STUDIA MATHEMATICA, T. XXIX. (1968)

On normed lattices topologically isomorphic to some Orlicz space L_Φ^*

by

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1. Introduction. Let μ be a non-atomic, completely additive measure on a set Ω with $\mu(\Omega)=1$.

The *Orlicz space* $L_\Phi^*(\Omega, \mu)$ consists of all real-valued functions $x(t)$, μ-measurable on Ω, such that

$$(1)\qquad \varrho(ax)=\int_\Omega \Phi(a|x(t)|)\,d\mu<+\infty \text{ for some real number } a>0,$$

where Φ is an N-function which satisfies (Δ_2)-condition (1). Then, the space L_Φ^* is not only a Banach space with the norm (2)

$$\|x\|=\inf\{1/|\xi|;\ \varrho(\xi x)\leqslant 1\},$$

also becomes a conditionally complete vector lattice (3) by the usual ordering.

In the preceding paper [7], we gave a characterization of L_Φ^*. The purpose of the present paper is *to characterize L_Φ^* under the topological equivalence without containing the function Φ in the condition by which L_Φ^* is characterized.*

We shall easily see that an N-function has an equivalent N-function with the continuous derivative. Therefore, we shall assume in this section that Φ is continuously differentiable. Then, *the modular norm on L_Φ^* is*

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(1) A continuous convex function Φ is said to be *N-function* if

$$\lim_{\xi\to+0}\Phi(\xi)/\xi=0 \quad\text{and}\quad \lim_{\xi\to+\infty}\Phi(\xi)/\xi=+\infty \quad ([9],\ \text{p. } 9).$$

Φ is said to satisfy (Δ_2)-*condition* if there exist two real numbers $a>0$ and $\xi_0\geqq 0$ such that $\Phi(2\xi)\leqq a\Phi(\xi)$ for all $\xi\geqq\xi_0$. In this case, L_Φ^* can be defined as the totality of μ-measurable functions $x(t)$ such that $\varrho(x)<+\infty$. The functional ϱ on L_Φ^* in (1) is a *modular* in Nakano's terminology.

(2) This norm is called the *modular norm* or *Luxemburg norm*.

(3) A vector lattice R is said to be *conditionally complete*, if for $R\ni a_\lambda\geqslant 0$ $(\lambda\in\Lambda)$ there exists $a\in R$ such that $a=\bigcap_{\lambda\in\Lambda}a_\lambda$.

smooth and monotone (4). Furthermore, we can see that for each $x(t) \in L_\Phi^*$ there exists only one $\bar{x}(t) \in L_\Psi^*(\Omega, \mu)$, Ψ is the *complementary* N-function, for which the equality in the Young's inequality holds, i.e.,

(2) $$\int_\Omega x(t)\bar{x}(t)\,d\mu = \int_\Omega \Phi(|x(t)|)\,d\mu + \int_\Omega \Psi(|x(t)|)\,d\mu.$$

Indeed, $\bar{x}(t) = \varphi(|x(t)|)\operatorname{sgn} x(t)$ where φ is the derivative of Φ (cf. [12], Theorem 39.1, and [4], p. 64). Hence, we obtain a transformation T from L_Φ^* into L_Ψ^* through the correspondence $x(t) \to \bar{x}(t) = \varphi(|x(t)|)\operatorname{sgn} x(t)$.

This transformation T has the following properties:

(i) $0 \leqslant x \leqslant y$ implies $0 \leqslant Tx \leqslant Ty$,

(ii) $(Tx)[p] = T([p]x)$ for any projector $[p]$ (5).

(iii) $T(-x) = -Tx$.

Let R be a conditionally complete vector lattice, and $\bar{R}$ be its *conjugate space*, i.e., the totality of all linear functionals $\bar{x}$ on R for which

$$\inf_{\lambda \in \Lambda} |(x_\lambda, \bar{x})| = 0 \ (6)$$

for any system $\{x_\lambda; \lambda \in \Lambda\}$ in R with $x_\lambda \downarrow_{\lambda \in \Lambda} 0$. A transformation T from R into $\bar{R}$, with conditions (i)-(iii) is said to be *conjugately similar* ([12], p. 254).

Recently, the present author and Yamamuro [5] have shown the following theorem:

Let R be a conditionally complete vector lattice possessing a norm with $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$, which has at least two linearly independent elements and its conjugate norm be strictly convex. If there exists a one-to-one conjugately similar transformation T from R into its conjugate $\bar{R}$ with the condition

(3) $$(x, Tx) = \|x\|\cdot\|Tx\| \qquad (x \in R),$$

then R is of L_p-type ($p > 1$).

(4) The norm on the normed space X is said to be *smooth*, if at every point of the unit surface of X there is only one supporting hyperplane of the unit sphere of X. This is equivalent to the Gateaux differentiability of the norm [8]. The norm on the normed lattice X is said to be *monotone*, if $0 < x < y$ implies $\|x\| < \|y\|$ for $x, y \in X$. If Φ satisfies the (Δ_2)-condition and $\Phi(\xi) > 0$ for each $\xi > 0$, then the modular norm is monotone ([2], Theorem 3.3).

(5) For the support F of an element $p(t) \in L_\Phi^*$, the *projector* $[p]$ is defined by $[p]x(t) = \chi_F x(t)$, where χ_F is the characteristic function of F. In a conditionally complete vector lattice R, the *projector* $[p]$ ($p \in R$) is defined by $[p]x = \bigcup_{n=1}^{\infty} (x \frown n|p|)$ if $x > 0$, and $[p]x = [p]x^+ - [p]x$ for any $x \in R$, where $x^+ = x \smile 0$ $x^- = (-x)^+$ and $|x| = x^+ + x^-$. For $\bar{x} \in L_\Psi^*$, $\bar{x}[p]$ is a linear functional on R such that $(y, \bar{x}[p]) = ([p]y, \bar{x})$ for all $y \in L_\Phi^*$. See also footnote (6).

(6) $(y, \bar{x})$ means the value of $\bar{x} \in \bar{R}$ at $y \in R$.

In the Orlicz space L_Φ^*, a similar behavior to L_p-space may be seen. For $x \in L_\Phi^*$ with $\|x\| = 1$, we denote by x^* the element in the conjugate space of L_Φ^*, with the norm 1, for which the equality in Hölder's inequality holds, i.e., $(x, x^*) = \|x\|\cdot\|x^*\|$. This x^* determines uniquely for x, because of the smoothness of the norm on L_Φ^*.

