является математическим ожиданием $E(x^*)$ распределения F^{x^*} (или случайной величины x^*). Ясно, что

$$\mu\{\omega: \omega \in \Omega, |x^*(\omega)| \geqslant |E(x^*)|\} = \sigma(x^*) > 0$$

для любого фиксированного линейного функционала $x^* \in X^*$. Однако, как показывает пример 1, условие $\sigma(x^*) > 0$ не выполняется, вообще говоря, равномерно в X^*. Именно в требовании этой равномерности и заключается достаточное для интегрируемости по Петтису условие (3), выраженное таким образом в тех терминах, в которых дается само определение интеграла Петтиса.

Пусть теперь x — нормальный случайный элемент со значениями в X. Имея в виду рассматриваемый сейчас вопрос, можно считать, не ограничивая общности, что пространство X вещественно. Тогда $x^*(\omega)$ будет при всех $x^* \in X^*$ действительной гауссовской случайной величиной и очевидно, что

$$\mu\{\omega: \omega \in \Omega, x^*(\omega) \geqslant E(x^*)\} = \mu\{\omega: \omega \in \Omega, x^*(\omega) \leqslant E(x^*)\} = \tfrac{1}{2}.$$

Отсюда получаем $\sigma(x^*) \geqslant \frac{1}{2}$. Следовательно, простым следствием теоремы 1 является следующая

Теорема 2. *Существует математическое ожидание любого нормального случайного элемента со значениями в произвольном сепарабельном пространстве Банаха.*

Частные случаи этой теоремы были получены ранее непосредственными рассуждениями в работах Мурье [3] (случай сепарабельного пространства Гильберта) и автора [4], [5] (случай пространств l_p, $1 \leqslant p < \infty$).

Литература

[1] Э. Хилле и Р. Филлипс, *Функциональный анализ и полугруппы* Москва 1962.

[2] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[3] E. Mourier, *Eléments aléatories dans un espace de Banach*, Ann. Inst. H. Poincaré 13 (1953), стр. 161-244.

[4] N. Vakhania, *Sur une propriété des répartitions normales de probabilités dans les espaces l_p ($1 < p < \infty$) et H*, Comptes Rend. Acad. Sc. Paris 260 (1965), стр. 1334-1336.

[5] Н. Н. Вахания, *О характеристических функционалах для случайных последовательностей*, Труды ВЦ АН Грузинской ССР, V: 1 (1965), стр. 5-32.

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On the general form of subalgebras of codimension 1 of B-algebras with a unit

by

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Let X be a Banach algebra with unit e. It is known [1] that

0.1. *Every subalgebra X_0 of X of codimension 1 such that $e \notin X_0$ is a set of all zeros of a linear and multiplicative functional on X, and conversely.*

In this paper we shall give some theorems on the general form of subalgebras of X such that $e \in X_0$ and codim $X_0 = 1$. Their class for a given X will be denoted by $\mathfrak{K}(X)$.

1. It is easy to verify that

1.0.1. Lemma. *$X_0 \in \mathfrak{K}(X)$ if and only if there exists an $x_0^* \in X^*$ such that*

1. $X_0 = \{x \in X: x_0^*(x) = 0\}$.
2. $x_0^*(e) = 0$, $x_0^* \neq 0$.
3. *If* $x_0^*(x) = 0$, *then* $x_0^*(x^2) = 0$.

Indeed, if $X_0 \in \mathfrak{K}(X)$, then X_0 is a zero-set of linear functional such that $x_0^* \in X^*$ and since X_0 is a subalgebra with e of X, conditions 2 and 3 must hold.

Inversely, if $x_0^* \in X^*$, then the set

$$X_0 = \{x \in X: x_0^*(x) = 0\}$$

is a linear subspace of X, codim $X_0 = 1$ and $e \in X_0$ in view of 2, and if $x, y \in X$, then

$$x_0^*(xy) = \tfrac{1}{4}x_0[(x+y)^2] - \tfrac{1}{4}x_0^*[(x-y)^2] = 0,$$

whence $xy \in X$ and $X_0 \in \mathfrak{K}(X)$.

It follows from Lemma 1.0.1 that every subalgebra of the class $\mathfrak{K}(X)$ is determined by a linear functional on X which has properties 2 and 3.

[1] М. А. Наймарк, *Нормированные кольца*, Москва 1956, p. 183.

Let $\mathfrak{R}(X)$ be a set of all functionals which have properties 2 and 3, and let

$$V(X) = \{x^* \in X^* \colon x^* \neq 0, \bigvee_{g \in \mathfrak{M}(X)} \bigwedge_{x, y \in X} x^*(xy) = x^*(x)g(y) + x^*(y)g(x)\},$$

where $\mathfrak{M}(X)$ is a set of all linear and multiplicative functionals on X.

The first section is devoted to the proof of the following

1.1. Theorem. *$x^* \in \mathfrak{R}(X)$ if and only if there exist $g_1, g_2 \in \mathfrak{M}(X)$, $g_1 \neq g_2$, such that $x_0^* = c(g_1 - g_2)$, where c is a constant different from 0, or $x_0^* \in V(X)$.*

The proof of this theorem can be based on the following lemmas.

