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On normed lattices
topologically isomorphic to some Orlicz space \( L^\varphi \)

by

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1. Introduction. Let \( \mu \) be a non-atomic, completely additive measure on a set \( \Omega \) with \( \mu(\Omega) = 1 \).

The Orlicz space \( L^\varphi(\Omega, \mu) \) consists of all real-valued functions \( x(t) \), \( \mu \)-measurable on \( \Omega \), such that

\[
\int_{\Omega} \varphi(a|x(t)|) \, d\mu < +\infty \quad \text{for some real number } a > 0,
\]

where \( \varphi \) is an \( N \)-function which satisfies \((\Delta_2)\)-condition (1). Then, the space \( L^\varphi \) is not only a Banach space with the norm (2)

\[
\|x\| = \inf \{ \|f\|; \varphi(\|f\|) \leq \|x\| \},
\]

also becomes a conditionally complete vector lattice (3) by the usual ordering.

In the preceding paper [7], we gave a characterization of \( L^\varphi \). The purpose of the present paper is to characterize \( L^\varphi \) under the topological equivalence without containing the function \( \Phi \) in the condition by which \( L^\varphi \) is characterized.

We shall easily see that an \( N \)-function has an equivalent \( X \)-function with the continuous derivative. Therefore, we shall assume in this section that \( \Phi \) is continuously differentiable. Then, the modular norm on \( L^\varphi \) is

\[
(1) \quad \lim_{t \to \pm \infty} \Phi(\|f\|) = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} \Phi(\|f\|) = +\infty \quad ((9), p. 9).
\]

(2) This norm is called the modular norm or Luxemburg norm.

(3) A continuous convex function \( \Phi \) is said to be \( X \)-function if

\[
\Phi(f(\|t\|) = 0 \quad \text{and} \quad \Phi(\|f\|) = +\infty \quad ((9), p. 9).
\]

(4) A vector lattice \( E \) is said to be conditionally complete, if for \( E \cdot a \geq 0 \text{ for } a \in A \) there exists \( a \in E \) such that \( A = \bigcap_{a \in A} g_a \).
smooth and monotone (9). Furthermore, we can see that for each \( x(t) \in L^*_p \) there exists only one \( \mathbb{E}(t) \in L^*_p(\Omega, \mu) \), \( \Psi \) is the complementary \( N \)-function, for which the equality in the Young’s inequality holds, i.e.,

\[
\int \mathbb{E}(t) \Psi(t) d\mu = \int \Phi(|x(t)|) d\mu + \int \Psi(|x(t)|) d\mu.
\]

Indeed, \( \mathbb{E}(t) = \varphi(|x(t)|) \text{sgn} x(t) \) where \( \varphi \) is the derivative of \( \Phi \) (cf. [13], Theorem 39.1, and [21], p. 64). Hence, we obtain a transformation \( T \) from \( L^*_p \) into \( L^*_p \) through the correspondence \( x(t) \rightarrow \mathbb{E}(t) = \varphi(|x(t)|) \text{sgn} x(t) \).

This transformation \( T \) has the following properties:

1. \( 0 \leq x \leq y \) implies \( 0 \leq T x \leq T y \),
2. \( (T x)[p] = T([p] x) \) for any projector \([p]\) (9),
3. \( T(-x) = -T x \).

Let \( R \) be a conditionally complete vector lattice, and \( \bar{R} \) be its conjugate space, i.e., the totality of all linear functionals \( f \) on \( R \) for which

\[
\inf_{x \in R} |f(x)| = 0
\]

for any system \( (s_n; \lambda \in A) \) in \( R \) with \( s_n \downarrow 0_{R} \). A transformation \( T \) from \( R \) into \( \bar{R} \), with conditions (i)-(iii) is said to be conditionally similar \((12), (13), p. 294)\).

Recently, the present author and Yamaudro [5] have shown the following theorem:

Let \( R \) be a conditionally complete vector lattice possessing a norm with \( |x| \leq |y| \) implies \( ||x|| \leq ||y|| \), which has at least two linearly independent elements and its conjugate norm be strictly convex. If there exists a one-to-one conjugately similar transformation \( T \) from \( R \) into its conjugate \( \bar{R} \) with the condition

\[
(x, T x) = ||x|| \cdot ||T x|| \quad (x \in R),
\]

then \( R \) is of \( L_p \)-type \((p > 1)\).

(9) The norm on the normed space \( X \) is said to be smooth, if at every point of the unit sphere of \( X \) there is only one supporting hyperplane of the unit sphere of \( X \). This is equivalent to the Gateaux differentiability of the norm \([8]\). The norm on the normed lattice \( X \) is said to be monotone, if \( 0 \leq x \leq y \) implies \( ||x|| \leq ||y|| \) for \( x, y \in X \). If \( \Phi \) satisfies the \((\lambda)\)-condition and \( \Phi > 0 \) for each \( \lambda > 0 \), then the modular norm is monotone \((13), (13), p. 33)\).