Then, we shall be able to see the following property:

For any step element x in L_Φ^* (i.e., a simple function), with the norm 1, and for any sub-step projector $[p]$ of x (i.e., a projector satisfying $[p]x(t) \equiv$ const. for the simple function x), the equality in the Hölder's inequality in the form

(*) $$([p]x, x^*[p]) = \|[p]x\|\cdot\|x^*[p]\|$$

holds.

Indeed, let s in L_Φ^* be the function $s(t) \equiv 1$ on Ω a.e. In general, for the conjugately similar transformation T,

(4) $$L_\Phi^* \ni x,\ \|x\| = 1, \to Tx = \varphi(|x|)\operatorname{sgn} x(t) \in L_\Psi^*,$$

the relation

(5) $$(x, Tx) = \|Tx\| \ (7)$$

holds and hence we have $x^* = Tx/\|Tx\|$. Now, expressing x in (*) by a form

$$x(t) = \xi[p]s(t) + \sum_{i=1}^{n} \xi_i[p_i]s(t),$$

where $[p]$ and $[p_i]$ $(i = 1, 2, \ldots, n)$ are mutually orthogonal projectors (8), we have, by the property (ii),

$$Tx(t) = \operatorname{sgn}\xi\,\varphi(|\xi|)T[p]s(t) + \sum_{i=1}^{n} \varphi(|\xi_i|)T[p_i]s(t)\cdot \operatorname{sgn}\xi_i$$

so that

$$x^*[p] = \frac{\varphi(|\xi|)T[p]s\cdot\operatorname{sgn}\xi}{\|Tx\|}$$

and further

$$\left(\frac{[p]s}{\|[p]s\|}, \varphi\left(\frac{1}{\|[p]s\|}\right)T[p]s\right) = \left(\frac{[p]s}{\|[p]s\|}, T\frac{[p]s}{\|[p]s\|}\right)$$
$$= \left\| T\frac{[p]s}{\|[p]s\|}\right\| = \varphi\left(\frac{1}{\|[p]s\|}\right)\|T[p]s\|,$$

namely, (*) is satisfied.

(7) This fact is obtained from (2) and [14], Theorem 3.2.1.

(8) Projectors $[p]$ and $[q]$ are called *mutually orthogonal* if $[p][q] \equiv [|p| \frown |q|] = 0$.

To show that the property (*) is a characteristic property of L_Φ^* under the topological equivalence, we shall prepare in the next section.

2. Throughout this section, let R be a normed lattice which has the following properties:

(i) *R is non-atomic and conditionally σ-complete* ([9]),

(ii) *the norm $\|\cdot\|$ on R is semi-continuous, i.e.,*

$$0 \leqq x_n \uparrow_{n=1}^{\infty} x \,(x_n, x \in R) \text{ implies } \|x_n\| \uparrow_{n=1}^{\infty} \|x\|,$$

(iii) *the norm on R is smooth and monotone,*

(iv) *R has a positive complete element s with $\|s\| = 1$, i.e., no element in R is orthogonal to s,*

(v) $\sup(\sum\|[p_k]s\|) = +\infty$, *where $\{[p_k]s\}$ is any orthogonal partition of s, and also there exists a positive integer k_0 such that for any $[p]$ orthogonal partitions $[p]s = \sum_{i=1}^{k_0}[q_i]s$, with $\|[q_1]s\| = \|[q_2]s\| = \ldots = \|[q_{k_0}]s\|$, imply $\|[q_i]s\| \leqq \|[p]s\|/2$ for $i = 1, 2, \ldots, k_0$.*

Remark. It is easily verified that the Orlicz space L_Φ^* in section 1 satisfies property (v) from the facts that Φ satisfies (Δ_2)-condition and $\Phi(1/\|[p]s\|) = 1/\mu(F)$, where F is the support of $p(t) \in L_\Phi^*$.

An element x in R is called a *step element*, if x is of a form $\sum_{i=1}^{n} \xi_i[p_i]s$ for certain orthogonal system $\{[p_i]; i = 1, 2, \ldots, n\}$ of projectors in R. For a step element x, we shall call a *sub-step projector* of x the projector $[p]$ such that $[p]x = \xi[p]s$ for some real number ξ.

We denote again the main notation used in this paper.

$\bar{R}$ is the conjugate space of R; S is the unit surface of R, i.e., the set $\{x \in R; \|x\| = 1\}$; E is the set of all step elements in R; $(x, \bar{y})$ means the value of $\bar{y} \in \bar{R}$ at $x \in R$; x^* means, for $x \in S$, the element on the unit surface of $\bar{R}$ for which the equality in the Hölder's inequality holds, i.e., $(x, x^*) = \|x\|\cdot\|x^*\|$; $x^*[p]$, for any projector $[p]$ in R and $x \in S$, denotes the element of $\bar{R}$ such that $(y, x^*[p]) = ([p]y, x^*)$ for all $y \in R$.

For mutually orthogonal elements $a_i \in S$ $(i = 1, 2, \ldots, n)$, the functions

$$\xi_k = f_k(\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n) \qquad (k = 1, 2, \ldots, n)$$

([9]) R is said to be *non-atomic*, if every non-zero element in R is divided into two non-zero elements orthogonally. R is said to be *conditionally σ-complete*, if for $R \ni a_n \geqslant 0$ $(n = 1, 2, \ldots)$ there exists $a \in R$ such that $a = \bigcap_{n=1}^{\infty} a_n$. A normed lattice satisfying (i)-(iii) comes to a conditionally complete vector lattice, because (ii) and (iii) imply the continuity of the norm [3] and furthermore this fact and (i) imply the desired result ([12], Theorem 30.7).

which are defined by the relation

$$\|\xi_1 a_1 + \xi_2 a_2 + \ldots + \xi_n a_n\| = 1 \quad \text{and} \quad \xi_i \geqslant 0 \quad (i = 1, 2, \ldots, n)$$

are called the *represented functions* of an n-dimensional *indicatrix* $C(a_1, a_2, \ldots, a_n)$ ([10]) of R.

Moreover, Greek letters $\xi, \eta, \ldots$ denote the real numbers or real functions and small Latin letters $a, b, x, \ldots$ denote the elements in R.

We shall first give two lemmas concerning the properties of the indicatrix, which connect with [6] and [8].

