overset{I}{L} = \left|\frac{D(\overset{J}{\varkappa})}{D(\overset{I}{\varkappa})}\right|\circ\pi\,\overset{J}{L}.$$

3. The above identities are used in the following proof of correctness of the definition (3). We define

$${}_I\{\mathscr{L}_u\}_{X_\perp} := \sum_k (-1)^k\, \overline{\overset{I}{X}{}^0_\perp}\, \overline{\frac{\partial \overset{I}{L}}{\partial \overset{I}{v}_k}}\, d\overset{I}{\xi}{}^1\wedge \overset{k}{\widehat{\ldots}}\wedge d\overset{I}{\xi}{}^n.$$

Taking into account that $\overset{I}{X}{}^i_\perp = 0$, $i = 1, \ldots, n$, we will prove that

(7) $${}_I\{\mathscr{L}_u\}_{X_\perp} = {}_J\{\mathscr{L}_u\}_{X_\perp} \quad \text{on} \quad \overset{I}{U}\cap\overset{J}{U}\cap D_u.$$

$${}_I\{\mathscr L_u\}_{X_\perp} = \sum_k (-1)^k\, \overline{\overset{J}{X}{}^0_\perp}\, A_{IJ} A_{JI}\, \overline{\frac{\partial \overset{I}{\xi}{}^k}{\partial \overset{J}{\xi}{}^i}\left(\frac{\partial}{\partial \overset{J}{v}{}^i}\left|\frac{D(\overset{J}{\varkappa})}{D(\overset{I}{\varkappa})}\right|\circ\pi\,\overset{J}{L}\right)}\, d\overset{I}{\xi}{}^1\wedge\overset{k}{\widehat{\ldots}}\wedge d\overset{I}{\xi}{}^n$$

$$= \sum_{k,l}(-1)^k\, \overline{\overset{J}{X}{}^0_\perp}\, \frac{\partial\overset{I}{\xi}{}^k}{\partial\overset{J}{\xi}{}^i}\left|\frac{D(\overset{J}{\varkappa})}{D(\overset{I}{\varkappa})}\right|\frac{\partial \overset{J}{L}}{\partial\overset{J}{v}_i}\left|\frac{D(\overset{I}{\xi}{}^1,\overset{k}{\widehat{\ldots}},\overset{I}{\xi}{}^n)}{D(\overset{J}{\xi}{}^1,\overset{l}{\widehat{\ldots}},\overset{J}{\xi}{}^n)}\right| d\overset{J}{\xi}{}^1\wedge\overset{l}{\widehat{\ldots}}\wedge d\overset{J}{\xi}{}^n.$$

$$A_{lk} := (-1)^{k+l}\left|\frac{D(\overset{I}{\xi}{}^1,\overset{k}{\widehat{\ldots}},\overset{I}{\xi}{}^n)}{D(\overset{J}{\xi}{}^1,\overset{l}{\widehat{\ldots}},\overset{J}{\xi}{}^n)}\right|$$

is the algebraic complement of the element $a_{lk} := \partial\overset{I}{\xi}{}^k/\partial\overset{J}{\xi}{}^l$ in the matrix $A := D(\overset{I}{\varkappa})/D(\overset{J}{\varkappa})$. The following identity is valid:

$$\sum_k a_{ik}A_{lk} = \delta_{il}\det A.$$

This finish the proof of (7). As a consequence we have that $N_{X_\perp}(\mathscr{L}_u)$ does not depend on a choice of a partition of unity and that $N_{X_\perp}(\mathscr{L}_u) = {}_I\{\mathscr{L}_u\}_{X_\perp}$ on $\overset{I}{U}\cap D_u$.

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