(9) For the support \( \mathcal{E}(t) \) of an element \( x(t) \in L^*_p \), the projector \([p]\) is defined by \([p] x(t) = x_p \text{sgn} x(t)\), where \( x_p \) is the characteristic function of \( \mathcal{E} \). In a conditionally complete vector lattice \( R \), the projector \([p]\) is defined by \([p] x = \sum_{\|x\|} (x \wedge x_p)\) if \( x > 0 \), and \([p] x = \sum_{\|x\|} (x \wedge -x_p)\) for any \( x \in R \), where \( x^* = x \wedge 0 > -x^* = (\neg x)^* \text{ and } |x| = x^* + x^* \). For \( x \in L^*_p \), if \( \Phi \) is a linear functional on \( R \) such that \( (y, \Phi x) = \int \Phi(y) x d\mu \) for all \( y \in L^*_p \). See also footnote (9).

(9) \( (y, \Phi) \) means the value of \( \Phi x \) at \( y eR \).

In the Orlicz space \( L^*_p \), a similar behavior to \( L_p \)-space may be seen. For \( x \in L^*_p \) with \( ||x|| = 1 \), we denote by \( x^* \) the element in the conjugate space of \( L'_p \), with the norm 1, for which the equality in Hölder’s inequality holds, i.e., \( (x, x^*) = ||x|| \cdot ||x^*|| \). This \( x^* \) determines uniquely for \( x \), because of the smoothness of the norm on \( L'_p \).

Then, we shall be able to see the following property:

For any step element \( x \in L^*_p \) (i.e., a simple function), with the norm 1, and for any subprojector \([p]\) of \( x \) (i.e., a projector satisfying \([p] x(t) = \text{const. for the simple function } x\) the equality in Hölder’s inequality in the form

\[
[(p) y, x^*([p])] = ||[p]\|| \cdot ||x^*([p])||
\]

holds.

Indeed, let \( x \in L^*_p \) be the function \( k(t) = 1 \) on \( \Omega \) a.e. In general, for the conjugately similar transformation \( T \),

\[
\begin{align*}
L^*_p \ast [p], [p] = 1, & \rightarrow T x = \varphi([p]) \text{sgn} x([p]) \ast L^*_p, \\
\end{align*}
\]

the relation

\[
(x, T x) = ||T x||
\]

holds and hence we have \( x^* = T x ||T x|| \). Now, expressing \( x \) in (*) by a form

\[
(x, T x) = \varphi(\sum_{i=1}^{n} \xi_i([p]) \ast \xi_i([p]) T x + \sum_{i=1}^{n} \varphi(\sum_{i=1}^{n} \xi_i([p]) \ast \xi_i([p])) T x),
\]

where \([p]\) and \([p_i]\) (\( i = 1, 2, \ldots, n \)) are mutually orthogonal projectors \((9)\), we have, by the property (ii),

\[
T x = \varphi(\sum_{i=1}^{n} \xi_i([p]) \ast \xi_i([p])) T x + \sum_{i=1}^{n} \varphi(\sum_{i=1}^{n} \xi_i([p]) \ast \xi_i([p])) T x
\]

so that

\[
||[p] x|| = \varphi(\sum_{i=1}^{n} \xi_i([p]) \ast \xi_i([p])) ||T x||
\]

and further

\[
\begin{align*}
\left( [p] x, \varphi \left( \frac{1}{||[p] x||} T x \right) \right) &= \left( [p] x, \varphi \left( \frac{1}{||[p] x||} T x \right) \right) \\
&= \left( \frac{1}{||[p] x||} T x, \varphi \left( \frac{1}{||[p] x||} T x \right) \right) \\
&= \frac{1}{||[p] x||} ||T x||
\end{align*}
\]

namely, (*) is satisfied.

(9) This fact is obtained from (9) and [14], Theorem 3.2.1.

(9) Projectors \([p]\) and \([q]\) are called mutually orthogonal if \([p] [q] = [p] \wedge [q] = 0 \).
To show that the property (*) is a characteristic property of \( L^*_s \) under the topological equivalence, we shall prepare in the next section.

2. Throughout this section, let \( R \) be a normed lattice which has the following properties:

(i) \( R \) is non-atomic and conditionally \( \sigma \)-complete (\( ^{24} \)),
(ii) the norm \( \| \cdot \| \) on \( R \) is semi-continuous, i.e.,
\[
0 \leq a_n \uparrow a (a_n, a \in R) \quad \text{implies} \quad \| a_n \| \uparrow \| a \|, \\
\]
(iii) the norm on \( R \) is smooth and monotone,
(iv) \( R \) has a positive complete element \( s \) with \( \| s \| = 1 \), i.e., no element in \( R \) is orthogonal to \( s \),
(v) \( \sup \{ \| p \| : p \} = +\infty \), where \( \{ p \} \) is any orthogonal partition of \( s \), and also there exists a positive integer \( k \), such that for any \( \{ p \} \) orthogonal partitions \( \{ p \} = \sum_{i=1}^{k} [p_i s] \), with \( \| [p_1 s] \| = \| [q_1 s] \| = \ldots = \| [q_k s] \| \), imply \( \| [p_i s] \| \leq \| [p_j s] \| \) for \( i = 1, 2, \ldots, k \).

Remark. It is easily verified that the Orlicz space \( L^*_s \) in section 1 satisfies property (v) from the facts that \( \Phi \) satisfies (\( \Delta_2 \)) condition and \( \Phi(1/\| [p] \|) = 1/\mu(F) \), where \( F \) is the support of \( [p] \).

An element \( x \) in \( R \) is called a step element, if \( x \) is of the form \( \sum_{i=1}^{k} x_i [p_i] s \) for certain orthogonal subsystem \( \{ p_i \} \) \( i = 1, 2, \ldots, n \) of projectors in \( R \). For a step element, we shall call \( \sum_{i=1}^{k} x_i [p_i] s \) the projector \( p \) such that \( \{ p \} = \{ p_i \} \) for some real number \( x \).