Lemma A. *Each represented function $\xi_k = f_k(\xi_1, \ldots, \xi_n)$ of an n-dimensional indicatrix $C(a_1, \ldots, a_n)$ of R is partially differentiable with respect to the variable $\xi_i (i \neq k)$. Here, the differentiation at the end point in the domain of f_k means the one-side differentiation.*

Proof. Since the norm on R is smooth, when we denote the right and left derivatives by $D^+f_k(\xi_i)$ and $D^-f_k(\xi_i)$ respectively, we have

$$\big(a_i + (D^+f_k(\xi_i))a_k, x^*\big) = \big(a_i + (D^-f_k(\xi_i))a_k, x^*\big) = 0$$

by the same method as that in [6], Lemma 2, and [8], where

$$\sum_{j=1}^{n} \xi_j a_j = x \in S.$$

If $(a_k, x^*) = 0$, we have $(y, x^*) = 1$ for $y = \sum_{j \neq k} \xi_j a_j$ so that

$$1 = \|x^*\| \geqslant \left(\frac{y}{\|y\|}, x^*\right) > 1$$

provided that $\xi_k \neq 0$, because $\|x\| = \|\xi_k a_k + y\| > \|y\|$ by the monotony of the norm. This is impossible and consequently we have

$$\frac{\partial f_k}{\partial \xi_i} = -\frac{(a_i, x^*)}{(a_k, x^*)}. \tag{6}$$

It is obvious that $\partial f_k/\partial \xi_i = 0$ at $\xi_i = 0$.

Lemma B. *For a represented function $\xi_k = f_k(\xi_1, \ldots, \xi_n)$, let us assume that $x = \xi_1 a_1 + \ldots + f_k(\xi_i) a_k + \ldots + \xi_n a_n \in S$ and that ξ_i is variable and ξ_j $(j \neq i$ and $j \neq k)$ are fixed. Then (a_i, x^*) is non-decreasing function in $\xi_i \geqslant 0$.*

Proof. It is enough to prove the case in which the indicatrix is 2-dimensional, by reason of which the proof in the n-dimensional case is essentially the same as that in 2-dimensional case. Let $\eta = \eta(\xi)$ be a represented function of an indicatrix $C(a, b)$ of R with respect to $a, b \in S$ with $a \frown b = 0$.

([10]) The notion of an *indicatrix* has been introduced in [11], p. 342.

By definition, for $0 < \xi < 1$,

(7) $$\left(a, (\xi a+\eta(\xi) b)^*\right) = \lim_{\varepsilon\to 0} \frac{\|(\xi+\varepsilon)a+\eta(\xi)b\|-1}{\varepsilon}.$$

First, we shall prove, for each small $\varepsilon > 0$, that a function

$$g(\xi, \varepsilon) = \|(\xi+\varepsilon)a+\eta(\xi)b\|-1$$

is non-decreasing in $0 \leqslant \xi \leqslant 1$. Put, for $0 < \xi < 1$,

$$D_\xi^+ g(\xi, \varepsilon) = \lim_{\delta\to+0} \frac{1}{\delta}\{\|(\xi+\delta+\varepsilon)a+\eta(\xi+\delta)b\|-\|(\xi+\varepsilon)a+\eta(\xi)b\|\}.$$

Taking enough small δ and ε_1 with $0 < \delta < \varepsilon_1 < \varepsilon$, by virtue of Lemma A the derivative $\eta'(\xi)$ exists and is non-increasing by the concavity of $\eta(\xi)$.

Accordingly, we have for some $0 < \theta < 1$

$$g(\xi+\delta, \varepsilon) = \|(\xi+\varepsilon)a+\eta(\xi)b+\delta\{a+\eta'(\xi+\theta\delta)b\}\|-1$$
$$\geqslant \|(\xi+\varepsilon)a+\eta(\xi)b+\delta\{a+\eta'(\xi+\varepsilon_1)b\}\|-1$$

and hence

(8) $$D_\xi^+ g(\xi, \varepsilon) \geqslant \lim_{\delta\to+0}\frac{1}{\delta}[\|(\xi+\varepsilon)a+\eta(\xi)b+\delta\{a+\eta'(\xi+\varepsilon_1)b\}\|-\|(\xi+\varepsilon)a+\eta(\xi)b\|] = \left(a+\eta'(\xi+\varepsilon_1)b, c^*\right),$$

where

$$c = \frac{(\xi+\varepsilon)a+\eta(\xi)b}{\|(\xi+\varepsilon)a+\eta(\xi)b\|}.$$

Putting

$$\lambda = \frac{\xi+\varepsilon}{\|(\xi+\varepsilon)a+\eta(\xi)b\|} \quad \text{and} \quad \mu = \frac{\eta(\xi)}{\|(\xi+\varepsilon)a+\eta(\xi)b\|},$$

the point (λ, μ) is on the indicatrix $C(a, b)$. When we take again ε_1 such that $0 < \varepsilon_1 < \varepsilon(1-\xi)/(1+\varepsilon)$, then, on account of $\|(\xi+\varepsilon)a+\eta(\xi)b\| \leqslant 1+\varepsilon$, it follows that $\xi+\varepsilon_1 < (\xi+\varepsilon)/(1+\varepsilon) \leqslant \lambda$. Consequently, we have, by (8) and Lemma A, $D_\xi^+ g(\xi, \varepsilon) \geqslant \left(a+\eta'(\lambda)b, c^*\right) = 0$ which shows $g(\xi, \varepsilon)$ is non-decreasing in $0 \leqslant \xi \leqslant 1$. Therefore, by (7), $\left(a, (\xi a+\eta(\xi)b)^*\right)$ is non-decreasing in $0 \leqslant \xi \leqslant 1$.

3. Theorem. *Let R be the normed lattice which has properties* (i)-(v) *in the preceding section. If R satisfies the following condition:*

for any step element $x \in S\cdot E$ and for any sub-step projector $[p]$ of x,

(*) $$([p]x, x^*[p]) = \|[p]x\|\cdot\|x^*[p]\|$$

holds, then R is topologically isomorphic to some Orlicz space L_Φ^* the modular norm on which has properties (ii), (iii) *and* (v).

The central part of the proof of Theorem is to construct a function Φ which determined the Orlicz space L_Φ^*. Therefore, we shall begin to give the lemmas by which Φ is constructed and its properties are proposed.

In what follows, suppose that R satisfies the condition in Theorem.

