On Peano derivatives in $L^p(E_n)$

by

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1. It is known ([2], p. 270) that if a finite-valued function $f$ on the real line has everywhere on a set $E$ a one-sided derivative, then $f$ has a finite two-sided derivative almost everywhere on $E$. Some analogous results are here obtained for Peano derivatives in the metric $L^p$ in $n$-dimensional Euclidean space.

**Notation and Definitions.** By $x, t, \ldots$ we denote respectively points $(x_1, x_2, \ldots, x_n)$, $(t_1, t_2, \ldots, t_n), \ldots$ of the (real) $n$-dimensional Euclidean space $E_n$. $n = 1, 2, \ldots$ As usual

$$|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}, \quad x + t = (x_1 + t_1, x_2 + t_2, \ldots, x_n + t_n),$$

$$\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \quad \lambda \text{ a scalar.}$$

The functions $f = f(x)$ we deal with are real-valued and measurable. Extensions of results to complex-valued functions are immediate. If $p = \infty$ and $D$ is a subset of $E_n$,

$$\left\{ \int_D |f(x)|^p \, dx \right\}^{1/p}$$

denotes $\text{ess sup} \{ |f(x)| : x \in D \}$.

By $L^p(E_n), 1 \leq p < \infty$, we denote the class of functions $f$ such that

$$\|f\|_p = \left\{ \int_{E_n} |f(x)|^p \, dx \right\}^{1/p} < \infty$$

($\alpha$ denoting, for $1 \leq p < \infty$, the element of volume in $E_n$).

Let $a = (a_1, a_2, \ldots, a_n)$, the $a_i$'s being non-negative integers. Let $k_j$ be a real number. The degree of the term $k_1 a_1 x_1^{a_1} \ldots x_n^{a_n}$ is defined to be $a_1 + a_2 + \ldots + a_n$ if $k_j \neq 0$, and $-\infty$ if $k_j = 0$. The degree of a polynomial $P = P(x)$ is now defined in the usual way.

By a cone we mean a union of rays issuing from a point (the vertex). We require that this union be a measurable set with non-void interior. $C = C(x)$ denotes a cone with vertex $x$; $B_h = B_h(x)$ denotes the (closed) ball with center $x$ and radius $h$ ($h > 0$); $C_h = C_h(x)$ denotes the conical
sector formed by the intersection of $O(x)$ and $B_h(x)$. By a right circular cone of angular magnitude $\theta$ we mean a cone $O = O(x)$ consisting of the union of all rays issuing from $x$ and forming an angle $\leq \theta$ with a fixed ray (the axis of the cone). We require that $0 < \theta < \pi/2$. A right circular cone $O = O(x)$ is characterized by its vertex $x$ and the vector $v$ whose direction is that of the axis of the cone and whose magnitude $|v|$ is equal to the angular magnitude of the cone. By the direction of a right circular cone we mean the direction of its axis.

For $n = 1$ the cone $O = O(x)$ is either the half-line $\{t : t > 0\}$, with associated vector $v = (1)$; or the half-line $\{t : t < 0\}$, with associated vector $v = (-1)$. The cone $O$, thus defined for $n = 1$, is, for convenience, considered to be a right circular cone, coinciding with its axis, with non-vanishing interior, and angular magnitude $1 = |v|$.

For $1 \leq p < \infty$ let $u := -n/p$ with the last inequality denoting $u > 0$ if $p = \infty$. Given a point $x$ in $E_u$, a function $f$ is said to belong to the class $\xi_\alpha(x)$ if it is in $L^p(E_u)$ and there exists a polynomial $P = P_1(t)$ of degree equal to or less than $u$ such that for the balls $B_h = B_h(x)$ we have

$$
\left( \int_{B_h} \left| f - P_1(t) \right|^p dt \right)^{1/p} = o(h^u) \quad \text{as} \quad h \to 0.
$$

Analogously, a function $f$ is said to belong to $T_\alpha(x)$ if $f$ is in $L^p(E_u)$ and there exists a polynomial $P = P_2(t)$ of degree less than $u$ such that

$$
\left( \int_{B_h} \left| f - P_2(t) \right|^p dt \right)^{1/p} \leq A h^u \quad (0 < h < \infty),
$$

with $A$ independent of $h$.

These definitions --- of $\xi_\alpha(x)$ and $T_\alpha(x)$ --- were introduced by Calderón and Zygmund [1] and the purpose of this paper is to extend one of their results concerning these classes of functions.

