assume $\mathscr{F} = \{I_n\}$ without affecting the set of points covered by $\mathscr{F}$. Now, if $E = (\bigcup I_n)^c$, then E is closed. Moreover $E \cap I \neq \emptyset$.

Suppose $E \cap I$ has an isolated point x_0. Then there is an open interval $I' \subseteq I$ such that $x_0 \in I'$ and $I' \cap E - \{x_0\} = \emptyset$. Let $G_n(x) = pF'(x)$ on I_n, and

$$E_{mn} = \{x\colon x \in I_m \cap I_n \text{ and } G_n(x) \neq G_m(x)\}.$$

Then, by Theorem 1, $|E_{mn}| = 0$ for all m and n. Define

$$G(x) = \begin{cases} G_n(x) & \text{if} \quad x \in I_n - \bigcup E_{mn}, \\ 0 & \text{if} \quad x \in (\bigcup I_n)^c \cup (\bigcup E_{mn}). \end{cases}$$

Then, clearly $G(x) = pF'(x)$ on I'. But $I' \cap E \neq \emptyset$, a contradiction.

Now suppose $E \cap I$ has no isolated point. Then there is an open $I' \subseteq I$ such that $pF'(x)$ exists on $P = I' \cap E \neq \emptyset$. Let $G_P(x) = pF'(x)$ on P. Let $H = E \cup (\bigcup E_{mn})$. Define

$$G(x) = \begin{cases} G_n(x) & \text{if} \quad x \in I_n - H, \\ G_P(x) & \text{if} \quad x \in P, \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly $G(x) = pF'(x)$ on I'. Hence $I' \in \mathscr{F}$. But $I' \cap E \neq \emptyset$, a contradiction. Thus $\mathscr{F}$ covers I. Define

$$f(x) = \begin{cases} G_n(x) & \text{if} \quad x \in I_n - (\bigcup E_{mn}), \\ 0 & \text{otherwise.} \end{cases}$$

Then f is well-defined and $pF'(x) = f(x)$ on I.

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The Stone-Čech operator and its associated functionals

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1.1. Introduction. The object of this work is to provide a realization of a certain Hilbert space of vector-valued sequences and to show how the structure obtained applies to a class of functionals on the space $\mathscr{L}(H)$. We use the symbol H to denote a separable Hilbert space, $\mathscr{L}(H)$ to denote the space of bounded linear transformations thereon, and m to denote the space of bounded complex-valued sequences.

1.1.1. Definition. A *generalized limit* is a bounded linear functional L on m which preserves the ordinary notion of convergence. That is, if $\lim(a_n) = a$, then $L((a_n)) = a$.

Generalized limits may be characterized as those continuous functionals which satisfy

1) $a_n \geqslant 0$ for all n implies $L((a_n)) \geqslant 0$.
2) $L((1)) = 1$, where $(1) = (1, 1, 1, \ldots)$.
3) If $a_n = b_n$ for $n \geqslant K$, then $L((a_n)) = L((b_n))$.

A stronger requirement than 3) is the *translation invariant* property:

4) $L((a_{n+1})) = L((a_n))$,

which we will assume only in special cases. The existence of generalized limits satisfying 1)-4) was proved by Banach [1].

1.2. Extensions and measures. It is well known that each completely regular topological space X possesses a Stone-Čech compactification βX with the property that X is densely embeddable in βX and every continuous function f mapping X into a compact space S possesses a continuous extension $f^\beta\colon \beta X \to S$. In particular, each bounded continuous complex-valued function has such an extension, and the correspondence $f \to f^\beta$ is an isometric isomorphism between $C_b(X)$ and $C(\beta X)$. Applying this to m (where the integers N are given the discrete topology), we see that m is isomorphic to $C(\beta N)$, that each sequence $(a_n) \in m$ has a continuous extension a^β defined in βN, and that

$$\sup_{n \in N} |a_n| = \sup_{t \in \beta N} |a^\beta(t)|.$$

This permits us to associate with each generalized limit L a measure λ on βN so that

$$L((a_n)) = \int_{\beta N} a^\beta(t)\,d\lambda(t).$$

It is easy to see that a measure λ on βN, in order to determine a generalized limit, must be a positive regular Borel measure, with $\lambda(\beta N) = 1$, whose support is contained in $\beta N - N$.

1.3. The space H_L. In the sequel we shall have occasion to deal with sequences (x_n) of vectors whose terms lie in a fixed Hilbert space H. If (x_n) and (y_n) are both bounded sequences in H, then their inner product sequence $((x_n, y_n))$ belongs to m. It makes sense, therefore, to consider L applied to such sequences, but rather than write $L\big(((x_n, y_n))\big)$, we will write simply $L(x_n, y_n)$. Likewise for $(a_n) \in m$ we write $L(a_n)$ instead of $L\big((a_n)\big)$. Finally, if z is a scalar or a vector, (z) will denote the constant sequence whose every term is z.