We denote again the main notation used in this paper.

\( R \) is the conjugate space of \( R \); \( S \) is the unit surface of \( R \), i.e., the set \( \{ x \in R : \| x \| = 1 \} ; B \) is the set of all step elements in \( R \); \( x, y \) means the value of \( xy \) at \( x \in R \); \( x^* \) means, for \( x \in R \), the element on the unit surface of \( R \) for which the equality in the Hölder’s inequality holds, i.e., \( (x, x^*) = \| x \| \cdot \| x^* \| \); \( y^* \) for any projector \( p \) in \( R \) and \( x \in R \), denotes the element of \( R \) such that \( y^* (p) = \{ p \} y, x^* \) for all \( y \in R \).

For mutually orthogonal elements \( a_i \in S \) \( i = 1, 2, \ldots, n \), the functions
\[
\xi_k = f_k (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n) \quad (k = 1, 2, \ldots, n)
\]
are called the represented functions of an n-dimensional indicatrix \( C(a_1, a_2, \ldots, a_n) \) \( ^{(26)} \) of \( R \).

Moreover, Greek letters \( \xi, \eta, \gamma, \ldots \) denote the real numbers or real functions and small Latin letters \( a, b, x, \ldots \) denote the elements in \( R \).

We shall first give two lemmas concerning the properties of the indicatrix, which connect with \( \xi \) and \( \eta \).

**Lemma A.** Each represented function \( \xi_k = f_k (\xi_1, \ldots, \xi_n) \) of an n-dimensional indicatrix \( C(a_1, a_2, \ldots, a_n) \) of \( R \) is partially differentiable with respect to the variable \( \xi_k (\pi \neq k) \). Here, the differentiation at the end point in the domain of \( f_k \) means the one-side differentiation.

**Proof.** Since the norm on \( R \) is smooth, when we denote the right and left derivatives by \( D^+ f_k (\xi_1, \ldots, \xi_n) \) and \( D^- f_k (\xi_1, \ldots, \xi_n) \) respectively, we have
\[
\left\{ a_i + (D^+ f_k (\xi_1, \ldots, \xi_n)) a_i, a^* \right\} = \left\{ a_i + (D^- f_k (\xi_1, \ldots, \xi_n)) a_i, a^* \right\} = 0
\]
by the same method as in \( \xi \), Lemma 2, and \( \xi \), where
\[
\sum_{k=1}^{n} \xi_k a_k = x \in S
\]
If \( (a_i, a^*) = 0 \), we have \( (y, a^*) = 1 \) for \( y = \sum \xi_k a_k \) so that
\[
1 = \| a^* \| = \left( \frac{y}{\| y \|}, a^* \right) > 1
\]
provided that \( \xi_k \neq 0 \), because \( \| y \| = \sum \xi_k a_k + \| y \| > \| y \| \) by the monotony of the norm. This is impossible and consequently we have
\[
\frac{\partial f_k}{\partial \xi_k} = \left( a_i, a^* \right)
\]
(6)

It is obvious that \( \frac{\partial f_k}{\partial \xi_k} = 0 \) at \( \xi_k = 0 \).

**Lemma B.** For a represented function \( \xi_k = f_k (\xi_1, \ldots, \xi_n) \), let us assume that \( a_i \xi_1 + \ldots + f_k (\xi_1) a_{i_k} + \ldots + \xi_k a_k \in S \) and that \( \xi_k \) is variable and \( \xi_j \) \( (j \neq k) \) is fixed. Then \( a_i, x^* \) is non-decreasing function in \( \xi_k \) \( \xi_k \neq 0 \).

**Proof.** It is enough to prove the case in which the indicatrix is 2-dimensional, by reason of which the proof in the n-dimensional case is essentially the same as that in 2-dimensional case. Let \( \eta = \eta (\xi) \) be a represented function of an indicatrix \( C(a, b) \) of \( R \) with respect to \( a, b \in S \) with \( a \cap b = 0 \).

\( ^{(26)} \) The notion of an indicatrix has been introduced in \( \xi \), p. 342.
By definition, for $0 < \xi < 1$,

$$
\{ a, (fa + \eta(\xi)b)^* \} = \lim_{\varepsilon \to 0} \frac{\| (\xi + \varepsilon)a + \eta(\xi)b \| - 1}{\varepsilon}.
$$

First, we shall prove, for each small $\varepsilon > 0$, that a function

$$
g(\xi, \varepsilon) = \| (\xi + \varepsilon)a + \eta(\xi)b \| - 1
$$

is non-decreasing in $0 \leq \xi \leq 1$. Put, for $0 < \xi < 1$,

$$
D_{\varepsilon}^{\xi} g(\xi, \varepsilon) = \lim_{\delta \to 0} \frac{1}{\delta} \left( \left\| (\xi + \delta + \varepsilon)a + \eta(\xi + \delta)b \right\| - \left\| (\xi + \varepsilon)a + \eta(\xi)b \right\| \right).
$$

Taking enough small $\delta$ and $\varepsilon$, with $0 < \delta < \varepsilon_1 < \varepsilon$, by virtue of Lemma A the derivative $\eta'(\xi)$ exists and is non-increasing by the concavity of $\eta(\xi)$.