1.2. Lemma. *If*

$$x_0 \in \mathfrak{R}(X), \quad x_0 \in X, \quad x_0^*(x_0) \neq 0 \quad \text{and} \quad 3\left(\frac{x_0^*(x_0^2)}{2}\right)^2 = x_0^*(x_0)x_0^*(x_0^3),$$

then for every $y \in X$ and every positive integer k

$$x_0^*(y^k) = \begin{cases} kx_0^*(y)\left(\dfrac{x_0^*(y^2)}{2x_0(y)}\right)^{k-1} & \text{if } x_0^*(y) \neq 0, \\ 0 & \text{if } x_0^*(y) = 0. \end{cases} \tag{1.2.1}$$

Proof. We shall show at first that if $x_0^* \in \mathfrak{R}(X)$ and $x_0^*(y) \neq 0$, then

$$\begin{aligned} (x_0^*(y))^2 x_0^*(y^{k+2}) \\ = x_0^*(y)x_0^*(y^2)x_0^*(y^{k+1}) + [x_0^*(y)x_0^*(y^3) - (x_0^*(y^2))^2]x_0^*(y^k) \end{aligned} \tag{1.2.2}$$

for $k = 0, 1, 2, \ldots$

Indeed, since the equation

$$x_0^*[x_0^*(y)y^k - x_0^*(y^k)y] = 0$$

holds for $k = 0, 1, 2, \ldots$ and, in particular,

$$x_0^*[x_0(y)y^2 - x_0^*(y^2)y] = 0,$$

we have

$$x_0^*[(x_0^*(y)y^k - x_0^*(y^k)y)(x_0^*(y)y^2 - x_0^*(y^2)y)] = 0$$

and (1.2.2) is a simple consequence of the last equality.

We can now proof the lemma.

If $x_0^*(y) = 0$, then evidently $x_0^*(y^k) = 0$ $(k = 0, 1, 2, \ldots)$ and it is sufficient to consider the case where $x_0^*(y) \neq 0$.

The equation

$$x_0^*(x_0^k) = kx_0^*(x_0)\left(\frac{x_0^*(x_0^2)}{2x_0^*(x_0)}\right)^{k-1} \tag{1.2.3}$$

which is obtained from (1.2.1) for $y = x_0$, will be proved by induction.

It is evident that

$$x_0^*(x_0) = 1 \cdot x_0^*(x_0)\left(\frac{x_0^*(x_0^2)}{2x_0^*(x_0)}\right)^0,$$

$$x_0^*(x_0^2) = 2 \cdot x_0^*(x_0)\left(\frac{x_0^*(x_0^2)}{2x_0^*(x_0)}\right)^1$$

and, from our assumptions,

$$x_0^*(x_0^3) = 3x_0^*(x_0)\left(\frac{x_0^*(x_0^2)}{2x_0^*(x_0)}\right)^2.$$

Let us suppose that the considered equality holds for $k = 1, 2, \ldots, n$, $n \geqslant 3$. Then from (1.2.2) and the last equality it follows that

$$\begin{aligned} x_0^*(x_0^{n+1}) &= \frac{x_0^*(x_0^2)}{x_0^*(x_0)}x_0^*(x_0^n) + \frac{x_0^*(x_0^3)}{x_0^*(x_0)}x_0^*(x_0^{n-1}) - \left(\frac{x_0^*(x_0^2)}{x_0^*(x_0)}\right)^2 x_0^*(x_0^{n-1}) \\ &= (n+1)x_0^*(x_0)\left(\frac{x_0^*(x_0^2)}{2x_0^*(x_0)}\right)^n \end{aligned}$$

and equality (1.2.3) holds for $k = 0, 1, 2, \ldots$

For any $y \neq x_0$ such that $x_0^*(y) \neq 0$ it is sufficient to show that

$$3\left(\frac{x_0^*(y^2)}{2}\right)^2 = x_0^*(y)x_0^*(y^3).$$

From the equalities

$$x_0^*[(x_0^*(x_0)y - x_0^*(y)x_0)^2] = 0, \quad x_0^*\left(x_0^2 - \frac{x_0^*(x_0^2)}{x_0^*(x_0)}x_0\right) = 0,$$

$$x_0^*(x_0^*(x_0)y - x_0^*(y)x_0) = 0,$$

$$x_0^*\left(x_0^*(x_0)x_0 y - \frac{(x_0^*(x_0))^2 x_0^*(y^2) + (x_0^*(y))^2 x_0^*(x_0^2)}{2x_0^*(x_0)x_0^*(y)}x_0\right) = 0,$$

we obtain

$$x_0^*(x_0^2 y) = x_0^*(y)\left(\frac{x_0^*(x_0^2)}{2x_0^*(x_0)}\right)^2 + 2x_0^*(x_0)\frac{x_0^*(x_0^2)}{2x_0^*(x_0)}\,\frac{x_0^*(y^2)}{2x_0^*(y)},$$

$$x_0^*(x_0 y^2) = x_0^*(x_0)\left(\frac{x_0^*(y^2)}{2x_0^*(y)}\right)^2 + x_0^*(y^2)\frac{x_0^*(x_0^2)}{2x_0^*(x_0)}.$$