For any $\xi \geqslant 0$, we define a function $f(\xi)$ as

(9) $$f(\xi) = \sup\left\{\frac{\|x^*[p]\|}{\|s^*[p]\|};\ x\in S\cdot E \text{ and } [p]x = \xi[p]s\right\}$$
$$= 0 \quad \text{if} \quad \xi = 0.$$

Remark. $0 \neq \|[p]x\|$ and $x \in S$ imply $([p]x, x^*) \neq 0$. Indeed, if $([p]x, x^*) = 0$, then $(x-[p]x, x^*) = 1$ and hence $\|x-[p]x\| = 1$ contradicting to the monotony of the norm.

Lemma 1. *There exists a positive constant β such that for arbitrary $x_0 \in S\cdot E$, with $x_0 = \xi[p_0]s+\sum_{i=1}^{n}\xi_i[p_i]s$, $0 < \|\xi[p_0]s\| \leqslant 1$ and $\xi > 0$,*

(10) $$\frac{\|x_0^*[p_0]\|}{\|s^*[p_0]\|} \leqslant f(\xi) \leqslant \beta\frac{\|x_0^*[p_0]\|}{\|s^*[p_0]\|}.$$

Proof. The left side inequality is obvious from the definition of $f(\xi)$. Suppose that $S\cdot E \ni x = \xi[p]s+\sum_{i=1}^{n}\xi_i[p_i]s$. Then, we have

$$\left(\frac{[p]s}{\|[p]s\|}, \frac{x^*[p]}{\|x^*[p]\|}\right) = 1$$

and hence, by virtue of the smoothness of R,

(11) $$\frac{x^*[p]}{\|x^*[p]\|} = \left(\frac{[p]s}{\|[p]s\|}\right)^* = \frac{s^*[p]}{\|s^*[p]\|},$$

because $\|x^*[p]\| \neq 0$ from the above remark.

Therefore, for any $0 \neq [q] \leqslant [p]$, $x^*[q]/\|x^*[p]\| = s^*[q]/\|s^*[p]\|$ and consequently

(12) $$\frac{\|x^*[q]\|}{\|s^*[q]\|} = \frac{\|x^*[p]\|}{\|s^*[p]\|} \quad \text{for every } 0 \neq [q] \leqslant [p]$$

Next, we shall prove that there exist two positive constant A and B such that for every elements x and y in $S\cdot E$

(13) $$A \leqslant \frac{\|x^*[p]\|}{\|y^*[q]\|} \leqslant B,$$

where $\xi > 0$, $x = \xi[p]s+u$, $y = \xi[q]s+v$, $[p]u = [q]v = 0$, $\|[p]s\| = \|[q]s\|$ and $0 \neq u, v \in E$.

If such constants do not exist, on account of (12), there exist some elements x_n and y_n in $S\cdot E$, which satisfy the following relations:

$$0 < x_n = \xi_n[p_n]s + \eta_n b_n, \quad b_n \in S\cdot E, \quad [p_n]b_n = 0,$$

$$0 < y_n = \xi_n[q_n]s + \zeta_n d_n, \quad d_n \in S\cdot E, \quad [q_n]d_n = 0,$$

$$1 \geqslant \|\xi_n[p_n]s\| = \|\xi_n[q_n]s\| = t_n\downarrow_{n=1}^{\infty}, \quad \xi_n > 0$$

and

$$\|y_n^*[q_n]\| = g(n)\|x_n^*[p_n]\| \quad \text{with} \quad g(n)\uparrow_{n=1}^{\infty} + \infty.$$

For simplicity, we put $[p_n]s/\|[p_n]s\| = a_n$, $[q_n]s/\|[q_n]s\| = c_n$, $X_n = (a_n, x_n^*)$ and $Y_n = (c_n, y_n^*)$. For the represented functions $\eta = \eta_n(t)$ and $\zeta = \zeta_n(t)$ of the indicatrices $C(a_n, b_n)$ and $C(c_n, d_n)$ respectively, as is shown in (6), we have then

$$\left[\frac{d\eta_n}{dt}\right]_{t=t_n} = -\frac{(b_n, x_n^*)}{X_n} \quad \text{and} \quad \left[\frac{d\zeta_n}{dt}\right]_{t=t_n} = -\frac{(d_n, y_n^*)}{Y_n}$$

and hence

$$0 \leqslant \left[-\frac{d\zeta_n}{dt}\right]_{t=t_n} = \frac{1-g(n)}{\zeta_n(t_n)\cdot X_n\cdot g(n)} + \frac{\eta_n(t_n)}{\zeta_n(t_n)}\left[-\frac{d\eta_n}{dt}\right]_{t=t_n},$$

because $t_n X_n + \eta_n(t_n)(b_n, x_n^*) = t_n Y_n + \zeta_n(t_n)(d_n, y_n^*) = 1$ and $Y_n = g(n)X_n$ by condition (*) in the theorem.

On the other hand, it is easily seen that for enough large n,

$$1-t_n \leqslant \eta_n(t_n), \quad \zeta_n(t_n) \leqslant 1, \quad 0 \leqslant \left[-\frac{d\eta_n}{dt}\right]_{t=t_n} \leqslant 1, \quad 0 < Y_n = X_n\cdot g(n)$$

and $\lim_{n\to\infty}(1-g(n)) = -\infty$. Consequently, we have

$$0 \leqslant \left[-\frac{d\zeta_n}{dt}\right]_{t=t_n} \leqslant \frac{1-g(n)}{(1-t_n)\,Y_n} + \frac{1}{1-t_n} < 0$$

for enough large n, which is impossible. Thus, there exists a constant $A > 0$ satisfying (13) We may be able to prove similarly the existence of B in (13). By (12), (13) and the definition of $f(\xi)$, we can see that Lemma 1 is verified for $\beta = B$.