Accordingly, we have for some $0 < \theta < 1$

$$
g(\xi + \delta, \varepsilon) = \left\| (\xi + \delta)a + \eta(\xi)b + \delta(a + \eta'(\xi + \delta)b) \right\| - 1
\geq \left\| (\xi + \delta)a + \eta(\xi)b + \theta(a + \eta'(\xi + \varepsilon_1)b) \right\| - 1
= (a + \eta'(\xi + \varepsilon_1)b, \sigma'),
$$
and hence

$$
D_{\varepsilon}^{\xi} g(\xi, \varepsilon)
\geq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \left\| (\xi + \varepsilon)a + \eta(\xi)b \right\| - \left\| (\xi + \varepsilon)a + \eta(\xi)b \right\| \right)
= (a + \eta'(\xi + \varepsilon_1)b, \sigma'),
$$
where

$$
e = \frac{(\xi + \varepsilon)a + \eta(\xi)b}{\| (\xi + \varepsilon)a + \eta(\xi)b \|}.
$$

Putting

$$
\lambda = \frac{\xi + \varepsilon}{\| (\xi + \varepsilon)a + \eta(\xi)b \|} \quad \text{and} \quad \mu = \frac{\eta(\xi)}{\| (\xi + \varepsilon)a + \eta(\xi)b \|},
$$

the point $(\lambda, \mu)$ is on the indicatrix $O(a, b)$. When we take again $\varepsilon_1$ such that $0 < \varepsilon_1 < \varepsilon(1 - \xi)/(1 + \varepsilon)$, then, on account of $\| (\xi + \varepsilon)a + \eta(\xi)b \| \leq 1 + \varepsilon$, it follows that $\xi + \varepsilon < (\xi + \varepsilon)/(1 + \varepsilon) \leq \xi$. Consequently, we have, by (8) and Lemma A, $D_{\varepsilon}^{\xi} g(\xi, \varepsilon) \geq (a + \eta'(\xi)b, \sigma') = 0$ which shows $g(\xi, \varepsilon)$ is non-decreasing in $0 \leq \xi \leq 1$. Therefore, by (7), $\{ a, (fa + \eta(\xi)b)^* \}$ is non-decreasing in $0 \leq \xi \leq 1$.

3. THEOREM. Let $R$ be the normed lattice which has properties (i)-(v) in the preceding section. If $R$ satisfies the following condition:

for any step element $a \in S \cdot E$ and for any sub-step projector $[p]$ of $a$,

$$
(*) \quad \left\| [p]a \right\| \leq \left\| [p]a \right\| \cdot \left\| a^*(p) \right\|,
$$

holds, then $R$ is topologically isomorphic to some Orlicz space $L^p$, the modular norm on which has properties (ii), (iii), (iv), and (v).

The central part of the proof of Theorem is to construct a function $\Phi$ which determined the Orlicz space $L^p$. Therefore, we shall begin to give the lemmas by which $\Phi$ is constructed and its properties are proposed.

In what follows, suppose that $R$ satisfies the condition in Theorem.

For any $\xi \geq 0$, we define a function $f(\xi)$ as

$$
f(\xi) = \sup \left\{ \frac{\| [p]a \|}{\| [p]a \|} \right\} \quad \text{for any} \quad a \in S \cdot E \quad \text{and} \quad [p]a = [p]a.
$$

and hence

$$
f(\xi) = 0 \quad \text{if} \quad \xi = 0.
$$

Remark. $0 \neq (p)a$ and $a \in S$ imply $(p)a = 0$. Indeed, if $(p)a = 0$, then $a = (p)a = 0$ and hence $\| a \| = 1$ contradicting the monotony of the norm.

**Lemma 1.** There exists a positive constant $\beta$ such that for arbitrary $a \in S \cdot E$, with $a = \xi[p]a + \sum_{i=1}^{n} \xi_i[p_i]a$, $0 < \| [p]a \| < 1$ and $\xi > 0$,

$$
\frac{\| a^*(p) \|}{\| [p]a \|} \leq f(\xi) \leq \frac{\| a^*(p) \|}{\| [p]a \|}.
$$

Proof. The left side inequality is obvious from the definition of $f(\xi)$. Suppose that $S \cdot E/a = \xi[p]a + \sum_{i=1}^{n} \xi_i[p_i]a$. Then, we have

$$
\left( \frac{[p]a}{\| [p]a \|} \right)^* = \frac{a^*(p)}{\| [p]a \|},
$$

and hence, by virtue of the smoothness of $E$,

$$
\frac{a^*(p)}{\| [p]a \|} = \frac{\| [p]a \|}{\| [p]a \|} \cdot \frac{\| a^*(p) \|}{\| [p]a \|},
$$

because $\| a^*(p) \| = 0$ from the above remark.

Therefore, for any $0 \neq [q] \subset [p]$, $\| a^*(p) \| = \| a^*(q) \| = \| [p]a \|$ and consequently

$$
\frac{\| a^*(q) \|}{\| [p]a \|} = \frac{\| a^*(p) \|}{\| [p]a \|} \quad \text{for every} \quad 0 \neq [q] \subset [p].
$$

Next, we shall prove that there exist two positive constant $A$ and $B$ such that for every elements $x$ and $y$ in $S \cdot E$,

$$
A \leq \frac{\| x \|}{\| y \|} \leq B,
$$

where $\xi > 0$, $x = \xi[p]a + u$, $y = \xi[q]a + v$, $[p]a = [q]a = 0$, $\| [p]a \| = \| [q]a \|$, and $0 \neq u, v \in E$.  