1.3.1. Definition. $H_L'' = \{x = (x_n) \mid x_n \in H$ for all n, and $\sup(\|x_n\|) < \infty\}$.

We make H_L'' into a vector space by defining operations termwise. We define an inner product by

$$[x, y] = L(x_n, y_n),$$

where L is a generalized limit. Thus we have a vector space with a positive semi-definite bilinear form thereon. We write $K = \{x \in H_L'' \mid L(\|x_n\|^2) = 0\}$, and set $H_L' = H_L''/K$. Then H_L' is a pre-Hilbert space, which we complete to form H_L. The construction and an application of this space are found in [2]. It is immediate that H_L contains a copy of H. The map $x \to (x)$ is an isometric embedding of H in H_L. We denote its image by (H).

Calkin [3] started with the set Q'' of sequences which converge weakly to zero in H. By the same process as above he constructs a space Q which he shows has dimension c. Evidently $Q \subset H_L$, so that $\dim H_L \geqslant c$. We can use his argument in the proof of the following remark.

1.3.2. Proposition. *The dimension of H_L is c.*

Proof. For $x \in H_L''$ we have $x_i = \Sigma_n (x_i, e_n) e_n$, where (e_n) is a sequence of basis elements in H. (We will reserve the symbol (e_n) for such a sequence.) Thus each x determines a matrix $a_{in} = (x_i, e_n)$ of complex numbers. There are c such matrices. When we factor by K and complete to form H_L, the cardinality of the resulting set is at most $c^{\aleph_0} = c$. Thus $\dim H_L \leqslant c$.

1.4. Vector sequence extensions. Each bounded sequence (x_n) of vectors in H may be viewed as a continuous function on N whose range is contained in a weakly compact set B. By the remarks of 1.2, there exists a weakly continuous extension function $x^\beta\colon \beta N \to B$. Evidently,

$$\sup_{t \in \beta N} \|x^\beta(t)\| \leqslant \sup \|x_n\|,$$

and the reverse inequality is immediate from the fact that x^β extends (x_n). This suggests the possibility of employing the Stone extension to define an operator on H_L which would map it in an isometric fashion onto a Hilbert space of functions, thus representing H_L in a more convenient fashion. The natural candidate for the range space is $L_2(\lambda, \beta N, H)$, the space of norm-square λ-integrable functions from βN to H. The inner product is

$$(f \mid g) = \int_{\beta N} \big(f(t), g(t)\big)_H\, d\lambda(t),$$

and λ would be the measure determined by L. As the following example shows, however, the Stone extension cannot be fully isometric. Let $y = (y_n) \in Q''$, so that $y_n \to 0$ (weakly) in H. Then for each $w \in H$, $a_n = (y_n, w)$ defines a sequence which converges to zero. The extension a^β must vanish on $\beta N - N$. Indeed, if there were $t_0 \in \beta N - N$ with $|a^\beta(t_0)| > \varepsilon > 0$, there would be an open set S about t_0 with $|a^\beta(t)| > \varepsilon$ whenever $t \in S$. Since N is dense in S, there are infinitely many integers in S, and thus $|a^\beta(n_i)| = |a_{n_i}| \geqslant \varepsilon$ for a subsequence n_i, and this is impossible.

Now, by weak continuity, $a^\beta(t) = \big(y^\beta(t), w\big)$ so that $\big(y^\beta(t), w\big) = 0$ for all w and all $t \in \beta N - N$. It follows that $y^\beta = 0$ on $\beta N - N$ and hence is zero λ-almost everywhere. If $y = (e_n)$, we have $[y, y] = L(e_n, e_n) = L(1) = 1$, while $(y^\beta \mid y^\beta) = 0$.

1.4.1. Lemma. $L_2(\lambda, \beta N, H) \cong L_2(\lambda, \beta N) \otimes H \cong \Sigma \oplus L_2(\lambda, \beta N)$ *where $L_2(\lambda, \beta N)$ is the usual square-integrable complex-valued functions on βN, and the last summation is a countable direct sum.*

Proof. Let $M_n = \{f e_n \mid f \in L_2(\lambda, \beta N)\}$. The sets M_n can be identified with the spaces $\langle e_n \rangle \otimes L_2(\lambda, \beta N)$ by the map $f e_n \leftrightarrow \langle e_n \rangle \otimes f$. (Here and in the sequel, $\langle S \rangle$ denotes the closed linear span of the set $S \subset H$.) Thus

$$\begin{aligned} H \otimes L_2(\lambda, \beta) &= [\Sigma_i \oplus \langle e_n \rangle] \otimes L_2(\lambda, \beta N) \\ &= \Sigma_i \oplus [\langle e_n \rangle \otimes L_2(\lambda, \beta N)] = \Sigma_n \oplus M_n. \end{aligned}$$

In order to show that $L_2(\lambda, \beta N, H) = \Sigma_n \oplus M_n$, we have to show that if $G \in L_2(\lambda, \beta N, H)$ and $G \perp M_n$ for all n, then $G = 0$ a.e. For such a G:

$$\int_{\beta N} \big(f(t) e_n, G(t)\big)\, d\lambda(t) = \int_{\beta N} f(t) \big(e_n, G(t)\big)\, d\lambda(t) = 0$$

for every $f \in L_2(\lambda, \beta N)$ and every n. This says $\big(e_n, G(t)\big) = 0$ almost everywhere for every n, and hence, by completeness of the basis set

(e_n), $G(t) = 0$ a.e. The correspondence for $\Sigma\oplus L_2(\lambda, \beta N)$ follows from the obvious identification between M_n and $L_2(\lambda, \beta N)$.