It is easy to see that

$$\begin{aligned} (x_0^*(x_0))^3 x_0^*(y^3) &= 3(x_0^*(x_0))^2 x_0^*(y)x_0^*(y^2 x_0) - 3x_0^*(x_0)(x_0^*(y))^2 x_0^*(yx_0^2) + \\ &+ (x_0^*(y))^3 x_0^*(x_0^3) = (x_0^*(x_0))^3\left(\frac{x_0^*(y^2)}{2x_0^*(y)}\right)^2 \end{aligned}$$

and the proof is complete.

1.3. Lemma. *If $x_0^*\in\mathfrak{R}(X)$ and, for some $x_0\in X$,*

$$x_0^*(x_0)\neq 0, \quad 3\left(\frac{x_0^*(x_0^2)}{2}\right)^2 = x_0^*(x_0)x_0^*(x_0^3),$$

then the functional

$$g(x)=\begin{cases}\dfrac{x_0^*(xx_0)}{x_0^*(x_0)} & \text{if} \quad x_0^*(x)=0,\\[2ex] \dfrac{x_0^*(x^2)}{2x_0^*(x)} & \text{if} \quad x_0^*(x)\neq 0,\end{cases}$$

belongs to $\mathfrak{M}(x)$ and for every $x, y\in X$ the following equality holds:

$$x_0^*(xy) = x_0^*(x)g(y)+x_0^*(y)g(x).$$

Proof. It is evident that the equality $g(\lambda x) = \lambda g(x)$ holds for every $x\in X$ and every λ.

It is easy to see that if $x_0^*(y_0)\neq 0$, then

$$\frac{x_0^*(xy_0)}{x_0^*(y_0)} = \frac{x_0^*(xx_0)}{x_0^*(x_0)}$$

for every x such that $x_0^*(x) = 0$. Let $x_0^*(x) = x_0^*(y) = 0$; then

$$g(x) = \frac{x_0^*(xx_0)}{x_0^*(x_0)}, \quad g(y) = \frac{x_0^*(yx_0)}{x_0^*(x_0)}, \quad g(x+y) = \frac{x_0^*\big((x+y)x_0\big)}{x_0^*(x_0)}$$

and

$$g(x+y) = g(x)+g(y).$$

If $x_0^*(x) = 0, x_0^*(y)\neq 0$, then $x_0^*(x^2) = 0, x_0^*(x+y)\neq 0$ and

$$g(x)+g(y) = \frac{x_0^*(xx_0)}{x_0^*(x_0)}+\frac{x_0^*(y^2)}{2x_0^*(y)} = \frac{x_0^*(xy)}{x_0^*(y)}+\frac{x_0^*(y^2)}{2x_0^*(y)}$$

$$= \frac{x_0^*(x^2)+2x_0^*(xx_0)+x_0^*(y^2)}{2x_0^*(y)} = \frac{x_0^*\big((x+y)^2\big)}{x_0^*(x+y)} = g(x+y).$$

Now let $x_0^*(x)\neq 0, x_0^*(y)\neq 0$; then

1° for $x_0^*(x+y)\neq 0$

$$g(x)+g(y) = \frac{x_0^*(x^2)}{2x_0^*(x)}+\frac{x_0^*(y^2)}{2x_0^*(y)} = \frac{x_0^*(x^2)x_0^*(y)+x_0^*(y^2)x_0^*(x)}{2x_0^*(x)x_0^*(y)}$$

$$= \frac{x_0^*(x^2)+x_0^*(y^2)+\dfrac{\big(x_0^*(x)\big)^2x_0^*(y^2)+\big(x_0^*(y)\big)^2x_0^*(x^2)}{x_0^*(x)x_0^*(y)}}{2\big(x_0^*(x)+x_0^*(y)\big)}$$

$$= \frac{x_0^*\big((x+y)^2\big)}{2x_0^*(x+y)} = g(x+y);$$

2° for $x_0^*(x+y) = 0, x_0^*(x) = -x_0^*(y)$

$$g(x)+g(y) = \frac{x_0^*(x^2)}{2x_0^*(x)}+\frac{x_0^*(y^2)}{2x_0^*(y)} = \frac{x_0^*(x^2)x_0^*(y)+x_0^*(y_0^*)x_0^*(x)}{2x_0^*(x)x_0^*(y)}$$

$$= \frac{x_0^*(y^2)x_0^*(x)-x_0^*(x^2)x_0^*(x)}{2x_0^*(x)x_0^*(y)} = \frac{x_0^*\big((y-x)(y+x)\big)}{2x_0^*(y)}$$

$$= \frac{x_0^*\big(x_0(y+x)\big)}{2x_0^*(x_0)} = g(x+y).$$

Hence $g(x+y) = g(x)+g(y)$ for every $x, y\in X$.