**Remarks.** Since $\xi_\alpha(x) \subseteq L^p(E_u)$, condition (1) implies condition (2) for an $f \in T_\alpha(x)$. Hence $\xi_\alpha(x) \subseteq T_\alpha(x)$. Let $f \in \xi_\alpha(x)$. Let $f = P_1(t) + R_k(x)$, a term of the polynomial $P_1(t)$; $Q = Q_2(t) = P_2(x + t) - P_2(x)$, a term of the polynomial $Q_2(t) = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n$. Then $\alpha_1 \alpha_2 \ldots \alpha_n, k = \alpha_1 \alpha_2 \ldots \alpha_n$ is a Peano derivative in $L^p$ of order $\alpha$ of $f$ at $x$.

In what follows $n, p, u(n = 1, 2, \ldots; 1 \leq p \leq \infty; u \geq -n/p)$ are fixed.

**Theorem 1.** Let $f$ be a function in $L^p(E_u)$ and let $E$ be any subset (measurable or not) of $E_u$. Suppose that for each point $x$ in $E$ there is a polynomial $P = P(t)$ of degree less than or equal to $u$ and a cone $O = O(x)$ with vertex at $x$ such that for the conical sectors $C_h = C_h(x)$ we have

$$
\left( \int_{C_h} \left| f - P(t) \right|^p dt \right)^{1/p} = o(h^u) \quad \text{as} \quad h \to 0.
$$

Then $f$ is in $\xi_\alpha(x)$ for almost every $x$ in $E$.

A similar result holds with $O$ instead of $p$.

**Theorem 2.** Let $f \in L^p(E_u)$ and let $E = E_u$. Suppose that for each $x$ in $E$ there is a polynomial $P = P_2(t)$ of degree less than $u$, and a cone $O = O(x)$ such that for the conical sectors $C_h = C_h(x)$, we have

$$
\left( \int_{C_h} \left| f - P_2(t) \right|^p dt \right)^{1/p} = o(h^u) \quad \text{as} \quad h \to 0.
$$

Then $f \in T_\alpha(x)$ for almost all $x$ in $E$.

If for a given $p$ ($1 < p \leq \infty$) and positive integer $u$, $f$ is in $T_\alpha(x)$ for all $x$ in a measurable set $E$, then $f$ is in $\xi_\alpha(x)$ for almost every $x$ in $E$ (see [1], p. 173). In view of this we obtain the following corollary of Theorem 2:

**Corollary 1.** Suppose that in Theorem 2; $1 < p \leq \infty$, $E$ is a measurable set, $u$ is a positive integer --- and (4) holds for all $x$ in $E$. Then $f$ is in $\xi_\alpha(x)$ for almost every $x$ in $E$.

If (3) holds for a cone $C$ (with non-vanishing interior), then (3) also holds for some right circular cone $C' \subseteq C$ with the same vertex as $C$. A similar statement holds for (4). We therefore limit consideration to right circular cones and in the sequel cones are understood to be right circular cones. We proceed with the proofs of Theorems 1 and 2.

**2. Lemma.** Given the cone $O = O(x)$. For the corresponding ball $B_h = B_h(x)$ and conical sector $C_h = C_h(x)$, and any polynomial $P = P(t)$ of degree not exceeding $u$, we have

$$
\left( \int_{B_h} \left| P(t) \right|^p dt \right)^{1/p} \leq A \left( \int_{C_h} \left| P(t) \right|^p dt \right)^{1/p} \quad (0 < h < \infty),
$$

where $A$ is a constant independent of $P$, $h$, and $x$ (but depending upon $n, p, u$, and the angular magnitude of the cone $C$).

**Proof.** We may assume that $P \neq 0$. Let

$$
g(P, h) = \left( \int_{B_h} \left| P(t) \right|^p dt \right)^{1/p} / \left( \int_{C_h} \left| P(t) \right|^p dt \right)^{1/p}.
$$

Multiplying $P$ by a constant, we may further assume that the sum of the absolute values of the coefficients of $P$ equals 1. If we identify each polynomial $P$ with the (appropriately ordered) set of its coefficients, our collection of polynomials forms a compact set $S$. For fixed $h > 0,$
Now let \( p = \infty \). Let \( G_0(z) = G_0(z) \cap B_h(z_0) \). Denote by \( G_h - G_0(z_i) \) the set of points which are in \( G_h \) but not in \( G_0(z_i) \). We have \( |G_h - G_0(z_i)| \to 0 \) as \( z_i \to 0 \). Hence, 

\[
\sup_{t \in C_h} |f(t) - P(t)| \leq \frac{1}{h^n}.
\]

Thus \( x = 0 \) is in \( E_h \); \( E_h \) is closed, and the set \( B(v, m, l) = \bigcup_{h=1}^{\infty} E_h \) is measurable \((1 < p \leq \infty)\).