1.4.2. LEMMA. *The (strongly continuous) functions of the form* $\sum_{i=0}^{K} f_i(t)e_i$, *where each* $f_i \in C(\beta N)$, *are dense in* $L_2(\lambda, \beta N, H)$.

Proof. The proof is simply a two stage approximation, using the fact that $C(\beta N)$ is dense in $L_2(\lambda, \beta N)$ and $L_2(\lambda, \beta N, H) \simeq \Sigma\oplus L_2(\lambda, \beta N)$.

1.4.3. LEMMA. *If* $x \in H_L''$, *then* $x^\beta \in L_2(\lambda, \beta N, H)$.

Proof. We know that x^β is norm bounded. We also observe that $\|x^\beta(t)\|^2$ is the monotone limit of the functions $\sum_{i=1}^{K} |(x^\beta(t), e_i)|^2$, each of which is continuous, hence measurable. Thus the integral $\int_{\beta N} \|x^\beta(t)\|^2 d\lambda(t)$ exists and is finite.

Now we wish to define a manifold of sequences which are controlled well enough for the Stone extension to operate isometrically on them. The definition is independent of any generalized limit, but the manifold will ultimately be closed in H_L and the closure will depend on L.

1.4.4. Definition. $U = \{x \in H_L'' \mid \forall \varepsilon > 0, \exists K \ni. \sup_n \sum_{i=K}^{\infty} |(x_n, e_i)|^2 < \varepsilon\}$.

It will become apparent that the definition is independent of the particular choice of basis.

1.4.5. LEMMA. (a) *U is a submanifold of H_L'' containing* (H).

(b) *If $x \in U$ and $y \in H_L''$, then the series* $(x_n, y_n) = \Sigma_i (x_n, e_i)(e_i, y_n)$ *converges uniformly in n.*

(c) *A necessary and sufficient condition that $\langle U\rangle = H_L$ is that H be finite-dimensional.*

Proof. (a) That U is a submanifold is immediate. The constant sequences are clearly U-sequences and these are precisely the ones that we identify with H.

(b) If $\|y_n\| < M$ for all n, we have

$$\sum_{i=K}^{\infty} |(x_n, e_i)(e_i, y_n)| \leqslant \Big[\sum_{i=K}^{\infty} |(x_n, e_i)|^2\Big]^{1/2} \Big[\sum_{i=K}^{\infty} |(y_n, e_i)|^2\Big]^{1/2}$$
$$\leqslant \Big[\sum_{i=K}^{\infty} |(x_n, e_i)|^2\Big]^{1/2} M,$$

so that the tail of the original series can be made uniformly small.

(c) If H has dimension D, write

$$x_n = \sum_{i=1}^{D} (x_n, e_i) \quad \text{for every } (x_n).$$

This is evidently a U-sequence, so $H_L = \langle U\rangle$.

Conversely, if H is infinite-dimensional, let $y = (e_n)$. Then $\|y\| = 1$, and if $x = (x_n) \in U$,

$$\lim_n (x_n, y_n) = \lim_n \Sigma_i (x_n, e_i)(e_i, y_n)$$
$$= \sum_i \lim_n (x_n, e_i)(e_i, y_n) \qquad \text{(by (b))}$$
$$= 0.$$

Thus $[x, y] = L(x_n, y_n) = \lim (x_n, e_n) = 0$ and $y \in U^\perp$.

The same argument as above applies to any sequence which converges weakly to zero, and so we see that $Q \subseteq U^\perp$. We will naturally assume H is infinite-dimensional.

1.5. THEOREM. *The following statements are equivalent:*

(a) $x = (x_n) \in U$.

(b) (x_n) *is contained in a strongly compact set in* H.

(c) x^β: $\beta N \to H$ *is strongly continuous.*

(d) $\|x^\beta(t)\|^2 = \Sigma_i |(x^\beta(t), e_i)|^2$ *converges uniformly on* βN.

(In the above, we are viewing the correspondence $x \to x^\beta$ as defined on individual bounded sequences, and not on elements of H_L' or H_L.)

Proof. (a) is equivalent to (b) by a well known characterization of strongly compact sets in Hilbert space.