The equality $g(x^2) = \big(g(x)\big)^2$ for x such that $x_0^*(x) = 0$ is a consequence of the equality

$$x_0^*\big[x\big(x_0^*(xx_0)x_0-x_0^*(x_0)xx_0\big)\big] = 0.$$

If $x_0^*(x)\neq 0$, then from Lemma 1.2 we have

$$x_0^*(x^4) = 4x_0^*(x)\left(\frac{x_0^*(x^2)}{2x_0^*(x)}\right)^3,$$

whence $g(x^2) = \big(g(x)\big)^2$ for $x_0^*(x^2)\neq 0$.

If $x_0^*(x)\neq 0, x_0^*(x^2) = 0$, then

$$g(x) = \frac{x_0^*(x^2)}{2x_0^*(x)} = 0, \quad g(x^2) = \frac{x_0^*(x^2x_0)}{x_0^*(x_0)} = \frac{x_0^*(x^3)}{x_0^*(x)} = \frac{3x_0^*(x)\left(\dfrac{x_0^*(x^2)}{2x_0^*(x)}\right)^2}{x_0^*(x)} = 0.$$

Consequently, the equality $g(x^2) = \big(g(x)\big)^2$ holds for every $x\in X$, and the functional g is linear and multiplicative.

We shall complete the proof if we show that

$$x_0^*(x^2) = 2x_0^*(x)g(x).$$

If $x_0^*(x) = 0$, then $x_0^*(x^2) = 0$ and $x_0^*(x^2) = 0 = 2x_0^*(x)g(x)$.

If $x_0^*(x)\neq 0$, then

$$2x_0^*(x)g(x) = 2x_0^*(x)\frac{x_0^*(x^2)}{2x_0^*(x)} = x_0^*(x^2),$$

q. e. d.

1.4. Lemma. *If $x_0^*\in\mathfrak{R}(X), x_0\in X, x_0^*(x_0)\neq 0$ and $x\in X, x_0^*(x)\neq 0$, then*

$$\frac{4x_0^*(x)x_0^*(x^3)-3\big(x_0^*(x^2)\big)^2}{\big(x_0^*(x)\big)^4} = \frac{4x_0^*(x_0)x_0^*(x_0^3)-3\big(x_0^*(x_0^2)\big)^2}{\big(x_0^*(x_0)\big)^4}.$$

Proof. In the proof of Lemma 1.2 we obtained the following equality:

$$(x_0^*(x_0))^3 x_0^*(x^3) = \frac{3}{4}\,\frac{(x_0^*(x_0))^3(x_0^*(x^2))^2}{x_0^*(x)} - \frac{3}{4}\,\frac{(x_0^*(x_0^2))^2(x_0^*(x))^3}{x_0^*(x_0)} + (x_0^*(x))^3 x_0^*(x_0^3).$$

Consequently

$$\frac{4x_0^*(x)x_0^*(x^3)-3(x_0^*(x^2))^2}{(x_0^*(x))^4}$$

$$= \frac{3(x_0^*(x_0))^3(x_0^*(x^2))^2}{(x_0^*(x_0))^3(x_0^*(x))^4} - \frac{3(x_0^*(x_0^2))^2(x_0^*(x))^4}{(x_0^*(x_0))^4(x_0^*(x))^4} + \frac{4(x_0^*(x))^4 x_0^*(x_0^3)}{(x_0^*(x_0))^3(x_0^*(x))^4} - \frac{3(x_0^*(x^2))^2}{(x_0^*(x))^4}$$

$$= \frac{4x_0^*(x_0)x_0^*(x_0^3)-3(x_0^*(x_0^2))^2}{(x_0^*(x_0))^4},$$

q. e. d.

Let $x \in X$ and $x_0^*(x) \neq 0$. Let us write

$$\lambda(x) = \frac{4x_0^*(x)x_0^*(x^3)-3(x_0^*(x^2))^2}{(x_0^*(x))^4}.$$

1.5. Lemma. *If $x_0^* \in \mathfrak{R}(X)$, $x_0 \in X$, $x_0^*(x_0) \neq 0$, then the functionals*

$$g_1(x) = \begin{cases} \dfrac{x_0^*(xx_0)}{x_0^*(x_0)} & \text{if } x_0^*(x) = 0, \\[2ex] \dfrac{x_0^*(x^2)+\sqrt{\lambda(x_0)}\,(x_0^*(x))^2}{2x_0^*(x)} & \text{if } x_0^*(x) \neq 0, \end{cases}$$

$$g_2(x) = \begin{cases} \dfrac{x_0^*(xx_0)}{x_0^*(x_0)} & \text{if } x_0^*(x) = 0, \\[2ex] \dfrac{x_0^*(x^2)-\sqrt{\lambda(x_0)}\,(x_0^*(x))^2}{2x_0^*(x)} & \text{if } x_0^*(x) \neq 0 \end{cases}$$

(where $\sqrt{\lambda(x_0)}$ is a fixed root of the constant $\lambda(x_0)$) are linear, multiplicative and $g_1 \neq g_2$ iff

$$3\left(\frac{x_0^*(x_0^2)}{2}\right)^2 \neq x_0^*(x_0)x_0^*(x_0^3).$$

Proof. The equalities

$$g_i(x+y) = g_i(x)+g_i(y), \qquad g_i(x^2) = (g_i(x))^2, \qquad i = 1, 2,$$

are trivial in the case of $x_0^*(x) = 0$, $x_0^*(y) = 0$.