LEMMA 2. *The function $f(\xi)$ defined in (9) is a real-valued, non-decreasing function in $\xi \geqslant 0$ and more $f(\xi) > 0$ for $\xi > 0$.*

Proof. It is evident, by Lemma 1, that $f(\xi)$ is real-valued and $f(\xi) > 0$ for $\xi > 0$ from the remark for the definition of $f(\xi)$. Suppose $0 < \xi_1 < \xi_2$. We choose a projector $[p]$ such that $0 < \|\xi_i[p]s\| \leqslant 1$ $(i = 1, 2)$. Moreover, we consider $x_i \in S\cdot E$ $(i = 1, 2)$ such that

$$x_i = \xi_i[p]s + \sum_{i=1}^{n}\eta_k[q_k]s + \eta(\xi_i)[q]s,$$

where $[p]$, $[q_k]$ $(k = 1, 2, \ldots, n)$ and $[q]$ are mutually orthogonal. Then, if we fix η_k $(k = 1, 2, \ldots, n)$, the represented functions $\eta(\xi_i)\|[q]s\| = h_i(\xi_i, \eta_1, \ldots, \eta_n)$ are differentiable at $\xi_i (i = 1, 2)$ respectively; namely, derivatives $\eta'(\xi_i)$ exist by Lemma A. Therefore, by Lemma B, we have

$$\left(\frac{[p]s}{\|[p]s\|}, x_1^*\right) \leqslant \left(\frac{[p]s}{\|[p]s\|}, x_2^*\right).$$

Hence, in virtue of the condition (*), we have

$$\frac{\|x_1^*[p]\|}{\|s^*[p]\|} \leqslant \frac{\|x_2^*[p]\|}{\|s^*[p]\|} \quad \text{and more} \quad \frac{\|x^*[p_1]\|}{\|s^*[p_1]\|} = \frac{\|x^*[p_2]\|}{\|s^*[p_2]\|}$$

if $0 < \|\xi_i[p_j]s\| \leqslant 1$ and $[p_2] \leqslant [p_1]$, for each element $x = \xi_1[p_1]S + \sum\zeta_k[r_k]s \in S\cdot E$ by (12). Accordingly, when we choose $0 < \eta < 1$ such that $y = \xi_2[p_2]s + \eta\cdot\sum\zeta_k[r_k]s$ belongs in $S\cdot E$ (this is possible, on account of (12), by taking $[p_2]$ such that $\|[p_2]s\|$ is enough small) we obtain, by Lemma 2,

$$\frac{\|x^*[p_1]\|}{\|s^*[p_1]\|} = \frac{\|x^*[p_2]\|}{\|s^*[p_2]\|} \leqslant \frac{\|y^*[p_2]\|}{\|s^*[p_2]\|}.$$

Therefore, we have

$$f(\xi_1) = \sup\left\{\frac{\|x^*[p]\|}{\|s^*[p]}; x \in S\cdot E \text{ and } [p]x = \xi_1[p]s\right\}$$

$$\leqslant \sup\left\{\frac{\|y^*[p]\|}{\|s^*[p]\|}; y \in S\cdot E \text{ and } [p]y = \xi_2[p]s\right\} = f(\xi_2).$$

LEMMA 3. *For the convex function*

$$M(\xi) = \int_0^{\xi} f(t)\,dt,$$

there exists a convex function $\Phi(\xi)$, equivalent to $M(\xi)$, such that

(i) *the derivative of $\Phi(\xi)$ is continuous,*

(ii) $\lim_{\xi\to+0}\Phi(\xi)/\xi = 0$ *and* $\lim_{\xi\to+\infty}\Phi(\xi)/\xi = +\infty$,

(iii) $\Phi(\xi) > 0$ *for* $\xi > 0$.

Proof. Putting

$$\Phi(\xi) = \int_0^{\xi}\frac{M(t)}{t}\,dt \quad \text{for} \quad \xi \geqslant 0, \tag{14}$$

we have

$$\tfrac{1}{2}M(\tfrac{1}{2}\xi) \leqslant \Phi(\xi) \leqslant M(\xi) \quad \text{for} \quad \xi \geqslant 0$$

so that $\Phi(\xi)$ is equivalent to $M(\xi)$. It is evident that Φ satisfies (i) and (iii). In order to prove that Φ satisfies (ii), it will suffice to prove that

$$\lim_{\xi\to+0} f(\xi) = 0 \quad \text{and} \quad \lim_{\xi\to+\infty} f(\xi) = +\infty.$$

For $0<\xi<1$, choosing $x_\xi = \xi[p]s+\eta(\xi)[q]s \in S\cdot E$, with $[p][q] = 0$, we have

$$\lim_{\xi\to+0} f(\xi) = \lim_{\xi\to+0} \frac{\|x_\xi^*[p]\|}{\|s^*[p]\|} = \lim_{\xi\to+0} \frac{([p]s, x_\xi^*)}{\|s^*[p]\|}$$
$$= \left([p]s, \left(\frac{[q]s}{\|[q]s\|}\right)^*\right) \Big/ \|s^*[p]\| = 0.$$

Next, taking $x_\xi = \xi[p_\xi]s \in S\cdot E$ for each $\xi>1$, it follows from the property (v) for R that $\lim_{\xi\to+\infty}\|s^*[p_\xi]\| = 0$. Therefore, we have, by Lemma 1,

$$\lim_{\xi\to+\infty} f(\xi) \geqslant \lim_{\xi\to+\infty} \frac{\|x_\xi^*[p_\xi]\|}{\|s^*[p_\xi]\|} = \lim_{\xi\to+\infty} \frac{1}{\|s^*[p_\xi]\|} = +\infty.$$

4. The proof of Theorem. We shall make use of the spectral theory of H. Nakano [12; §§ 8-13 and §§ 20-23] and [13; Chap. III]. Therefore, we restate at the moment several results obtained by H. Nakano.

Let $\mathscr{E}$ be the proper space of R, i.e., the compact Hausdorff space consisting of all maximal ideals (11) *$\mathscr{P}$ of projectors in R with a neighbourhood system $\mathscr{J} = \{U_{[x]}; x\in R\}$, where $U_{[x]} = \{\mathscr{P}\in\mathscr{E};\ [x]\in\mathscr{P}\}$. Then, each $U_{[x]}$ is both open and closed in $\mathscr{E}$, and $\mathscr{J}$ forms a Boolean algebra with respect to the set operation, i.e.,*

$$U_{[x]} \cup U_{[y]} = U_{[|x|\cup|y|]} \quad \textit{and} \quad U_{[x]}U_{[y]} = U_{[x][y]} \quad [12;\ \text{p. } 32].$$

For $x\in R$, the function $(x/s, \mathscr{P})$ on $\mathscr{E}$ is defined by

$$\left(\frac{x}{s}, \mathscr{P}\right) = \begin{cases} \lambda & \text{if} \quad \mathscr{P}\in \prod_{\varepsilon>0} (U_{[x_{\lambda+\varepsilon}]} - U_{[x_{\lambda-\varepsilon}]}), \\ +\infty & \text{if} \quad \mathscr{P}\in \prod_{-\infty<\lambda<+\infty} (\mathscr{E} - U_{[x_\lambda]}), \\ -\infty & \text{if} \quad \mathscr{P}\in \prod_{-\infty<\lambda<+\infty} U_{[x_\lambda]}, \end{cases}$$

where $[x_\lambda] = [(\lambda s - x)^+]$, and is called the *relative spectrum* [13; Theorem 23.3].