(b) implies (c). If (x_n) is contained in a *strongly* compact set S, there is a *strongly* continuous extension x^α: $\beta N \to S$. For each $w \in H$, and $n \in N$

$$(x^\alpha(n), w) = (x_n, w) = (x^\beta(n), w).$$

Thus the continuous functions $(x^\alpha(t), w)$ and $(x^\beta(t), w)$ coincide on the integers and hence on βN. We have $x^\alpha = x^\beta$, and in this case x^β is strongly continuous.

(c) implies (d). The function $\|x^\beta(t)\|^2$ is continuous and is the monotone pointwise limit of the continuous partial sums of the series. Dini's Theorem shows that the convergence must be uniform.

(d) implies (a). For each $\varepsilon > 0$, we have a K so that

$$\sup_{t \in \beta N} \sum_{i=K}^{\infty} |(x^\beta(t), e_i)|^2 < \varepsilon.$$

In particular, on the subset $N \subset \beta N$

$$\sup_{n \in N} \sum_{i=K}^{\infty} |(x^\beta(n), e_i)|^2 = \sup_{n \in N} \sum_{i=K}^{\infty} |(x_n, e_i)|^2 < \varepsilon,$$

whence $(x_n) \in U$.

1.5.1. COROLLARY. *The definition of U is independent of any choice of basis.*

1.5.2. COROLLARY. *The Stone extension maps $U \subset H_L''$ onto a dense subset of $L_2(\lambda, \beta N, H)$.*

Proof. The strongly continuous functions are dense in $L_2(\lambda, \beta N, H)$ and each, when restricted to N evidently determines a U-sequence. Since the extension is unique, we are done.

Now we will show that the Stone extension induces a well defined operator from $H_L \to L_2(\lambda, \beta N, H)$. To extend the map algebraically from H_L'' to H_L' we need to show that if $L(\|x_n\|^2) = 0$, then $x^\beta = 0$ almost everywhere. To get from H_L' to H_L we need boundedness on the former space. Both of these requirements are fulfilled by the following theorem:

1.5.3. THEOREM. *The Stone extension mapping H_L'' into $L_2(\lambda, \beta N, H)$ is norm reducing.*

Proof. First, if $x \in U$ and $y \in H_L''$, then we have $(x, y)^\beta(t) = (x^\beta(t), y^\beta(t))$. That is to say, the extension of the inner product sequence (x_n, y_n) is just the inner product of the two vector extension functions. This derives from the fact that $(x^\beta(t), y^\beta(t))$ is a continuous complex-valued function. For, let $s \in \beta N$ be fixed. Then for $t \in \beta N$:

$$\begin{aligned}&|(x^\beta(s), y^\beta(s)) - (x^\beta(t), y^\beta(t))|\\ &\qquad \leqslant |(x^\beta(t) - x^\beta(s), y^\beta(t))| + |(x^\beta(s), y^\beta(t) - y^\beta(s))|\\ &\qquad \leqslant \|x^\beta(t) - x^\beta(s)\| \cdot \|y^\beta(t)\| + |(x^\beta(s), y^\beta(t) - y^\beta(s))|.\end{aligned}$$

By strong continuity of x^β, and by the boundedness of y^β, the norm expression above can be made arbitrarily small. Also, since y^β is weakly continuous, the last term on the right can be made small.

Thus, whenever $x \in U, y \in H_L''$ we have

$$(*) \qquad [x, y] = L(x_n, y_n) = \int_{\beta N} (x^\beta(t), y^\beta(t))\, d\alpha(t) = (x^\beta | y^\beta)$$

and in particular $x \to x^\beta$ is isometric on U. Moreover

$$\|y^\beta\| = \sup_{\|f\|=1} |(f|y^\beta)| = \sup_{\substack{\|x^\beta\|=1\\ x \in U}} |(x^\beta|y^\beta)| = \sup_{\|x^\beta\|=1} \left| \int_{\beta N} (x^\beta(t), y^\beta(t))\, d\alpha(t) \right|$$

$$= \sup_{\substack{\|x^\beta\|=1\\ x \in U}} |L(x_n, y_n)| \leqslant \|x\| \cdot \|y\| = \|y\|.$$

Now we can define the Stone extension on H_L' because if $L(\|y_n\|^2) = 0$, then $\|y^\beta\| = 0$ as well and so $y^\beta = 0$ a.e. Since the map is bounded on H_L', we can extend it by continuity to the completion H_L. The map which results from this two-stage process will still be denoted by $x \to x^\beta$, and we will call it the *Stone-Čech operator*.

Equation $(*)$ says that the Stone-Čech operator preserves inner products as long as one term is in U, or, equivalently, has strongly continuous extension.

