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On Peano derivatives in $L^p(E_n)$

by

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1. It is known ([2], p. 270) that if a finite-valued function f on the real line has everywhere on a set E a one-sided derivative, then f has a finite two-sided derivative almost everywhere on E. Some analogous results are here obtained for Peano derivatives in the metric L^p in n-dimensional Euclidean space.

NOTATION AND DEFINITIONS. By $x, t, \ldots$ we denote respectively points $(x_1, x_2, \ldots, x_n)$, $(t_1, t_2, \ldots, t_n), \ldots$ of the (real) n-dimensional Euclidean space E_n, $n = 1, 2, \ldots$ As usual

$$|x| = (x_1^2+x_2^2+\ldots+x_n^2)^{1/2}, \qquad x+t = (x_1+t_1, x_2+t_2, \ldots, x_n+t_n),$$

$$\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \qquad \lambda \text{ a scalar}.$$

The functions $f = f(x)$ we deal with are real-valued and measurable. Extensions of results to complex-valued functions are immediate. If $p = \infty$ and D is a subset of E_n,

$$\Big\{\int_D |f|^p\,dx\Big\}^{1/p}$$

denotes $\operatorname{ess\,sup}\{|f(x)| : x \in D\}$. By $L^p(E_n)$, $1 \leqslant p \leqslant \infty$, we denote the class of functions f such that

$$\|f\|_p = \Big\{\int_{E_n} |f|^p\,dx\Big\}^{1/p} < \infty$$

(dx denoting, for $1 \leqslant p < \infty$, the element of volume in E_n).

Let $a = (a_1, a_2, \ldots, a_n)$, the a_i's being non-negative integers. Let k_a be a real number. The *degree* of the term $k_a x^a = k_a x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is defined to be $a_1+a_2+\ldots+a_n$ if $k_a \neq 0$, and $-\infty$ if $k_a = 0$. The degree of a polynomial $P = P(x)$ is now defined in the usual way.

By a *cone* we mean a union of rays issuing from a point (the vertex). We require that this union be a measurable set with non-void interior. $C = C(x)$ denotes a cone with vertex x; $B_h = B_h(x)$ denotes the (closed) ball with center x and radius h $(h > 0)$; $C_h = C_h(x)$ denotes the conical sector formed by the intersection of $C(x)$ and $B_h(x)$. By a *right circular cone of angular magnitude* θ we mean a cone $C = C(x)$ consisting of the union of all rays issuing from x and forming an angle $\leqslant \theta$ with a fixed ray (the axis of the cone). We require that $0 < \theta < \pi/2$. A right circular cone $C = C(x)$ is characterized by its vertex x and the vector v whose direction is that of the axis of the cone and whose magnitude $|v|$ is equal to the angular magnitude of the cone. By the *direction* of a right circular cone we mean the direction of its axis.

For $n = 1$ the cone $C = C(x)$ is either the half-line $\{t: t \geqslant x\}$, with associated vector $v = (1)$; or the half-line $\{t: t \leqslant x\}$, with associated vector $v = (-1)$. The cone C, thus defined for $n = 1$, is, for convenience, considered to be a right circular cone, coinciding with its axis, with non-void interior, and angular magnitude 1 $(= |v|)$.

For $1 \leqslant p \leqslant \infty$ let $u \geqslant -n/p$, with the last inequality denoting $u \geqslant 0$ if $p = \infty$. Given a point x in E_n, a function f is said *to belong to the class* $t_u^p(x)$ if f is in $L^p(E_n)$ and there exists a polynomial $P = P_x(t)$ of degree equal to or less than u such that for the balls $B_h = B_h(x)$ we have

$$\left\{\frac{1}{|B_h|}\int_{B_h} |f-P|^p\,dt\right\}^{1/p} = o(h^u) \quad \text{as} \quad h \to 0. \tag{1}$$

Analogously, a function f is said *to belong to* $T_u^p(x)$ if f is in $L^p(E_n)$ and there exists a polynomial $P = P_x(t)$ of degree less than u such that

$$\left\{\frac{1}{|B_h|}\int_{B_h} |f-P|^p\,dt\right\}^{1/p} \leqslant Ah^u \quad (0 < h < \infty), \tag{2}$$

with A independent of h.

These definitions — of $t_u^p(x)$ and $T_u^p(x)$ — were introduced by Calderón and Zygmund [1] and the purpose of this paper is to extend one of their results concerning these classes of functions.

REMARKS. Since $t_u^p(x) \subset L^p(E_n)$, condition (1) implies condition (2) for an $f \in t_u^p(x)$. Hence $t_u^p(x) \subset T_u^p(x)$. Let $f \in t_u^p(x)$; $P_x(t)$ be its approximating polynomial satisfying (1); $Q = Q_x(t) = P_x(x+t)$; $K_\alpha t^\alpha$, a term of the polynomial Q; and $\alpha! = \alpha_1!\,\alpha_2!\ldots\alpha_n!$. Then $\alpha! K_\alpha$ is a *Peano derivative* in L^p of order α of f at x.