For this function, we can see that

LEMMA 4. (i) *$(x/s, \mathscr{P})$ is almost finite, i.e., finite in an open dense set in $\mathscr{E}$, and is continuous* (12) [13; Theorems 19.2 and 19.3];

(11) The set of projectors $\mathscr{P}$ is called an *ideal*, if (i) $\mathscr{P}\not\ni 0$, (ii) $\mathscr{P}\ni[x]$ and $[x]\leqslant[y]$ imply $[y]\in\mathscr{P}$, (iii) $\mathscr{P}\ni[x],[y]$ implies $[x][y]\in\mathscr{P}$, where $[x][y]$ means $[|x|\cap|y|]$.

(12) In the case $(x/s, \mathscr{P}_0) = +\infty$ (or $-\infty$), the continuity means that for any real number $\lambda>0$, there exists a nbd. $U_{[q]}\ni\mathscr{P}_0$ such that $(x/s, \mathscr{P})\geqslant\lambda$ (or $\leqslant -\lambda$) for all $\mathscr{P}\in U_{[q]}$.

(ii) *$(x/s, \mathscr{P}) = ([p]x/s, \mathscr{P})$ on $U_{[p][x]}$ for any projector $[p]$* [13; Theorem 18.4];

(iii) *the set $\{(x/s, \mathscr{P}); x\in R\}$ is linear and lattice isomorphic to R* [13; Theorem 18.5-Theorem 18.10].

For a bounded continuous function $f(\mathscr{P})$ on $U_{[p]}$, *the integral of $f(\mathscr{P})$ by $x\in R$*, denoted by $\int_{[p]} f(\mathscr{P})d\mathscr{P}x$, is defined as a limit of partial sums

$$\sum_{j=1}^{n_i} f(\mathscr{P}_{ij})[p_{ij}]x$$

for every sequence of orthogonal partitions $\{[p_{ij}]\}$ of $[p]$ such that

$$\operatorname*{Osc}_{\mathscr{P}\in U_{[p_{ij}]}} f(\mathscr{P}) \leqslant \varepsilon_i \quad (j=1,2,\ldots,n;\ i=1,2,\ldots), \quad \lim_{i\to\infty}\varepsilon_i = 0$$

and for arbitrary $\mathscr{P}_{ij}\in U_{[p_{ij}]}$.

For an unbounded continuous function $f(\mathscr{P})$ on $U_{[p]}$, if there exists an increasing sequence of bounded continuous functions $f_n(\mathscr{P})$ on $U_{[p]}$ such that

$$\lim_{n\to\infty} f_n(\mathscr{P}) = f(\mathscr{P}) \quad \text{and} \quad \lim_{n\to\infty}\int_{[p]} f_n(\mathscr{P})d\mathscr{P}x$$

exists, then we shall say that *$f(\mathscr{P})$ is integrable by x* on $U_{[p]}$ and denote this limit by $\int_{[p]} f(\mathscr{P})d\mathscr{P}x$. We have, as an integral representation,

LEMMA 5 [13; Theorems 21.1 and 21.2]. *For any $a\in R$, $(a/s, \mathscr{P})$ is integrable by s and*

$$a = \int_{[s]}\left(\frac{a}{s}, \mathscr{P}\right)d\mathscr{P}s.$$

Conversely, if a continuous function $f(\mathscr{P})$ is integrable by s and

$$b = \int_{[s]} f(\mathscr{P})d\mathscr{P}s,$$

then $f(\mathscr{P}) = (b/s, \mathscr{P})$ for all $\mathscr{P}\in\mathscr{E}$.

For $x\in R$ and $\bar a\in\bar R$, considering $([p]x, \bar a)$ as a *measure* of $U_{[p]}$, we can define *integral of continuous functions $f(\mathscr{P})$ on $U_{[p]}$ by $(x, \bar a)$, denoted by*

$$\int_{U_{[p]}} f(\mathscr{P})(d\mathscr{P}x, \bar a).$$

This integral has been introduced in [11; § 4 and § 5] and the following facts are obtained.

LEMMA 6 [11; § 4, Hilfsatz 4.22] [12; Theorem 21.10]. *For $x \in R$, $(x/s, \mathscr{P})$ is integrable by $(s, \bar a)$ for any $\bar a \in \bar R$, and we have*

$$([p]x, \bar a) = \int_{U_{[p]}} \left(\frac{x}{s}, \mathscr{P}\right)(d\mathscr{P}s, \bar a).$$

LEMMA 7. (i) *For $0 \neq \bar a \in \bar R$,*

$$\lim_{[p]\to\mathscr{P}} \frac{([p]x,\bar a)}{([p]s,\bar a)} = \left(\frac{x}{s},\mathscr{P}\right) \quad \textit{for} \quad \mathscr{P}\in U_{[x]}\cdot C_{\bar a}$$ [13],

namely,

$$\inf_{U_{[p]}\ni\mathscr{P}}\left\{\sup_{0\neq[y]\le[p]}\frac{([y]x,\bar a)}{([y]s,\bar a)}\right\} = \sup_{U_{[p]}\ni\mathscr{P}}\left\{\inf_{0\neq[y]\le[p]}\frac{([y]x,\bar a)}{([y]s,\bar a)}\right\} = \left(\frac{x}{s},\mathscr{P}\right)$$

[13; Theorem 54.3];

(ii) *for any $a, b\in R$, there exists*

$$\lim_{[p]\to\mathscr{P}}\frac{([p]x,\bar b)}{([p]x,\bar a)} = g(\mathscr{P}) \quad \textit{for} \quad \mathscr{P}\in U_{[x]}\cdot C_{\bar a}$$

and the limit is independent from $x\in R$ [13; Theorem 51.5].

The above limit $g(\mathscr{P})$ is denoted by $(\bar b/\bar a, \mathscr{P})$ and integrable by $(y,\bar a)$ for each $y\in R$ [13; Theorem 51.8].