1.5.4. THEOREM. *The Stone-Čech operator is a partial isometry from H_L onto $L_2(\lambda, \beta N, H)$. It is isometric on $\langle U \rangle$ and annihilates $U^\perp$.*

Proof. We know that $\langle U \rangle$ will be mapped isometrically onto $L_2(\lambda, \beta N, H)$ by the Stone-Čech operator. Equation $(*)$ will extend by continuity so that for arbitrary $x \in \langle U \rangle$ and $y \in H_L$ we have $(x^\beta|y^\beta) = [x, y]$. If, in particular, $y \in U^\perp$, we have $(x^\beta|y^\beta) = 0$ for every $x \in \langle U \rangle$. But this implies $y^\beta = 0$.

1.5.5. Remarks. We have seen that $H_L = \langle U \rangle \oplus U^\perp$, where $\langle U \rangle \cong L_2(\lambda, \beta N, H)$ and $Q \subseteq U^\perp$. Although the dimension of $U^\perp$ is always c, the dimension of $\langle U \rangle$ varies with L. If L is a point mass, i.e. $L(a_n) = a^\beta(t_0)$, where $(a_n) \in m$ and $t_0 \in \beta N - N$, then $\dim L_2(\lambda, \beta N) = 1$, and $L_2(\lambda, \beta N, H) \cong L_2(\lambda, \beta N) \otimes H \cong H$. That is to say, the only U-sequences are essentially constant. On the other hand, if L is translation invariant, Douglas [5] has shown that $\langle U \rangle$ is non-separable.

2.1. The representation $A \to A^0$. Let $A \in \mathscr{L}(H)$. For each $x = (x_n) \in H_L''$, we define $A^0(x_n) = (Ax_n)$. Thus A^0 maps H_L'' into H_L'' and it is immediate that $A \to A^0$ determines a $*$-isomorphism of $\mathscr{L}(H)$ into $\mathscr{L}(H_L)$. The definition is identical to the one given by Berberian [2] and by Calkin [3], except that the latter applied it to Q only.

2.1.1. PROPOSITION. (H), $\langle U \rangle$ *and Q are all reducing subspaces for A^0 when $A \in \mathscr{L}(H)$.*

Proof. The proposition is a statement of the fact that each operator (and its adjoint) preserves constant sequences, compact sets, and weak convergence to zero.

2.1.2. PROPOSITION. *A^0 maps H_L into $\langle U \rangle$ iff A is a compact operator.*

Proof. If A is compact, A maps bounded sets into pre-compact sets. Since every $(x_n) \in H_L''$ is contained in a bounded set, $A^0(x_n) = (Ax_n)$ will be contained in a compact set. Thus $A^0\colon H_L'' \to U$ and so $A^0\colon H_L \to \langle U \rangle$.

Conversely, suppose $A^0\colon H_L \to \langle U \rangle$. Let $x_n \to 0$ weakly in H. Then $A^0(x_n) = (Ax_n) \in U$ and $Ax_n \to 0$ weakly. But this implies $Ax_n \to 0$ strongly, because

$$\|Ax_n\|^2 = \sum_{i=1}^{K-1} |(Ax_n, e_i)|^2 + \sum_{i=K}^{\infty} |(Ax_n, e_i)|^2$$

and we may make the second term small by choosing K sufficiently large. The first term converges to zero as n increases, so $Ax_n \to 0$ strongly.

2.1.3. COROLLARY. *The restriction of A^0 to $U^\perp$ is zero when A is compact.*

2.2. On linear functionals. In [4], Dixmier showed that every continuous functional f on $\mathscr{L}(H)$ may be written $f = g+h$, where $g \in I_0^*$, the dual of the space of compact operators and h annihilates all compact operators. The elements of I_0^* are just the functionals continuous in the ultra-strong topology on $\mathscr{L}(H)$. Further, $\|f\| = \|g\|+\|h\|$, where $\|g\|$ is by definition $\sup\{|g(A)|: \|A\| = 1, A \text{ compact}\}$, and $\|h\|$ is just the norm of h as an element of $\mathscr{L}(H)^*$. Since $I_0^{**} = \mathscr{L}(H)$, $I_0^{***} = \mathscr{L}(H)^*$, and each functional on I_0 may be viewed as a functional on $\mathscr{L}(H)$ by the canonical embedding. This embedding is isomorphic, and so $\|g\|$ is the same whether we think of g in I_0^* or in $\mathscr{L}(H)^*$.

Dixmier completely analyzed the g's, but to date nothing is known about the structure of the h's. As an example of such a "compact annihilator" functional, we have $h(A) = L(Ae_i, e_i)$, where L is a generalized limit and (e_i) is the sequence of basis elements. If we write $e = (e_i)$, and form H_L, then we have $h(A) = [A^0 e, e]$. We now proceed to study functionals of the form $f(A) = [A^0 x, y]$ where $x, y \in H_L$, and $[\cdot,\cdot]$ is the inner product in that space.

By linearity, it suffices to consider the case where A is Hermitian, and by polarization it is enough to consider symmetric functionals $[A^0 x, x]$.