In what follows $n, p, u\,(n = 1, 2, \ldots; 1 \leqslant p \leqslant \infty; u \geqslant -n/p)$ are fixed.

THEOREM 1. *Let f be a function in $L^p(E_n)$ and let E be any subset (measurable or not) of E_n. Suppose that for each point x in E there is a polynomial $P = P_x(t)$ of degree less than or equal to u and a cone $C = C(x)$ with vertex at x such that for the conical sectors $C_h = C_h(x)$ we have*

$$\left\{\frac{1}{|C_h|}\int_{C_h} |f-P|^p\,dt\right\}^{1/p} = o(h^u) \quad \text{as} \quad h \to 0. \tag{3}$$

Then f is in $t_u^p(x)$ for almost every x in E.

A similar result holds with O instead of o:

THEOREM 2. *Let $f \in L^p(E_n)$ and let $E \subset E_n$. Suppose that for each x in E there is a polynomial $P = P_x(t)$ of degree less than u, and a cone $C = C(x)$ such that for the conical sectors $C_h = C_h(x)$, we have*

$$\left\{\frac{1}{|C_h|}\int_{C_h} |f-P|^p\right\}^{1/p} = O(h^u) \quad \text{as} \quad h \to 0. \tag{4}$$

Then $f \in T_u^p(x)$ for almost all x in E.

If for a given p $(1 < p \leqslant \infty)$ and positive integer u, f is in $T_u^p(x)$ for all x in a measurable set E, then f is in $t_u^p(x)$ for almost every x in E (see [1], p. 175). In view of this we obtain the following corollary of Theorem 2:

COROLLARY 1. *Suppose that in Theorem 2: $1 < p \leqslant \infty$, E is a measurable set, u is a positive integer — and (4) holds for all x in E. Then f is in $t_u^p(x)$ for almost every x in E.*

If (3) holds for a cone C (with non-void interior), then (3) also holds for some right circular cone $C' \subset C$ with the same vertex as C. A similar statement holds for (4). We may therefore limit consideration to right circular cones and in the sequel *cones are understood to be right circular cones*. We proceed with the proofs of Theorems 1 and 2.

2. LEMMA 1. *Given the cone $C = C(x)$. For the corresponding ball $B_h = B_h(x)$ and conical sector $C_h = C_h(x)$, and any polynomial $P = P(t)$ of degree not exceeding u, we have*

$$\left\{\frac{1}{|B_h|}\int_{B_h} |P|^p\,dt\right\}^{1/p} \leqslant A\left\{\frac{1}{|C_h|}\int_{C_h} |P|^p\,dt\right\}^{1/p} \quad (0 < h < \infty), \tag{5}$$

where A is a constant independent of P, h, and x (but depending upon n, p, u, and the angular magnitude of the cone C).

Proof. We may assume that $P \not\equiv 0$. Let

$$g(P, h) = \left\{\frac{1}{|B_h|}\int_{B_h} |P|^p\,dt\right\}^{1/p} \Big/ \left\{\frac{1}{|C_h|}\int_{C_h} |P|^p\,dt\right\}^{1/p}.$$

Multiplying P by a constant, we may further assume that the sum of the absolute values of the coefficients of P equals 1. If we identify each polynomial P with the (appropriately ordered) set of its coefficients, our collection of polynomials forms a compact set S. For fixed $h > 0$,

$g(P, h)$ is a continuous function on S and assumes a maximum, say $A = A(h) > 0$. Let $k > 0$ and $Q(t) = P\left(\frac{k}{h} t\right)$. Then

$$g(P, k) = \left\{\frac{1}{|B_h|} \int_{B_h} |Q|^p dt\right\}^{1/p} \Big/ \left\{\frac{1}{|C_h|} \int_{C_h} |Q|^p dt\right\}^{1/p} \leqslant A(h).$$

Thus $A(k) \leqslant A(h)$; and by symmetry, $A(k) = A(h)$. Hence A is independent of h (and P). Similarly, we may show that A is also independent of x and the direction of the cone C.

3. LEMMA 2. *Let $f \in L^p(E_n)$. Consider cones $C = C_v(x)$, $x \in E_n$, where v is a vector giving the common direction and angular magnitude of the cones. Let m and l be positive numbers. Fix v, m, l and let $E(v, m, l)$ denote the set of all x in E_n for which there exists some polynomial $P = P_x(t)$ of degree not exceeding u, such that for the conical sector $C_h = C_v(x) \cap B_h(x)$ we have*

$$\left\{\frac{1}{|C_h|} \int_{C_h} |f-P|^p dt\right\}^{1/p} \leqslant \frac{1}{m} h^u, \quad 0 < h \leqslant 1/l. \tag{6}$$

Then $E(v, m, l)$ is a measurable set.