LEMMA 8 [11; 4, Hilfsatz 4.23] [12; Theorem 21.11]. *If $f(\mathscr{P})$ is integrable by $(b,\bar a)$ in $U_{[p]}$, then $f(\mathscr{P})(b/s,\mathscr{P})$ is integrable by $(s,\bar a)$ in $U_{[p]}$ and*

$$\int_{U_{[p]}} f(\mathscr{P})(d\mathscr{P}b,\bar a) = \int_{U_{[p]}} f(\mathscr{P})\left(\frac{b}{s},\mathscr{P}\right)(d\mathscr{P}s,\bar a).$$

Now, we consider such a completely additive measure μ on $(\mathscr{E};\mathscr{J})$ as

(15) $$\mu(U_{[p]}) = ([p]s, s^*).$$

Suppose that $x\in S\cdot E$ with $x=\sum_{i=1}^n \xi_i[p_i]s$ where $[p_i]s$ are mutually orthogonal and $\xi_i\neq 0$ $(i=1,2,\ldots,n)$. For such a ξ_i, we take a projector $[p]$ such that

$$0<\|\xi_i[p]s\|\le 1 \quad\text{and}\quad [p]\le[p_i].$$

By Lemma 1 and condition (*), we have

(16) $$\frac{([p]x, x^*[p])}{([p]s, s^*[p])} \le |\xi_i|f(|\xi_i|) \le \beta\frac{([p]x,x^*[p])}{([p]s,s^*[p])}.$$

(13) $C_{\bar a} = \mathscr{E} - \bigcup_{\bar a[p]=0} U_{[p]}$.

Therefore, we have, by Lemma 7,

$$\lim_{[p]\to\mathscr{P}}\frac{([p]x,x^*[p])}{([p]s,s^*[p])} = \left(\frac{x}{s},\mathscr{P}\right)\left(\frac{x^*}{s^*},\mathscr{P}\right) \quad\text{for}\quad \mathscr{P}\in U_{[p]}\cdot\mathscr{E},$$

so that

(17) $$\left(\frac{x}{s},\mathscr{P}\right)\left(\frac{x^*}{s^*},\mathscr{P}\right)\le|\xi_i|f(|\xi_i|)\le\beta\left(\frac{x}{s},\mathscr{P}\right)\left(\frac{x^*}{s^*},\mathscr{P}\right) \quad\text{for}\quad \mathscr{P}\in U_{[p_i]}.$$

On the other hand, $(x/s,\mathscr{P}) = \xi_i$ for $\mathscr{P}\in U_{[p_i]}$ $(i=1,2,\ldots,n)$ and by Lemma 8

$$\int_{\mathscr{E}}\left(\frac{x}{s},\mathscr{P}\right)\left(\frac{x^*}{s^*},\mathscr{P}\right)(d\mathscr{P}s,s^*) = \int_{\mathscr{E}}\left(\frac{x}{s},\mathscr{P}\right)(d\mathscr{P}s,x^*) = (x,x^*)=1.$$

Consequently, on account of (15) and (17), we have

(18) $$1\le\int_{\mathscr{E}}\left|\left(\frac{x}{s},\mathscr{P}\right)\right| f\left(\left|\left(\frac{x}{s},\mathscr{P}\right)\right|\right)d\mu\le\beta.$$

For $\Phi(\xi)$ finding in Lemma 3, there exist two constants $0<\gamma<\delta<+\infty$ such that $|\xi|f(|\xi|)\le\Phi(\delta\cdot|\xi|)$ and $\Phi(\gamma|\xi|)\le|\xi|f(|\xi|)$, because $|\xi|f(|\xi|)$ is equivalent to $M(|\xi|)$. Consequently,

(19) $$1\le\int_{\mathscr{E}}\Phi\left(\delta\left|\left(\frac{x}{s},\mathscr{P}\right)\right|\right)d\mu \quad\text{and}\quad \int_{\mathscr{E}}\Phi\left(\gamma\left|\left(\frac{x}{s},\mathscr{P}\right)\right|\right)d\mu\le\beta$$

for any $x\in S\cdot E$.

In [8], we prove that for any $0\neq x\in R$ there exists a sequence of step elements $x_n\in E$ such that $0\le x_n\uparrow_{n=1}^{\infty}|x|$. Hence, by the Lebesgue's bounded sequence theorem,

$$\lim_{n\to\infty}\int_{\mathscr{E}}\Phi\left(\left(\frac{x_n}{s},\mathscr{P}\right)\right)d\mu = \int_{\mathscr{E}}\Phi\left(\left|\left(\frac{x}{s},\mathscr{P}\right)\right|\right)d\mu$$

so that, from (19),

(20) $$1\le\lim_{n\to\infty}\int_{\mathscr{E}}\Phi\left(\frac{\delta}{\|x_n\|}\left(\frac{x_n}{s},\mathscr{P}\right)\right)d\mu = \int_{\mathscr{E}}\Phi\left(\frac{\delta}{\|x\|}\left|\left(\frac{x}{s},\mathscr{P}\right)\right|\right)d\mu$$

and similarly

(21) $$\int_{\mathscr{E}}\Phi\left(\frac{\gamma}{\|x\|}\left|\left(\frac{x}{s},\mathscr{P}\right)\right|\right)d\mu\le\beta.$$

Thus, *the function space $\Lambda\equiv\{(x/s,\mathscr{P}); x\in R\}$* comes to a modulared space, with the modular

$$\varrho(x) = \int_{\mathscr{E}}\Phi\left(\left|\left(\frac{x}{s},\mathscr{P}\right)\right|\right)d\mu,$$

which is topologically isomorphic to R, that is, for the modular norm

$$\left|\left|\left|\left(\frac{x}{s}, \mathscr{P}\right)\right|\right|\right| = \inf_{\varrho(\xi x)\leqslant 1} \frac{1}{|\xi|},$$

it follows that $\|x\|/\delta \leqslant |||(x/s, \mathscr{P})||| \leqslant \beta\|x\|/\gamma$ for each $x \in R$.

Therefore, the modular norm $|||\cdot|||$ on Λ is continuous (by the continuity of $\|\cdot\|$ on R [14]) and Λ is non-atomic (by the non-atomicity of R). Consequently, the modular ϱ on Λ is *finite* ([1; p. 62] and [9; § 10]), i.e., $\varrho(x) < +\infty$ for every $x \in \Lambda$.