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If such constants do not exist, on account of (12), there exist some elements \(a_n\) and \(y_n\) in \(S \cdot E\) which satisfy the following relations:

\[
\begin{align*}
0 &< a_n = \xi_n[p_n] + \eta_n b_n, \quad b_n \in S \cdot E, \quad [p_n] b_n = 0, \\
0 &< y_n = \zeta_n[y_n] + \xi_n y_n, \quad d_n \in S \cdot E, \quad [y_n] d_n = 0, \\
1 &> \|a_n[p_n]\| = \|\eta_n[y_n]\| = \xi_n^{-1}, \quad \xi_n > 0
\end{align*}
\]

and

\[
\|\eta_n[y_n]\| = g(n) \|a_n[p_n]\| \quad \text{with} \quad g(n) \uparrow n + \infty.
\]

For simplicity, we put \([p_n] r / \|[p_n]\| = a_n\), \([y_n] r / \|y_n\| = y_n\). For the represented functions \(\eta = \eta_n(t)\) and \(\xi = \xi_n(t)\) of the indicatrix \(C(a_n, b_n)\) and \(C(c_n, d_n)\) respectively, as is shown in (6), we have then

\[
\left[ \frac{dy_n}{dt} \right]_{t=0} = - \frac{(a_n, y_n^n)}{X_n} \quad \text{and} \quad \left[ \frac{dx_n}{dt} \right]_{t=0} = - \frac{(d_n, y_n^n)}{Y_n}
\]

and hence

\[
0 \leq - \left[ \frac{dx_n}{dt} \right]_{t=0} = \frac{1-g(n)}{\xi_n(t)} \left( \frac{\eta_n(t)}{\xi_n(t)} \right) + \frac{d\eta_n}{dt} \left[ \frac{dy_n}{dt} \right]_{t=0},
\]

because \(1 + \eta_n(t) b_n + \xi_n(t) y_n = 1 + \xi_n(t) y_n = \xi_n(t) 1 = \xi_n(t) X_n + \xi_n(t) y_n = 1\) and \(Y_n = g(n) X_n\) by condition (5) in the theorem.

On the other hand, it is easily seen that for enough large \(n\),

\[
1 - t_n < \eta_n(t_n), \quad \xi_n(t_n) > 0, \quad 0 < - \left[ \frac{d\eta_n}{dt} \right]_{t=0} < 1, \quad 0 < Y_n = X_n g(n)
\]

and \(
\lim_{n \to \infty} \left[ \frac{d\eta_n}{dt} \right]_{t=0} = - \infty.
\)

Consequently, we have

\[
0 < - \left[ \frac{dx_n}{dt} \right]_{t=0} \leq \frac{1-g(n)}{(1-t_n) Y_n} + \frac{1}{1-t_n} < 0
\]

for enough large \(n\), which is impossible. Thus, there exists a constant \(A > 0\) satisfying (13). By (12), (13) and the definition of \(f(\xi)\), we can see that Lemma 2, with \(\beta = E\).

**Lemma 2.** The function \(f(\xi)\) defined in (9) is a real-valued, non-decreasing function in \(\xi \geq 0\) and more \(f(\xi) > 0\) for \(\xi > 0\).

**Proof.** It is evident, by Lemma 1, that \(f(\xi)\) is real-valued and \(f(\xi) > 0\) for \(\xi > 0\) from the remark for the definition of \(f(\xi)\). Suppose \(0 < \xi_i < \xi_j\). We choose a projector \([p]\) such that \(0 < \|\xi_i[p]\| \leq 1\) \((i = 1, 2)\). Moreover, we consider \(a_i \in S \cdot E\) \((i = 1, 2)\) such that

\[
\eta_i = \xi_i[p] + z \eta_i[y_i] + \eta_i[y_i],
\]

where \([p], [y_i]\) \((k = 1, 2, \ldots, n)\) and \([y]\) are mutually orthogonal. Then, if we fix \(a_i\) \((k = 1, 2, \ldots, n)\), the represented functions \(\eta_i(\xi)\|\eta_i[y_i]\| = \lambda_i(\xi_i, \eta_i, \ldots, \eta_i)\) are differentiable at \(\xi_i(1 = 1, 2)\) respectively; namely, derivatives \(\eta_i'(\xi)\) exist by Lemma A. Therefore, by Lemma B, we have

\[
\frac{\partial\|\eta_i[p]\|}{\partial\|\eta_i[y]\|} \leq \frac{\|\eta_i[p]\|}{\|\eta_i[y]\|}.
\]

Hence, in virtue of the condition (4), we have

\[
\frac{\partial\|\xi_i[p]\|}{\partial\|\xi_i[y]\|} \leq \frac{\|\xi_i[p]\|}{\|\xi_i[y]\|} \quad \text{and more} \quad \frac{\|\xi_i[p]\|}{\|\xi_i[y]\|} \leq \frac{\|\xi_i[p]\|}{\|\xi_i[y]\|}.
\]