Proof. Let k be a temporarily fixed positive integer and let $E_k = E(v, m, l, k)$ be defined in the same way as $E(v, m, l)$ except that the polynomials P are further restricted so that their coefficients do not exceed k in absolute value. Suppose that $x_i \in E_k$, $i = 1, 2, \ldots$, and that the x_i form a convergent sequence with limit $x = x_0$. For each i let $P = P_i$ be a polynomial satisfying condition (6) with $x = x_i$. The sequence of polynomials P_i, $i = 1, 2, \ldots$, contains a (coefficient-wise) convergent subsequence to which we now confine ourselves and which we again denote by P_i, $i = 1, 2, \ldots$ Let the polynomial P_0 be the limit of the P_i's. We may assume that $x_0 = 0$; and we write C_h for $C_h(0)$ and P for P_0.

Fix h $(0 < h \leqslant 1/l)$. Let $1 \leqslant p < \infty$. Using Minkowski's inequality, we obtain

$$\begin{aligned}\left\{\frac{1}{|C_h|} \int_{C_h} |f(t)-P(t)|^p dt\right\}^{1/p} &= \left\{\frac{1}{|C_h|} \int_{C_h} |f(t)-f(x_i+t)|^p dt\right\}^{1/p} + \\ &+ \left\{\frac{1}{|C_h|} \int_{C_h} |f(x_i+t)-P_i(x_i+t)|^p dt\right\}^{1/p} + \\ &+ \left\{\frac{1}{|C_h|} \int_{C_h} |P_i(x_i+t)-P(x_i+t)|^p dt\right\}^{1/p} + \\ &+ \left\{\frac{1}{|C_h|} \int_{C_h} |P(x_i+t)-P(t)|^p dt\right\}^{1/p} \\ &\leqslant o(1) + m^{-1} h^u + o(1) + o(1) \quad \text{as} \quad x_i \to 0 (= x_0).\end{aligned}$$

Now let $p = \infty$. Let $C_h(x_i) = C_v(x_i) \cap B_h(x_i)$. Denote by $C_h - C_h(x_i)$ the set of points which are in C_h but not in $C_h(x_i)$. We have $|C_h - C_h(x_i)| \to 0$ as $x_i \to 0$. Hence,

$$\begin{aligned}&\operatorname{ess\,sup}\{|f(t)-P(t)| : t \in C_h\} \\ &\leqslant \limsup_{i\to\infty} \operatorname{ess\,sup}\{|f(t)-P_i(t)| : t \in C_h(x_i)\} \leqslant \frac{1}{m} h^u.\end{aligned}$$

Thus $x = 0$ is in E_k, E_k is closed, and the set $E(v, m, l) = \bigcup_{k=1}^{\infty} E_k$ is measurable $(1 \leqslant p \leqslant \infty)$.

4. Proof of Theorem 1. Almost every point of the (measurable) set $E(v, m, l)$ defined as in Lemma 2 is a point of linear density of $E(v, m, l)$ in the direction of v ([2], p. 298). Suppose for convenience that the origin $x = 0$ is such a point of density. Let $C = C_v(0)$, $B_h = B_h(0)$, $C_h = C_h(0) = C_v(0) \cap B_h(0)$, $P = P_0(t)$ as in (6). Let A be a constant greater than $\csc|v|$, say $A = 1 + \csc|v|$. There is then a positive constant $h' = h(v, m, l, x)$, $x = 0$, such that for every h, $0 < h \leqslant h'$, we may choose a point $y = y(h)$ satisfying the following conditions:

(7) (a) y is in $E(v, m, l)$;

(b) y is outside the cone C and lies on the line containing the axis of C;

(c) the conical sector $C'_h = C_{|y|+h}(y)$ of radius $|y| + h$, vertex y, and angular magnitude and direction v contains the ball B_h (of radius h and center $x = 0$);

(d) $|y| \leqslant Ah$;

(e) $|y| + h \leqslant 1/l$.

Let the restriction $0 < h \leqslant h'$ remain. Let A now denote a positive constant, not necessarily the same from one occurrence to the next, independent of x, h, m and l (and depending at most upon n, u, p and $|v|$).