2.2.1. THEOREM. *$[A^0x, x] = [A^0x_1, x_1]+[A^0x_2, x_2]$, where $x_1 \in \langle U\rangle$, and $x_2 \in U^\perp$. The decomposition is the same as Dixmier's in that $g(A) = [A^0x_1, x_1]$ is ultra-strongly continuous, and $h(A) = [Ax_2, x_2]$ annihilates all compact operators.*

Proof. The decomposition exists because $\langle U\rangle$ is a reducing subspace for A^0 and the mixed terms vanish. The norm equality follows since for $f(A) = [A^0x, x]$ we have

$$\|f\| = \|x\|^2 = \|x_1\|^2+\|x_2\|^2 = \|g\|+\|h\|.$$

We have also seen in 2.1.3 that h annihilates all compact operators. What remains to be shown is that g is ultra-strongly continuous. Note that if we apply the basic decomposition theorem to g, we have $g = g_1+g_2$, where g_1 is ultra-strongly continuous. Since $\|g\| = \|g_1\|+\|g_2\|$, if we can show that $\|g\|$ is approximated by values of $|g(A)|$ when A is in the unit ball of the compacts we will be done. For then $\|g\| = \|g_1\|$ and $g_2 = 0$. This leads to the following lemma:

2.2.2. LEMMA. *Let $x = (x_i) \in U$, and let $f(A) = L(Ax_i, x_i)$. Then there is a sequence of norm one compact operators (A_j) so that $|f(A_j)| \to \|f\|$.*

Proof. A_j will be the projection on the space spanned by the first j basis elements. Then

$$x_i = \Sigma_n(x_i, e_n)e_n, \qquad A_jx_i = \sum_{n=1}^{j}(x_i, e_n)e_n,$$

$$(A_jx_i, x_i) = \sum_{n=1}^{j}|(x_i, e_n)|^2.$$

For j sufficiently large and $\delta > 0$,

$$\|x_i\|^2 - \sum_{n=1}^{j}|(x_i, e_n)|^2 < \delta$$

for all i. By continuity of L as a functional on m, we have

$$\big|\|x\|^2 - [A_j^0x, x]\big| = \big|L(\|x_i\|^2) - L(A_jx_i, x_i)\big| = \Big|L(\|x_i\|^2) - L_i\sum_{n=1}^{j}|(x_i, e_n)|^2\Big| < \varepsilon.$$

Since $\|A_j\| = 1$, and the A_j are all compact, we need only observe that $\|f\| = \|x\|^2$.

2.2.3. COROLLARY. *Let $g(A) = [A^0x, x]$, where $x \in \langle U\rangle$ (not necessarily a sequence). Then $\|g\|$ can be approximated, in the sense of the above lemma, by compact operators.*

Proof. The proof is accomplished by approximating the arbitrary element $x \in \langle U\rangle$ by a sequence $y = (y_i)$. The operators A_j above will work for both y and x.

The proof of 2.2.1 is completed now, since 2.2.3 implies the ultra-strong continuity of g.

2.3. On $\langle U\rangle$-functionals. If L is any generalized limit, and (x_i) is a bounded sequence in H, then $\varphi(y) = L(y, x_i)$ defines a bounded linear functional in H. Thus there exists a unique $w \in H$ so that $\varphi(y) = L(y, x_i) = (y, w)$. The correspondence $(x_i) \to w$ may be viewed as a mapping of $H_L'' \to (H) \subseteq H_L''$ provided we associate with the $w \in H$ above the constant sequence $(w) = (w, w, w, \ldots)$. Then $(x_i) \to (w)$ is linear and has norm one:

$$\|w\| = \|\varphi\| = \sup_{\|y\|=1}|L(y, x_i)| \leqslant \sup_{\|y\|=1}\|y\|[L(\|x_i\|^2)]^{1/2} = \|x\|.$$

We define $L^0(x_i) = (w)$ and call L^0 the extension of L to bounded vector sequences. Viewed in the context of its own H_L, L^0 annihilates $U^\perp$, acts in a termwise fashion on U, i.e.

$$L^0(x_i) = \sum_{n=1}^{\infty} L(x_i, e_n)$$

when $(x_i) \in U$, and is in fact the projection on (H). L^0 extends by continuity to $\langle U\rangle$.

Schatten [8] develops I_0^* as the ideal of trace class operators and shows that for every $g \in I_0^*$ there exists a unique $\mathfrak{A}$ belonging to the trace class so that $g(A) = \operatorname{tr}(A\mathfrak{A})$ for every $A \in I_0$ (and for every $A \in \mathscr{L}(H)$ if we regard g as defined thereon). We may now completely characterize the $\langle U\rangle$-functionals.