Therefore, for each element \(x = \xi_i[p] S + \eta_i[y], \|\xi_i[p]\| \leq 1\), \(p = [p]\), and \(\|\xi_i[p]\| \leq [p]\|\|\eta_i[y]\|\|\eta_i[y]\| = \xi_i[p]|[p]|\|\eta_i[y]\|\|\eta_i[y]\|)

we have

\[
\frac{\partial\|\xi_i[p]\|}{\partial\|\xi_i[y]\|} \leq \frac{\|\xi_i[p]\|}{\|\xi_i[y]\|} \leq \frac{\|\xi_i[p]\|}{\|\xi_i[y]\|}.
\]

**Lemma 3.** For the convex function

\[
M(\xi) = \int f(\xi) dt,
\]

there exists a convex function \(\Phi(\xi)\), equivalent to \(M(\xi)\), such that

(i) the derivative of \(\Phi(\xi)\) is continuous,

(ii) \(\lim \Phi(\xi)/\xi = 0\) and \(\lim \Phi(\xi)/\xi = +\infty\),

(iii) \(\Phi(\xi) > 0\) for \(\xi > 0\).

**Proof.** Putting

\[
\Phi(\xi) = \frac{1}{2} M(\xi)
\]

we have

\[
\frac{1}{2} M(\xi) \leq \Phi(\xi) \leq M(\xi) \quad \text{for} \quad \xi > 0.
\]
so that $\Phi(\xi)$ is equivalent to $M(\xi)$. It is evident that $\Phi$ satisfies (i) and (iii). In order to prove that $\Phi$ satisfies (ii), it will suffice to prove that
\[ \lim_{t \to 0} f(\xi) = 0 \quad \text{and} \quad \lim_{t \to \infty} f(\xi) = +\infty. \]

For $0 < \xi < 1$, choosing $x_\xi = \xi[x_\xi] + \eta(x)[\xi]x_\xi E$, with $[\xi][\eta] = 0$, we have
\[ \lim_{t \to 0} f(\xi) = \lim_{t \to 0} \|x_\xi\|^2 \|x_\xi\|^2 = \lim_{t \to 0} \|x_\xi\|^2 \|x_\xi\|^2 = \left( \|x_\xi\|^2 \left( \|x_\xi\|^2 \|x_\xi\|^2 \right) = 0. \]

Next, taking $x_{\xi} = \xi[x_{\xi}] + \eta(x)(\xi x_{\xi} E)$ for each $\xi > 1$, it follows from the property (v) for $R$ that $\|x_{\xi}^2\| = \|x_{\xi}^2\| = +\infty$. Therefore, we have, by Lemma 1,
\[ \lim_{t \to 1^+} f(\xi) = \lim_{t \to 1^+} \|x_{\xi}^2\| = \lim_{t \to 1^+} \|x_{\xi}^2\| = +\infty. \]

4. The proof of Theorem. We shall make use of the spectral theory of H. Nakano [12; §§ 8-13 and §§ 20-23] and [13; Chap. III]. Therefore, we state at the moment several results obtained by H. Nakano.

Let $\mathcal{S}$ be the proper space of $R$, i.e., the compact Hausdorff space consisting of all maximal ideals in the space of all projections in $R$ which form a Boolean algebra with respect to the set operation, i.e.,
\[ U_{\pi \mathcal{S}} U_{\eta \mathcal{S}} = U_{\pi \mathcal{S} \cap \eta \mathcal{S}} \quad \text{and} \quad U_{\pi \mathcal{S}} U_{\eta \mathcal{S}} = U_{\pi \mathcal{S} \cap \eta \mathcal{S}} \quad [12; \text{p. 32}]. \]

For $x \in R$, the function $(a|\pi, \mathcal{S})$ on $\mathcal{S}$ is defined by
\[ \left\{ \begin{array}{l}
\|x\|^2, \\
+\infty, \\
-\infty,
\end{array} \right. \begin{array}{l}
\text{if} \quad \mathcal{S} \subseteq \prod_{\pi \in \mathcal{S}} (U_{\pi \mathcal{S}} \cup \eta \mathcal{S}), \\
\text{if} \quad \mathcal{S} \subseteq \prod_{\pi \in \mathcal{S}} (U_{\pi \mathcal{S}} \cap \eta \mathcal{S}), \\
\text{if} \quad \mathcal{S} \subseteq \prod_{\pi \in \mathcal{S}} (U_{\pi \mathcal{S}} \cup \eta \mathcal{S}).
\end{array} \]
Lemma 6. For \(x \in E\), \((x)\), \((y)\), \((z)\) are integrable by \((x), (y), (z)\) for any \(a \in E\), and we have
\[
\left(\int_E x, \phi \right) = \int_{E \phi} (x) (y), \phi \right) (z).
\]

Lemma 7. \(i\). For \(0 \neq a \in E\),
\[
\lim_{p \to 0} \left(\int_{E \phi} (x), \phi \right) = \left(\frac{a}{a}, \phi \right) \text{ for } \phi \in C_0 (E).
\]

\(ii\). For any \(a, b \in E\), there exists
\[
\lim_{p \to 0} \left(\int_{E \phi} (x), \phi \right) = g(\phi) \text{ for } \phi \in C_0 (E)
\]
and the limit is independent from \(x \in E\) [Theorem 5.1.5].

The above limit \(g(\phi)\) is denoted by \((b, \phi)\) and integrable by \((y), \phi\) for each \(y \in E\) [Theorem 5.1.5].