We obtain (see (7c), (7d)),

$$|C'_h| \leqslant A|C_h| \leqslant A|B_h|. \tag{8}$$

We denote by $P_h = P_y(t)$ a polynomial of degree not exceeding u associated with the point $y = y(h)$ such that (6) holds, that is (see (7c), (7e), (7d)),

$$\left\{\frac{1}{|C'_h|} \int_{C'_h} |f-P_h|^p dt\right\}^{1/p} \leqslant \frac{1}{m} (|y|+h)^u \leqslant \frac{A}{m} h^u. \tag{9}$$

In view of the inclusions $C_h \subset B_h \subset C'_h$, we obtain from (8) and (9)

$$\left\{\frac{1}{|C_h|} \int_{C_h} |f-P_h|^p dt\right\}^{1/p} \leqslant \left\{\frac{|C'_h|}{|C_h|} \frac{1}{|C'_h|} \int_{C'_h} |f-P_h|^p dt\right\}^{1/p} \leqslant \frac{A}{m} h^u. \tag{10}$$

$$\left\{\frac{1}{|B_h|}\int\limits_{B_h}|f-P_h|^p dt\right\}^{1/p} \leqslant \left\{\frac{|C_h'|}{|B_h|}\,\frac{1}{|C_h'|}\int\limits_{C_h'}|f-P_h|^p dt\right\}^{1/p} \leqslant \frac{A}{m}h^u. \tag{11}$$

We now obtain from (7e), (6), (10) and Minkowski's inequality

$$\left\{\frac{1}{|C_h|}\int\limits_{C_h}|P-P_h|^p dt\right\}^{1/p} \leqslant \frac{A}{m}h^u. \tag{12}$$

It follows from Lemma 1 that

$$\left\{\frac{1}{|B_h|}\int\limits_{B_h}|P-P_h|^p dt\right\}^{1/p} \leqslant \frac{A}{m}h^u. \tag{13}$$

We obtain from (11) and (13), again using Minkowski's inequality,

$$\left\{\frac{1}{|B_h|}\int\limits_{B_h}|f-P|^p dt\right\}^{1/p} \leqslant \frac{A}{m}h^u, \quad 0 < h \leqslant h'. \tag{14}$$

Thus, for almost every x in $E(v, l, m)$, (14) holds with $P = P_x(t)$ as in (6), $B_h = B_h(x)$, the ball with center x and radius h, and $h' = h(v, l, m, x)$.

By appropriately shrinking, if need be, the cones $C = C(x)$ associated with the points x, $x \in E$, we may confine ourselves to an at most denumerable set V of direction and angular magnitude vectors v. Since

$$E \subset \bigcup_{v\in V}\bigcap_{m=1}^{\infty}\bigcup_{l=1}^{\infty} E(v, m, l), \tag{15}$$

and the polynomials P in (14) do not depend upon m and l, Theorem 1 follows.

5. The proof of Theorem 2 is similar to that of Theorem 1 and we present it in outline. We use notation based upon that contained in the statement and proof of Theorem 1. We consider now polynomials P of degree less than u. Lemmas 1 and 2 and their proofs remain valid with such polynomials P. Since $T_u^p(x) \subset L^p(E_n)$, the definition of $T_u^p(x)$ remains unchanged if (2) is replaced by

$$\left\{\frac{1}{|B_h|}\int\limits_{B_h}|f-P|^p dt\right\}^{1/p} = O(h^u) \quad \text{as} \quad h \to 0. \tag{2'}$$

Replacing m by $1/l$ in $E(v; m, l)$, we obtain as in the proof of Theorem 1 that

$$E \subset \bigcup_{v\in V}\bigcup_{l=1}^{\infty} E(v, l^{-1}, l),$$

from which Theorem 2 follows.

6. Analogues of Theorems 1, 2 and Corollary 1 hold for the case $p = \infty$, with sup replacing ess sup in the definition of $\|f\|_\infty$. With $E, P_x(t), C_h(x), B_h(x), u$ defined as in Theorem 1 we have respectively:

THEOREM 3. *Let f be a measurable function and E a subset (measurable or not) of E_n. Suppose that for each x in E*

$$\sup\{|f(t)-P_x(t)|: t\in C_h(x)\} = o(h^u) \quad \textit{as} \quad h \to 0. \tag{16}$$

Then for almost every x in E

$$\sup\{|f(t)-P_x(t)|: t\in B_h(x)\} = o(h^u) \quad \textit{as} \quad h \to 0. \tag{17}$$

THEOREM 4. *Theorem 3 holds with O replacing o in both hypothesis and conclusion.*

COROLLARY 2. *For u a positive integer and E a measurable set, Theorem 3 holds with O replacing o in the hypothesis.*

The proofs of Theorems 3 and 4 are essentially the same as those for Theorems 1 and 2 (except that Lemma 2 is proved now for open conical sectors and f measurable).

Corollary 2 immediately follows from Theorem 4 and the following result: If in (17) u is a positive integer and the estimate holds with O on a measurable set E, then it holds with o almost everywhere on E (see [3], Theorem 4.24, p. 76, for the case $n = 1$, and [1], sec. 3, for the case $n \geqslant 1$).

References

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