Let us suppose that $x_0^*(x) = 0$, $x_0^*(y) \neq 0$. Then we have $x_0^*(x+y) \neq 0$ and

$$g_1(x)+g_1(y) = \frac{x_0^*(xx_0)}{x_0^*(x_0)} + \frac{x_0^*(y^2)+\sqrt{\lambda(x_0)}\,(x_0^*(y))^2}{2x_0^*(y)}$$

$$= \frac{x_0^*(xy)}{x_0^*(y)} + \frac{x_0^*(y^2)+\sqrt{\lambda(y)}\,(x_0^*(y))^2}{2x_0^*(y)}$$

$$= \frac{x_0^*(x^2)+2x_0^*(xy)+x_0^*(y^2)+\sqrt{\lambda(x+y)}\,(x_0^*(x+y))^2}{2x_0^*(y)}$$

$$= \frac{x_0^*((x+y)^2)+\sqrt{\lambda(x+y)}\,(x_0^*(x+y))^2}{2x_0^*(x+y)} = g_1(x+y).$$

If $x_0^*(x) \neq 0$ and $x_0^*(y) \neq 0$, then

$$g_1(x)+g_1(y) = \frac{x_0^*(x^2)+\sqrt{\lambda(x)}\,(x_0^*(x))^2}{2x_0^*(x)} + \frac{x_0^*(y^2)+\sqrt{\lambda(y)}\,(x_0^*(y))^2}{2x_0^*(y)}$$

$$= \begin{cases} \dfrac{x_0^*((y-x)(y+x))}{x_0^*(y-x)} = \dfrac{x_0^*(x_0(x+y))}{x_0^*(x_0)} & \text{if } x_0^*(x+y) = 0, \\[2ex] \dfrac{x_0^*((x+y)^2)+\sqrt{\lambda(x+y)}\,(x_0^*(x+y))^2}{2x_0^*(x+y)} & \text{if } x_0^*(x+y) \neq 0. \end{cases}$$

Consequently, the functional g_1 is additive and since it is homogeneous, it is also linear.

We shall now show that $g_1(x^2) = (g_1(x))^2$.

If $x_0^*(x) = 0$, then the last equality evidently holds. Let $x_0^*(x) \neq 0$, $x_0^*(x^2) \neq 0$; then

$$(g_1(x))^2 = \left[\frac{x_0^*(x^2)+\sqrt{\lambda(x)}\,(x_0^*(x))^2}{2x_0^*(x)}\right]^2, \qquad g_1(x^2) = \frac{x_0^*(x^4)+\sqrt{\lambda(x^2)}\,(x_0^*(x^2))^2}{2x_0^*(x^2)}$$

and it is sufficient to show that

$$x_0^*[(x_0^*(x))^2x^4-2x_0^*(x)x_0^*(x^2)x^3+(x_0^*(x^2))^2x^2] = 0.$$

But this equality immediately follows from the equality

$$x_0^*[(x_0^*(x)x^2-x_0^*(x^2)x)^2] = 0,$$

which evidently holds.

If $x_0^*(x) \neq 0$ and $x_0^*(x^2) = 0$, then

$$(g_1(x))^2 = \tfrac{1}{4}\lambda(x)(x_0^*(x))^2 = \frac{x_0^*(x^3)}{x_0^*(x)} = \frac{x_0^*(x^2x_0)}{x_0^*(x_0)} = g_1(x)^2$$

and the proof of the multiplicativity of g_1 is complete.

Similarly, we can prove that g_2 is linear and multiplicative. Now we find

$$g_1(x_0)-g_2(x_0) = \frac{\left(x_0^*(x_0)\right)^2\sqrt{\lambda(x_0)}}{x_0^*(x_0)} = x_0^*(x_0)\sqrt{\lambda(x_0)}.$$

If

$$3\left(\frac{x_0^*(x_0^2)}{2}\right)^2 \neq x_0^*(x_0)x_0^*(x_0^3),$$

then $\lambda(x_0)\neq 0$ and $g_1(x_0)-g_2(x_0)\neq 0$.

1.6. LEMMA. *If $x_0^*\in\mathfrak{N}(X)$, $x_0\in X$, $x_0^*(x_0)\neq 0$ and*

$$3\left(\frac{x_0^*(x_0^2)}{2}\right)^2 \neq x_0^*(x_0)x_0^*(x_0^3),$$

then

$$(1.6.1)\qquad g_1(x_0)\neq g_2(x_0),\qquad x_0^*(x) = \frac{x_0^*(x_0)}{g_1(x_0)-g_2(x_0)}\left(g_1(x)-g_2(x)\right).$$

Proof. If $x_0^*(x)\doteq 0$, then $g_1(x) = g_2(x) = 0$ and equality (1.6.1) is trivial.