Lemma 8. If \(f(\phi)\) is integrable by \((b), \phi\) in \(E \phi\), then \(f(\phi)\) \((b), \phi\) is integrable by \((y), \phi\) in \(E \phi\) and
\[
\int_{E \phi} f(\phi) (x) \phi = \int_{E \phi} f(\phi) (x) \phi
\]

Now, we consider such a completely additive measure \(\mu\) on \((E, \mathcal{F})\)
\[
\mu(\mathcal{F}) = \left(\int_{E \phi} (x), \phi \right)
\]
and suppose that \(x \in E\) such that \(0 < \|x\| \leq 1\) and \(\|x\| \leq 1\).

By Lemma 1 and condition (\#), we have
\[
\left(\int_{E \phi} (x), \phi \right) \leq \|x\| f(\|x\|) \leq \beta \left(\int_{E \phi} (x), \phi \right)
\]

Therefore, we have, by Lemma 7,
\[
\lim_{p \to 0} \left(\int_{E \phi} (x), \phi \right) = \left(\frac{x}{x}, \phi \right) \text{ for } \phi \in C_0 (E),
\]

so that
\[
\text{(17)} \quad \int_{E \phi} \left(\frac{x}{x}, \phi \right) \phi \leq \|x\| f(\|x\|) \leq \beta \left(\frac{x}{x}, \phi \right) \phi, \text{ for } \phi \in C_0 (E).
\]

On the other hand, \((x)\), \((y)\), \((z)\) for \(\phi \in C_0 (E)\) \((i = 1, 2, \ldots, n)\) and by Lemma 8,
\[
\int_{E \phi} \left(\frac{x}{x}, \phi \right) \phi (x, y) = \int_{E \phi} \left(\frac{x}{x}, \phi \right) \phi (x, y) = \phi (x, y) = 1.
\]

Consequently, on account of (15) and (17), we have
\[
\text{(18)} \quad 1 \leq \int_{E \phi} \left(\frac{x}{x}, \phi \right) \phi \leq \beta.
\]

For \(\xi \in \mathcal{F}\) finding in Lemma 3, there exist two constants \(0 < \gamma \leq \delta < +\infty\) such that \([\xi]/f([\xi]) \leq \Phi(\delta|\xi|)\) and \(\Phi|\xi| \leq \|\xi|/f([\xi])\|\), because \(\|\xi|/f([\xi])\|\) is equivalent to \(\mathcal{M}(\|\xi|\)). Consequently,
\[
\text{(19)} \quad 1 \leq \int \Phi\left(\frac{x}{x}, \phi \right) \phi \|\mu \| \text{ and } \int \Phi\left(\frac{x}{x}, \phi \right) \phi \|\mu \| \leq \beta.
\]

For any \(x \in E\),

In [8], we prove that for any \(0 \neq a \in E\) there exists a sequence of step elements \(x_n \in E\) such that \(0 < \|x_n\| \leq \|x\|\). Hence, by the Lebesgue's bounded sequence theorem,
\[
\lim_{n \to \infty} \int \Phi\left(\frac{x}{x}, \phi \right) \phi \|\mu \| = \int \Phi\left(\frac{x}{x}, \phi \right) \phi \|\mu \|
\]

so that, from (19),
\[
\text{(20)} \quad 1 \leq \lim_{n \to \infty} \int \Phi\left(\frac{x}{x}, \phi \right) \phi \|\mu \| = \int \Phi\left(\frac{x}{x}, \phi \right) \phi \|\mu \| \leq \beta.
\]

Thus, the function space \(A = ((x), \phi); x \in E\) comes to a modulated space, with the modular
\[
\phi(x) = \int \Phi\left(\frac{x}{x}, \phi \right) \phi \|\mu \|,
\]
which is topologically isomorphic to \( R \), that is, for the modular norm
\[
\left\| \frac{p(x)}{q(x)} \right\| = \inf_{\alpha(x) \geq 1} \frac{1}{\alpha(x)} ,
\]
it follows that \( \| \alpha \| \leq \left( \frac{\| (\alpha, \beta) \|}{\| (\alpha(x), \beta) \|} \right) \) \( \beta \| \alpha(x) \| \) for each \( x \in R \).

Therefore, the modular norm \( \| \cdot \| \) on \( A \) is continuous (by the continuity of \( \| \cdot \| \) on \( R \) and \( A \) is non-atomic (by the non-atomicity of \( R \)). Consequently, the modular \( \varepsilon \) on \( A \) is finite \( (11; p. 62) \) and \( 9(\S 10) \), i.e., \( \varepsilon(x) < +\infty \) for every \( x \in A \).

These facts show that \( R \) is topologically isomorphic to a subspace \( A \) of the Orlicz space \( L_\Phi (\sigma, \mu) \), however, we can verify that \( \Phi \) satisfies the \( (A) \)-condition, so we know the inclusion \( L_\Phi (\sigma, \mu) \subset A \), by the same method as in the end of the proof of the theorem in \( 7; p. 150 \) and \( 589 \).