2.3.1. THEOREM. *Let $g(Ax_i, x_i)$, where $(x_i) \in U$. Then, if we represent $g(A) = \operatorname{tr}(A\mathfrak{A})$, and identify x with (x), we have $\mathfrak{A}x = L^0(x, x_i)x_i$ for each $x \in H$.*

Proof. For each $v, w \in H$, define the (compact) operator $\{v, w\}$ on H by $\{v, w\}z = (z, v)w$. Then [4] we have

$$(\mathfrak{A}w, v) = g(\{v, w\}) = L(\{v, w\}x_i, x_i)$$
$$= L\big((x_i, v)w, x_i\big) = L[(x_i, v)(w, x_i)].$$

Let $\mathfrak{B}$ be defined on H by $\mathfrak{B}z = L^0(z, x_i)x_i$. Then for every $v, w \in H$ we have

$$(\mathfrak{B}w, v) = \big(L^0(w, x_i)x_i, v\big) = L\big((w, x_i)x_i, v\big)$$
$$= L[(w, x_i)(x_i, v)] = (\mathfrak{A}w, v)$$

so that $\mathfrak{A} = \mathfrak{B}$ as claimed.

In order to extend the theorem to $\langle U\rangle$, one employs the weak integral $\int^{w}$ of a continuous vector-valued function in a Hilbert space and shows that L^0 on $\langle U\rangle = L_2(\lambda, \beta N, H)$ is given by

$$L^0(x) = \int_{\beta N}^{w} x^\beta(t)\,d\lambda(t).$$

Then

$$\mathfrak{A}z = \int_{\beta N}^{w} \big(z, x^\beta(t)\big)x^\beta(t)\,d\lambda(t).$$

The computations are almost identical in pattern with those above, but one uses the image under β of the representation $A \to A^0$ on $L_2(\lambda, \beta N, H)$, $A^0[f(t)] = [Af(t)]$ (a.e.). Thus for each $x \in \langle U\rangle$ in H_L, the functional $g(A) = [A^0x, x]$ may be represented by $\operatorname{tr}(A\mathfrak{A})$, where $\mathfrak{A}$ is given by the weak integral above.

2.4. On $U^\perp$-functionals. If $x \in U^\perp$, then the functional $h(A) = [A^0x, x]$ annihilates all compact operators. Otherwise put, the representation $A \to A^0$ (restricted to $U^\perp$) is a continuous homomorphism of $\mathscr{L}(H)$ whose kernel is precisely I_0. Calkin [3] produced the first explicit representation satisfying this latter criterion, and his space Q is a subspace of $U^\perp$. We are naturally concerned with the question of whether the spaces are equal. In one simple case they are the same.

2.4.1. Definition. A point $p \in \beta N - N$ is called a *P-point* if every countable intersection of neighbourhoods of p contains a neighbourhood of p.

For the following facts, the reader is referred to [6] and [7]. Assuming the continuum hypothesis, P-points exist in $\beta N - N$, in fact they are a dense subset. The definition implies that a continuous function which vanishes at a P-point must vanish on a neighborhood of the point. Finally, [6], the open and closed sets in $\beta N - N$ for a base for the open sets in $\beta N - N$, and every open and closed set A' in $\beta N - N$ may be represented as $A' = (\mathrm{cl}_{\beta N}A) - N$, where A is an infinite subset of N.

2.4.2. THEOREM. *Let L be a P-point mass, i.e. $L(a_n) = a^\beta(t_0)$ for some P-point $t_0 \in \beta N - N$. Then the spaces Q and $U^\perp$ determined by L coincide.*

Proof. First, suppose (z_n) is a *sequence* in $U^\perp$. More formally, (z_n) lies in an equivalence class in $U_L' \subset H_L$, one of whose representatives lies in H_L'. We will show that (z_n) is equivalent to a sequence which converges weakly to zero, and hence lies in $Q'' \subset Q$. Since $(z_n) \in U^\perp$, we know that the Stone-Čech extension annihilates it, that is $z^\beta(t_0) = 0$. Thus the functions $a_k(t) = \big(z^\beta(t), e_k\big)$ each vanish at t_0, and hence on a neighborhood D_k of t_0. The countable intersection of the D_k's contains a neighborhood of t_0, and so there exists an open and closed set A' about t_0, on which all the functions $a_k(t)$ vanish. Say that $A' = (\mathrm{cl}_{\beta N}A) - N$, and define

$$w_n = \begin{cases} z_n & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let $b_n = \|z_n - w_n\|^2$. Then

$$b_n = \begin{cases} 0 & \text{if } n \in A, \\ \|z_n\|^2 & \text{otherwise.} \end{cases}$$

The continuous function $b^\beta(t)$ clearly vanishes on the closure of A in βN, and thus $L(\|z_n - w_n\|^2) = L(b_n) = b^\beta(t_0) = 0$, since $t_0 \in A'$. This shows (z_n) and (w_n) are equivalent modulo L. Consider the sequence $c_n = (w_n, e_k)$ where e_k is a fixed basis vector. The continuous functions $w^\beta(t)$ and $z^\beta(t)$ coincide on A' and hence $\big(w^\beta(t), e_k\big)$ vanishes on A'. This shows $c^\beta(t) = 0$ whenever $t \in A'$. On the other hand, if $t_1 \in \beta N - N$ and $t_1 \notin A'$, then t_1 cannot belong to the closure of A in βN. There is thus an open set isolating t_1 from A and a net $\{n_a\} \to t_1$, $\{n_a\} \subset N - A$. Thus

$$c^\beta(t_1) = \lim_a c^\beta(n_a) = \lim_a \big(w^\beta(n_a), e_k\big) = (0, e_k) = 0.$$