Let $x_0^*(x)\neq 0$. Then, taking into account that $\lambda(x) = \lambda(x_0)$, we have

$$\frac{x_0^*(x_0)}{g_1(x_0)-g_2(x_0)}\left(g_1(x)-g_2(x)\right) = \frac{x_0^*(x_0)}{\dfrac{\sqrt{\lambda(x_0)}\left(x_0^*(x_0)\right)^2}{x_0^*(x_0)}}\cdot\frac{\sqrt{\lambda(x)}\left(x_0^*(x)\right)^2}{x_0^*(x)} = x_0^*(x),$$

q. e. d.

The proof of Theorem 1.1 is an immediate consequence of Lemmas 11.2-1.6, namely:

1° If $x_0^*\in\mathfrak{N}(X)$ and for every $x\in X$ the equality

$$3\left(\frac{x_0^*(x^2)}{2}\right)^2 = x_0^*(x)x_0^*(x^3)$$

holds, then from Lemmas 1.2 and 1.3 it follows that $x_0^*\in V(X)$.

2° If $x_0^*\in\mathfrak{N}(X)$ and there exists an $x_0\in X$ such that

$$3\left(\frac{x_0^*(x_0^2)}{2}\right)^2 \neq x_0^*(x_0)x_0^*(x_0^3),$$

then evidently $x_0^*(x_0)\neq 0$ and from Lemmas 1.4-1.6 it follows that

$$x_0^*(x) = c\left(g_1(x)-g_2(x)\right)$$

where

$$c = \frac{1}{\sqrt{\lambda(x_0)}}\neq 0\quad\text{and}\quad g_1, g_2\in\mathfrak{M}(X),\ g_1\neq g_2.$$

3° It is evident that if $x_0^* = c(g_1-g_2)$ where $g_1, g_2\in\mathfrak{M}(X)$, $g_1\neq g_2$, $c\neq 0$ or $x_0^*\in V(X)$, then $x_0^*\in\mathfrak{N}(X)$.

Theorem 1.1 gives a characterization of the class $\mathfrak{K}(X)$ by functionals of a certain special class $\mathfrak{N}(X)\subset X^*$.

There are two kinds of such functionals. A functional of the first class can be written in the form $c(g_1-g_2)$, where $g_1, g_2\in\mathfrak{M}(X)$, $g_1\neq g_2$, $c\neq 0$; a functional of the second class satisfies the equation

$$x_0^*(xy) = x_0^*(x)g(y)+x_0^*(y)g(x),$$

where g is a functional of $\mathfrak{M}(X)$.

The shape of functionals of the first kind does not require any further explanations. But it seems useful to consider in more detail the form of the second group of functionals, that is the functionals of the class $V(X)$.

Some results concerning the problem of the form of functionals where X is a B-algebra with one generator e_1 are given below. The problem of a full description of these functionals in the general case is reduced to the problem which is formulated at the end.

We begin with

2.1. THEOREM. *Let A be the algebra of all complex functions holomorphic in the disc $|z|<1$ and continuous for $|z|\leqslant 1$, with norm $x\in A$,*

$$\|x\| = \sup_{|z|\leqslant 1}|x(z)|,$$

and pointwise multiplication. Then

$$(2.1.1)\qquad V(A) = \{x^*\in A^*: \bigvee_{c\neq 0}\ \bigvee_{|z_0|<1}\ \bigwedge_{x\in A} x^*(x) = cx'(z_0)\}.$$

Proof. It is evident that

$$\{x^*\in A^*: \bigvee_{c\neq 0}\ \bigvee_{|z_0|<1}\ \bigwedge_{x\in A} x^*(x) = x'(z_0)\}\subset V(A).$$

Let $x^*\in V(A)$. Then there exists a $g_0\in\mathfrak{M}(A)$ such that for every $x, y\in A$

$$x^*(xy) = x^*(x)g_0(y)+x^*(y)g_0(x).$$

But $g_0(x) = x(z_0)$, where $|z_0|\leqslant 1$, whence

$$x^*\left((z^k)\right) = ckz_0^{k-1},\qquad k = 1, 2, \ldots,\qquad x^*\left((1)\right) = 0.$$

As $\|(z^k)\| = 1$, $k = 0, 1, 2, \ldots$, and $x^*\in A^*$, the sequence $\left(x^*((z^k))\right)$ is bounded. Consequently, $|z_0|<1$ and (2.1.1) follows from the fact that polynomials are dense in A, q. e. d.