In what follows, we shall only prove that \( \Phi \) satisfies the \( (A) \)-condition. By property \( (v) \) for \( R \), which is described in section 2, there exists a positive integer \( k_0 \) such that
\[
\| (q_i) \| \leq \frac{1}{\beta} \| (p) \| (i = 1, 2, \ldots, k_0)
\]
and for any orthogonal partition
\[
(p) = \sum_{i=1}^{k_0} (q_i)
\]
with \( \| (q_i) \| = \| (q_i) \| = \cdots = \| (q_k) \| \). (The possibility of such an orthogonal partition arises from the facts that \( R \) is non-atomic and has the continuous norm). We have therefore
\[
\left\| (p) \right\| = \left\| \sum_{i=1}^{k_0} (q_i) \right\| = \sum_{i=1}^{k_0} \left\| (q_i) \right\| = \sum_{i=1}^{k_0} \left\| (q_i) \right\| \leq \frac{1}{\beta} \left\| (p) \right\| \sum_{i=1}^{k_0} \left\| (q_i) \right\|
\]
and hence
\[
\left\| (p) \right\| = \frac{1}{\beta} \sum_{i=1}^{k_0} \left\| (q_i) \right\| .
\]

On the other hand, we have, by \( 13 \),
\[
A \leq \left\| \sum_{i=0}^{\infty} (q_i) \right\| \leq B
\]
for non-zero projectors \( x, y \) with \( \left\| x \right\| = 0 \) and \( \left\| (x) \right\| = \left\| (y) \right\| . \)

Accordingly, we have
\[
\left\| \sum_{i=0}^{\infty} \left\| (q_i) \right\| \right\| \leq \frac{h_0 B}{2} \left\| (q_i) \right\| (i = 1, 2, \ldots, k_0).
\]

For any \( \xi > 1 \), we take a projector \( (p) \) satisfying \( \| (p) \| = 1 \) and use \( (p) \) instead of \( (p(x)) \) in \( (22) \). Then, for the orthogonal partition
\[
(p) = \sum_{i=1}^{k_0} (q_i)
\]
with \( \| (q_i) \| = \| (q_i) \| = \cdots = \| (q_i) \| \), we have
\[
\left\| 2 \xi (q_i) \right\| \leq 1 \quad (i = 1, 2, \ldots, k_0).
\]

Therefore, considering \( \xi = 2 \xi (q_i) \), we have
\[
f(2 \xi (q_i)) = \frac{1}{\beta} \left\| \sum_{i=0}^{\infty} \left\| (q_i) \right\| \right\| \quad \text{(by Lemma 1)}
\]
and hence
\[
\left\| \sum_{i=0}^{\infty} \left\| (q_i) \right\| \right\| \leq \frac{1}{\beta} \left\| (q_i) \right\| \quad \text{(by (22))}
\]
and
\[
\left\| \sum_{i=0}^{\infty} \left\| (q_i) \right\| \right\| \leq \frac{2 \beta}{B_k} \left\| (q_i) \right\| \quad \text{(by Lemma 1)}
\]
and hence
\[
\left\| \sum_{i=0}^{\infty} \left\| (q_i) \right\| \right\| \leq \frac{2 \beta}{B_k} \left\| (q_i) \right\| \quad \text{(by (22))}
\]

Namely, we have
\[
f(2 \xi (q_i)) = \frac{2 \beta}{B_k} \left\| (q_i) \right\| \quad \text{for all } \xi > 1
\]
and hence \( M(\xi) \) in Lemma 3 satisfies the \( (A) \)-condition and consequently \( \Phi(\xi) \) satisfies also the \( (A) \)-condition, because \( \Phi(\xi) \) is equivalent to \( M(\xi) \).

Thus, the normed lattice \( R \) having the properties \((1)-(v)\) is topologically isomorphic to the Orlicz space \( L_\Phi (\sigma, \mu) \). The theorem is proved.

References

A norm satisfying the Bernstein condition

by

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In the research that recently culminated ([1], [3], [4]) in the proof that all separable infinite-dimensional Fréchet spaces are homeomorphic, one step (not however used in that proof) was the introduction into $c_0$ of a new norm, equivalent to the original norm but in addition satisfying the "Bernstein condition". Bessaga [2] gives a rather complicated construction and proof, communicated to him by Kadets. The purpose of the present note is to point out that the very simple norm

$$||x|| = ||x|| + \sum_{i=1}^{\infty} a_i |x_i|$$

for $x = (x_i) \in c_0$,

where $\sum a_i$ is any fixed convergent series of positive numbers, will serve the purpose equally well. In view of the inequalities

$$||x|| \leq ||x|| \leq \left(1 + \sum_{i=1}^{\infty} a_i \right) ||x||$$

it is obvious that $||\cdot||$ is an admissible norm, equivalent to $||\cdot||$, and it remains to be shown that it satisfies the Bernstein condition. Thus, we have to prove the following

**Theorem.** If $x_i \geq 0$, $y_i \geq 0$ ($i = 1, 2, \ldots$), $x_i \to 0$, $y_i \to 0$, and

$$\sup_{i \geq 0} x_i + \sum_{i=1}^{\infty} a_i x_i = \sup_{i \geq 0} y_i + \sum_{i=1}^{\infty} a_i y_i = \delta_j \quad (say)$$

for $j = 1, 2, \ldots$, then $x_i = y_i$ for all $i = 1, 2, \ldots$

**Proof.** Suppose not. If $k$ is the first index for which $x_k = y_k$, and say $x_k > y_k$, then the inequality $x_i \geq y_i$ cannot hold for all $i \neq k$, otherwise (1) would fail for $j = 1$. Hence there exist indices $m$ and $n$ such that

$$1 \leq m < n, \quad x_m > y_m, \quad x_n < y_n,$$

and $x_i = y_i$ for $m \leq i < n$. 


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