4. Proof of Theorem 1. Almost every point of the (measurable) set \( B(v, m, l) \) defined as in Lemma 2 is a point of linear density of \( \cal E(v, m, l) \) in the direction of \( v \) (see [2], p. 288). Suppose for convenience that the origin \( x = 0 \) is such a point of density. Let \( G = G_0(0), B_h = B_h(0), G_h = G_h(0) = G_0(0) \cap B_h(0), P = P(t) \) as in (6). Let \( A \) be a constant greater than \( \sup |v| \), say \( A = 1 + \sup |v| \). There is then a positive constant \( h' = h'(v, m, l, x) \), \( x = 0 \), such that for every \( h \), \( 0 < h' \leq h \), we may choose a point \( y = y(h) \) satisfying the following conditions:

\begin{enumerate}
  \item \( y \) is in \( B(v, m, l) \);
  \item \( y \) is outside the cone \( C \) and lies on the line containing the axis of \( C \);
  \item the conical sector \( G_h = G_{h, x}(y) \) of radius \( |y| + h \), vertex \( y \), and angular and direction magnitude \( |v| \) contains the ball \( B_h \) (of radius \( h \) and center \( x = 0 \));
  \item \( |y| < Ah \);
  \item \( |y| + h' \leq h/l \).
\end{enumerate}

Let the restriction \( 0 < h' \leq h \) remain. Let \( A \) now denote a positive constant, not necessarily the same from one occurrence to the next, independent of \( x, h, m, l \) and \( v \) (and depending at most upon \( x, u, p \) and \( |v| \)).

We obtain (see (6), (7d)),

\[
|G_h| \leq A |G_h| \leq A |B_h|.
\]

We denote by \( P_h = P_h(t) \) a polynomial of degree not exceeding \( u \) associated with the point \( y = y(h) \) such that (6) holds, that is (see (7c), (7e), (7d)),

\[
\frac{1}{|G_h|} \int |f - P_h|^p dt \leq \frac{1}{m} |y| + h \leq A |y| + h.
\]

In view of the inclusions \( G_h \subset B_h \subset G_h \), we obtain from (8) and (9)

\[
\frac{1}{|G_h|} \int |f - P_h|^p dt \leq \frac{1}{m} |y| + h \leq A |y| + h.
\]
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6. Analogues of Theorems 1, 2 and Corollary 1 hold for the case $p = \infty$, with sup replacing ess sup in the definition of $\|f\|_\infty$. With $E, P_x(t), C_0(x), B_h(x)$, $u$ defined as in Theorem 1 we have respectively:

**Theorem 3.** Let $f$ be a measurable function and $E$ a subset (measurable or not) of $E_0$. Suppose that for each $x$ in $E$

$$\sup \{ |f(t) - P_x(t)| : t \in C_0(x) \} = o(h^u) \quad \text{as} \quad h \to 0. $$

Then for almost every $x$ in $E$

$$\sup \{ |f(t) - P_x(t)| : t \in B_h(x) \} = o(h^u) \quad \text{as} \quad h \to 0. $$

**Theorem 4.** Theorem 3 holds with $O$ replacing $o$ in both hypothesis and conclusion.

Corollary 2. For $u$ a positive integer and $E$ a measurable set, Theorem 3 holds with $O$ replacing $o$ in the hypothesis.

The proofs of Theorems 3 and 4 are essentially the same as those for Theorems 1 and 2 (except that Lemma 2 is proved now for open conical sectors and $f$ measurable).

Corollary 2 immediately follows from Theorem 4 and the following result: If in (17) $u$ is a positive integer and the estimate holds with $O$ on a measurable set $E$, then it holds with $O$ almost everywhere on $E$ (see [3], Theorem 4.24, p. 76, for the case $u = 1$, and [2], sec. 3, for the case $u \geq 1$).

References