This shows that the sequence (c_n) has an extention $c^\beta(t)$ which vanishes on $\beta N - N$. It is easy to see that this implies $\lim(c_n) = 0$. Thus $\lim(w_n, e_k) = 0$ for each k, and since (w_n) is a bounded sequence, this proves $w_n \to 0$ weakly in H.

Now by construction, sequences (modulo the kernel of L) are dense in H_L. If P and R are the projections on $\langle U\rangle$ and $U^\perp$ respectively, it follows that the set $\{R(x_i)\}$ for (x_i) in H_L' is dense in $U^\perp$. But $R(x_i) = (x_i) - P(x_i)$ and for L a P-point mass, $\langle U\rangle \cong H$. In other words,

$P(x_i)$ is essentially constant and $R(x_i) = (x_i) - (w)$, $w \in H$. But then $R(x_i)$ is a sequence in $U^\perp$ and hence equivalent to an element of Q''. Thus the set Q'' is dense in $U^\perp$ and $U^\perp$ is the closure of the weakly convergent to zero sequences, i.e. $U^\perp = Q$.

In case L is any finite convex combination of point masses, the conclusion of the above theorem and its method of proof still hold. The $U^\perp$-sequences are still equivalent to sequences which converge weakly to zero, and the projection on $\langle U \rangle$ must yield sequences only, since every function in $L_2(\lambda, \beta N)$ is a.e. equivalent to a continuous function and $U = \langle U \rangle$.

It seems reasonable to suspect that a generalized limit for which $Q \neq U^\perp$ would be in some sense dual to the point mass limits, and this is the case. Professor R. Raimi has communicated to the author a construction of a translation invariant generalized limit L for which $Q \neq U^\perp$ in H_L. The following construction is somewhat simpler, although we can say little about the resulting limit.

2.4.3. THEOREM. *There exists a generalized limit L for which the spaces Q and $U^\perp$, regarded as subspaces of H_L are distinct.*

Proof. Let $\Sigma(i) = i - ([\sqrt{i}])^2 + 1$, where $[\cdot]$ denotes the greatest integer function. It is easy to check that the function Σ assumes every integer infinitely often. Let $y_i = e_{\Sigma(i)}$. Consider the following sets in m:

$$A = \{(a_i) \in m \mid \text{For some } e_n\colon (a_i) = ((y_i, e_n))\},$$
$$B = \{(b_i) \in m \mid \text{For some vector sequences } (x_i) \text{ with } x_i \to 0 \text{ (weakly)}, (b_i) = ((y_i, x_i))\}.$$

Thus A is the set of basis component sequences of (y_i), while B is a linear manifold. Let M be the closed linear span of A, B, and c_0, the scalar sequences converging to zero. We claim (1) lies in the complement of M. Suppose not. Then for $\varepsilon > 0$, there are numbers $k_1, \ldots, k_n$, a sequence (x_i) converging weakly to zero, and an element $(c_i) \in c_0$ so that

$$\sup_i \Big| \sum_{j=1}^{n} k_j (y_i, e_j) + (y_i, x_i) + c_i - 1 \Big| < \varepsilon.$$

Since $c_i \to 0$, we can choose n_0 so that

$$\sup_{i > n_0} \Big| \Big(y_i, \sum_{i=1}^{n} k_j e_j + x_i\Big) - 1 \Big| < 2\varepsilon.$$

Further, y_i assumes the value e_{n_0+n} for every i in an infinite set $S \subset N$, which we may assume lies beyond n_0. Then

$$\sup_{i \in S} |(y_i, x_i) - 1| = \sup_{i \in S} |(e_{n_0+n}, x_i) - 1| < 2\varepsilon.$$

Since ε was arbitrary and $x_i \to 0$ weakly, this is impossible.

Now let L be a bounded linear functional defined everywhere on m which vanishes on M and is equal to 1 on (1). We have immediately:

(a) $L(\|y_i\|^2) = L(\|e_{\Sigma(i)}\|^2) = L(1) = 1$.

(b) For each J: $L(|(y_i, e_J)|^2) = L(y_i, e_J) = 0$.

(c) If $x_i \to 0$ weakly, then $L(y_i, x_i) = 0$.

Now, viewing $y = (y_i)$ as an element of H_L (for indeed, L is a generalized limit) we see that (a) says $\|y\| = 1$, (b) implies $y \in U^\perp$, and (c) says that $y \in Q^\perp$.

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