2.2. COROLLARY. *$x_0^* \in \mathfrak{R}(A)$ if and only if there exist z_1 and z_2 such that $z_1 \neq z_2$, $|z_1| \leqslant 1$ and*

$$x_0^*(x) = \frac{x(z_1) - x(z_2)}{z_1 - z_2} x_0^*((z))$$

or there exists a $|z_0| < 1$ such that

$$x_0^*(x) = x_0^*((z)) x'(z_0).$$

2.3. THEOREM. *Let X be a B-algebra with unit e and with one generator e_1 and $z_0 \in \operatorname{Int} \operatorname{Sp}(e_1)$. Then there exists an $x_0^* \in X^*$ such that $x_0^*(e) = 0$, $x_0^*(e_1^k) = kz_0^{k-1}$, $k = 1, 2, \ldots$, and evidently $x_0^* \in V(x)$.*

Proof. For $z \in \operatorname{Sp}(e_1)$ let g_z be a functional such that $g_z \in \mathfrak{M}(X)$ and $g_z(e_1) = z$.

If $x \in X$, then the functional $h_x(z) = g_z(x)$ is continuous on $\operatorname{Sp}(e_1)$ and holomorphic on $\operatorname{Int}\operatorname{Sp}(e_1)$ since it is a Gelfand transform of x. Hence for $z_0 \in \operatorname{Int}\operatorname{Sp}(e_1)$ the function

$$x_0^*(x) = \left(\frac{dh_x(z)}{dz}\right)_{z=z_0}$$

satisfies the conditions of Theorem 2.3 and the proof is complete.

2.4. Definition. Let $z_0 \in \operatorname{Sp}(e_1)$ and let us suppose that for $x \in X$ there exists a

$$\lim_{\substack{z \to z_0 \\ z \in \operatorname{Sp}(e_1)}} \frac{g_z(x) - g_{z_0}(x)}{g_z(e_1) - g_{z_0}(e_1)},$$

where $g_z, g_{z_0} \in \mathfrak{M}(X)$ and $g_z(e_1) = z$, $g_{z_0}(e_1) = z_0$. This limit will be called a *derivative* of x at z_0 and we shall denote it by $x'(z_0)$.

2.5. COROLLARY. *If $z_0 \in \operatorname{Int}\operatorname{Sp}(e_1)$, then $x'(z_0)$ exists for every $x \in X$ and $e_1'(z_0) = 0$, $(e_1^k)'(z_0) = kz_0^{k-1}$, $k = 1, 2, \ldots$, $x'(z_0) \in V(X)$.*

The proof is an immediate consequence of Theorem 2.2.

2.6. COROLLARY. *Let $x^* \in V(X)$ and*

$$\frac{x^*(e_1^2)}{2x^*(e_1)} \in \operatorname{Int}\operatorname{Sp}(e_1).$$

Then for every $x \in X$ there exists an

$$x'\left(\frac{x^*(e_1^2)}{2x^*(e_1)}\right) \quad \text{and} \quad x^*(x) = x'\left(\frac{x^*(e_1^2)}{2x^*(e_1)}\right).$$

This remark is a consequence of Theorems 2.2 and 2.5.

It remains to consider the form of functionals $x^* \in V(X)$ such that

$$\frac{x^*(e_1^2)}{2x^*(e_1)} \notin \operatorname{Int}\operatorname{Sp}(e_1)$$

(evidently $x^*(e_1^2)/2x^*(e_1) \in \operatorname{Sp}(e_1)$).

It is easy to verify that there exists an $x^* \in V(X)$ such that the derivative

$$x'\left(\frac{x^*(e_1^2)}{2x^*(e_1)}\right)$$

does not exist for some $x \in X$ ([2]).

So the following problem arises:

PROBLEM 1. Let $x_0^* \in V(X)$. Do there exist a sequence of functionals g_n and a functional g_0 such that

(1) $$g_n, g_0 \in \mathfrak{M}(X), \quad g_n \neq g_0 \quad (n = 1, 2, \ldots),$$

(2) $$g_0(e_1) = \frac{x_0^*(e_1^2)}{2x_0^*(e_1)},$$

(3) $$g_n(x) \to g_0(x) \quad \text{for every } x \in X$$

and

$$x_0^*(x) = \lim_{n \to \infty} \frac{g_n(x) - g_0(x)}{g_n(e_1) - g_0(e_1)} x_0^*(e_1).$$

We shall now give some of the properties of the set $\mathfrak{R}(X)$.

2.7. THEOREM. *The set $\mathfrak{R}(X)$ is weakly sequentially closed.*

The proof is based on the following lemmas.

2.8. LEMMA. *Let $x_0^* \in \mathfrak{R}(X)$. There exists an $M > 0$ such that for every $x \in X$ the inequality $|x_0^*(x^2)| \leqslant M |x_0^*(x)| \, \|x\|$ holds.*

([2]) It is so for example in the algebra X of all functions $x \in A$ such that there exists a limit

$$\lim_{n \to \infty} \frac{x(z_n) - x(1)}{z_n - 1},$$

where (z_n) is a fixed sequence of numbers different from 1, converging to 1. Multiplication in X is as usual, and

$$\|x\| = \sup_{|z| \leqslant 1} |x(z)| + \sup_n \left| \frac{x(z_n) - x(1)}{z_n - 1} \right|.$$

It is easy to see that

$$x_0^*(x) = \lim_{n \to \infty} \frac{x(z_n) - x(1)}{z_n - 1} \in V(X)$$

but $x'(1)$ does not exist in the whole of X.