These facts show that R is topologically isomorphic to a subspace Λ of the Orlicz space $L_\Phi^*(\mathscr{E}, \mu)$, however, we can verify that Φ satisfies the (Δ_2)-condition, so we know the inclusion $L_\Phi^*(\mathscr{E}, \mu) \subset \Lambda$, by the same method as in the end of the proof of the theorem in [7; p. 150 and p. 580]. In what follows, we shall only prove that Φ satisfies the (Δ_2)-condition. By property (v) for R, which is described in section 2, there exists a positive integer k_0 such that

$$\|[q_i]s\| \leqslant \tfrac{1}{2}\|[p]s\| \quad (i = 1, 2, \ldots, k_0) \text{ for any projector } [p]$$

and for any orthogonal partition

$$[p]s = \sum_{i=1}^{k_0} [q_i]s$$

with $\|[q_1]s\| = \|[q_2]s\| = \ldots = \|[q_{k_0}]s\|$. (The possibility of such a orthogonal partition arises from the facts that R is non-atomic and has the continuous norm). We have therefore

$$\|[p]s\|\cdot\|s^*[p]\| = ([p]s, s^*[p]) = \sum_{i=1}^{k_0}([q_i]s, s^*[q_i])$$
$$= \sum_{i=1}^{k_0}\|[q_i]s\|\cdot\|s^*[q_i]\| \leqslant \tfrac{1}{2}\|[p]s\|\cdot\sum_{i=1}^{k_0}\|s^*[q_i]\|$$

and hence

$$\|s^*[p]\| \leqslant \tfrac{1}{2}\sum_{i=1}^{k_0}\|s^*[q_i]\|.$$

On the other hand, we have, by (13),

$$A \leqslant \frac{\|s^*[x]\|}{\|s^*[y]\|} \leqslant B$$

for non-zero projectors $[x], [y]$ with $[x][y] = 0$ and $\|[x]s\| = \|[y]s\|$. Accordingly, we have

$$(22) \qquad \|s^*[p]\| \leqslant \frac{k_0 B}{2}\|s^*[q_i]\| \quad (i = 1, 2, \ldots, k_0).$$

[14] See the footnote [9].

Now, for any $\xi > 1$, we take a projector $[p_\xi]$ satisfying $\|\xi[p_\xi]s\| = 1$ and use $[p_\xi]$ instead of $[p]$ in (22). Then, for the orthogonal partition

$$[p_\xi]s = \sum_{i=1}^{k_0}[q_i]s \quad \text{with} \quad \|[q_1]s\| = \|[q_2]s\| = \ldots = \|[q_{k_0}]s\|,$$

we have

$$\|2\xi[q_i]s\| \leqslant 1 \quad (i = 1, 2, \ldots, k_0).$$

Therefore, considering $x = 2\xi[q_1]s + \eta[r]s \in S\cdot E$, where $[r]$ is a projector with $[q_1][r] = 0$, we have

$$f(2\xi) \leqslant \beta\frac{\|x^*[q_1]\|}{\|s^*[q_1]\|} \quad \text{(by Lemma 1)}$$
$$\leqslant \beta\frac{1}{\|s^*[q_1]\|}$$
$$\leqslant \frac{2\beta}{Bk_0}\,\frac{1}{\|s^*[p_\xi]\|} \quad \text{(by (22))}$$
$$= \frac{2\beta}{Bk_0}\cdot\frac{\|(\xi[p_\xi]s)^*[p_\xi]\|}{\|s^*[p_\xi]\|}$$
$$\leqslant \frac{2\beta}{Bk_0}f(\xi) \quad \text{(by Lemma 1)}.$$

Namely, we have

$$f(2\xi) \leqslant \frac{2\beta}{Bk_0}f(\xi) \quad \text{for all } \xi > 1$$

and hence $M(\xi)$ in Lemma 3 satisfies the (Δ_2)-condition and consequently $\Phi(\xi)$ satisfies also the (Δ_2)-condition, because $\Phi(\xi)$ is equivalent to $M(\xi)$.

Thus, the normed lattice R having the properties (i)-(v) is topologically isomorphic to the Orlicz space $L_\Phi^*(\mathscr{E}, \mu)$. The theorem is proved.

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Reçu par la Rédaction le 28. 8. 1966

STUDIA MATHEMATICA, T. XXIX. (1968)

A norm satisfying the Bernstein condition

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In the research that recently culminated ([1], [3], [4]) in the proof that all separable infinite-dimensional Fréchet spaces are homeomorphic, one step (not however used in that proof) was the introduction into c_0 of a new norm, equivalent to the original norm but in addition satisfying the "Bernstein condition". Bessaga [2] gives a rather complicated construction and proof, communicated to him by Kadets. The purpose of the present note is to point out that the very simple norm

$$|||x||| = \|x\| + \sum_{i=1}^{\infty} a_i |x_i| \quad \text{for} \quad x = (x_i) \in c_0,$$

where $\sum a_i$ is any fixed convergent series of positive numbers, will serve the purpose equally well. In view of the inequalities

$$\|x\| \leqslant |||x||| \leqslant \Big(1 + \sum_{i=1}^{\infty} a_i\Big) \|x\|$$

it is obvious that $|||\cdot|||$ is an admissible norm, equivalent to $\|\cdot\|$, and it remains to be shown that it satisfies the Bernstein condition. Thus, we have to prove the following

THEOREM. *If* $x_i \geqslant 0$, $y_i \geqslant 0$ $(i = 1, 2, \ldots)$, $x_i \to 0$, $y_i \to 0$, *and*

$$(1) \qquad \sup_{i \geqslant j} x_i + \sum_{i=j}^{\infty} a_i x_i = \sup_{i \geqslant j} y_i + \sum_{i=j}^{\infty} a_i y_i = \delta_j \quad (say)$$

for $j = 1, 2, \ldots$, *then* $x_i = y_i$ *for all* $i = 1, 2, \ldots$

Proof. Suppose not. If k is the first index for which $x_k \neq y_k$, and say $x_k > y_k$, then the inequality $x_i \geqslant y_i$ cannot hold for all $i \neq k$, otherwise (1) would fail for $j = 1$. Hence there exist indices m and n such that

$$1 \leqslant m < n, \; x_m > y_m, \; x_n < y_n, \; \text{and} \; x_i = y_i \; \text{for} \; m < i < n.$$