2.9. Lemma. *If* $x_0^* \in \mathfrak{R}(X)$, *then*

$$\pi(x_0^*) = \inf\{M > 0: \bigwedge_{x \in X} |x_0^*(x^2)| \leqslant M |x_0^*(x)|\,\|x\|\} \leqslant \|2e_1 - x_0^*(e_1^2)e\| \cdot \|x_0^*\|$$

We shall omit the proofs of lemmas.

Proof of Theorem 2.7. Let $x_n^* \in \mathfrak{R}(X)$ and $x_n^* \underset{w}{\to} x_0^*$; then

$$x_0^*(e) = \lim_{n\to\infty} x_n^*(e) = 0 \quad \text{and} \quad x_0^*(e_1) = \lim_{n\to\infty} x_n^*(e_1).$$

From Lemmas 2.8 and 2.9 we have

$$|x_n^*(x^2)| \leqslant \pi(x_n^*)\,|x_n^*(x)|\,\|x\| \leqslant \|2e_1 - x_n^*(e_1^2)e\|\,\|x_n^*\|\,\|x\|\,|x_n^*(x)|$$
$$\leqslant \|2e_1 - x_n^*(e_1^2)e\| \cdot M \cdot \|x\|\,|x_n^*(x)|$$

Hence from the last inequality we obtain (as $n \to \infty$)

$$|x_0^*(x^2)| \leqslant \|2e_1 - x_0^*(e_1^2)e\| \cdot M\,|x_0^*(x)|\,\|x\| \leqslant M^*\,|x_0^*(x)|\,\|x\|.$$

Consequently, if $x_0^*(x) = 0$, then $x_0^*(x^2) = 0$ too, which completes the proof.

2.10. Corollary. *We have*

$$\overline{\left\{x^* \in X^*:\ x^*(x) = x^*(e_1)\frac{g_1(x) - g_2(x)}{g_1(e_1) - g_2(e_1)},\ g_1, g_2 \in \mathfrak{M}(X),\ g_1 \neq g_2\right\}} \subset \mathfrak{R}(X),$$

where the line on the left-hand side denotes weak sequential closure.

Now we can reformulate Problem 1 as follows:

Problem 1′. Does the equality

$$\mathfrak{R}(X) = \overline{\left\{x^* \in X^*:\ x^*(x) = \frac{g_1(x) - g_2(x)}{g_1(e_1) - g_2(e_1)},\ g_1, g_2 \in \mathfrak{M}(X),\quad g_1 \neq g_2\right\}}$$

hold?

Problem 1 (or the equivalent Problem 1′) can be generalized in the following manner.

Let X be an arbitrary B-algebra with unit. Let

$$U(X) = \{x^* \in R(X):\ x^* = c(g_1 - g_2),\ g_1, g_2 \in \mathfrak{M}(X),\ g_1 \neq g_2,\ c \neq 0\}$$

(evidently $U(X) = \left\{x_0^* \in \mathfrak{R}(X):\ \bigvee_{x_0 \in X} x_0^*(x_0) \neq 0,\ 3\left(\frac{x_0^*(x_0^2)}{2}\right)^2 \neq x_0^*(x_0)x_0^*(x_0^3)\right\}$).

Problem 2. Is it true that $\overline{U(X)} = \mathfrak{R}(X)$ (where $\overline{U(X)}$ denotes, as in the previous case, weak sequential closure $U(X)$)?

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A modern version of the E. Noether's theorems in the calculus of variations, I

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INTRODUCTION

A growing interest in problems connected with the Noether's theorems can be noticed in the last years. It is a result of a popularity of the Lagrange approach to physical theories.

In the period of time following the original paper of E. Noether [10] there were written many works developing its subject (Bessel-Hagen [2], Rosenfeld [11]) or treating about some mathematical foundations (de Donder [4]). Since 1950 we have had a lot of papers due as well to mathematicians as to physicists concerning those problems (Hill [9], Bergman and Schiller [1], Trautman [16], Fletcher [6], Schmutzer [12], Edelen [5], Funk [7], Steidel [13] and [14], Trautman [17] and [18], Demmig [3]).

In spite of the fact that the Lagrange formalism has a geometrical sense the authors use at every level of considerations a coordinate system for a description of geometrical objects. (One of such geometrical objects is a function, i.e. a scalar field, which being defined at the points of a space can not be considered as a function of their coordinates.) Such concept does not make easier to set off the gist of the matter and sometimes leads to misunderstandings.

As we will see, the notions of a differentiable manifold, a vector bundle and a jet-bundle are very useful in geometrical formulating of the variational problems [1].

A general variation of a functional defined on cross-sections of a finite-dimensional vector bundle M is considered, wherein the variation of the functional is induced by a variation of the cross-section, i.e. a one-parameter "smooth" family of cross-sections. It is easy to see that all such variations of a cross-section can be given by differentiable vector

(1) It is pointed out also in the recently appeared paper, *Noether equations and conservation laws* by A. Trautman in Commun. Math. Phys. 6 (1967), p. 248-261 (*added in proof*).