7. We shall still prove that the commutativity $(f*g)*h = f*(g*h)$ holds also, if one of distributions f, g, h is of bounded carrier, provided the convolution of two remaining ones exists. We have namely:

THEOREM 7. *If the convolution $f*g$ exists and h is a distribution of bounded carrier, then, in the equality $(f*g)*h = f*(g*h)$, all the convolutions exist and the equality holds.*

Proof. The existence of convolutions $(f*g)*h$ and $g*h$ follows, as a particular case, from Theorem 5. It remains to prove the existence of $f*(g*h)$ and the equality. Let $k_n = (f*\delta_n)*(g*\delta_n)$. Then $k_n \to f*g$ by the hypothesis that $f*g$ exists. Now, by Theorem 5 we have $k_n*h \to (f*g)*h$. Assume that h is a continuous function. Then

$$k_n*h = [(f*\delta_n)*(g*\delta_n)]*h = (f*\delta_n)*[(g*\delta_n)*h] = (f*\delta_n)*[(g*h)*\delta_n];$$

here, the second equality follows from the fact that $f*\delta_n$, $g*\delta_n$ and h are functions for which the convolutions $(|f*\delta_n|*|g*\delta_n|)*|h|$ and $(g*\delta_n)*h$ exist, and the last equality follows from the remark at the end of section 6. Since k_n*h converges, the convolution $f*(g*h)$ exists, by definition, and the equality $(f*g)*h = f*(g*h)$ holds. If h is not a continuous function, then the assertion follows by Theorem 3.

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On the uniqueness of the ideals of compact and strictly singular operators *

by

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The purpose of this note is to extend a result of [3]. In particular, it is shown there that the ideal of compact operators is unique in $[X]$, the bounded linear operators from X to X, where $X = lp$, $1 \leqslant p < \infty$ and c_0. An obvious question is do there exist other spaces for which this is true? We obtain a partial result in this direction by requiring our space to have two properties which lp and c_0 enjoy. In addition, using one of these properties, we show that the ideal of compact and strictly singular operators agree. The phrase "partial result" is used since we cannot at this time exhibit a space with the above-mentioned properties other than lp or c_0. However, the proofs given here have the advantage of treating all cases simultaneously, as opposed to what is done in [3].

We will assume that the reader is somewhat familiar with the theory of Schauder bases in Banach spaces. Results used from this theory may be found in [2].

1. Definition. $\{e_i\}$ is a *Schauder basis* for X if for each $x \in X$, $x = \sum_1^\infty a_i x_i$ uniquely. In this case $a_i = g_i(x)$, $g_i \in X^*$.

2. Definition. $\{z_k\}$ is said to be a *block basis* if

$$z_k = \sum_{a_k+1}^{a_{k+1}} a_i^{(k)} e_i, \qquad a_1 < a_2 < \ldots$$

If $\{e_i\}$ is a Schauder basis for X, then $\{z_k\}$ is a Schauder basis for $\overline{\mathrm{sp}}\{z_k\}$, [1].

3. Definition. We will say that a Banach space X with a basis has $(+)$ if given a block basis $\{z_k\}$, there exists $P\colon X \to \overline{\mathrm{sp}}\{z_k\}$, P a projection.

* This work was done while the author was an NDEA Fellow at The University of Maryland.

4. Lemma. *If Ω is a non-zero, two-sided ideal in $[X]$, X a Banach space, then Ω contains all operators with finite-dimensional range.*

The following is due to Maddaus [6].

5. Lemma. *Let X be a Banach space with a Schauder basis. Every compact operator in X is the uniform limit of finite-dimensional operators.*

6. Lemma. *Let $A \in [X]$, X a Banach space with a Schauder basis $\{e_k\}$. A sufficient condition for A to be compact is that*

$$\sum_1^\infty \|Ae_k\| < \infty.$$

Proof. Since X is separable, we may apply the condition for compactness given by Gelfand [4]. Let $\{f_j\} \subset X^*$ be such that $f_j \to 0$ in the weak* topology. We must show $f_j \to 0$ uniformly on the image, under A, of the unit ball S in X. Let $x \in S$. Then

$$x = \sum_1^\infty a_i e_i \quad \text{and} \quad Ax = \sum_1^\infty a_i Ae_i.$$

Let g_j be the coordinate functionals. There exists M such that $\|g_j\| \leqslant M$, hence, $|a_j| = |g_j(x)| \leqslant M$. Let $\varepsilon > 0$ be given, and pick $N(\varepsilon)$ such that

$$\sum_N^\infty \|Ae_n\| < \frac{\varepsilon}{2M}.$$

Pick $J(\varepsilon)$ large enough so that $|f_j(Ae_i)| < \varepsilon/2NM$ for $j > J$ and $i = 1, \ldots, N$. Then

$$|f_j(Ax)| \leqslant \Big|f_j\Big(\sum_1^N a_i Ae_i\Big)\Big| + \Big|f_j\Big(\sum_N^\infty a_i Ae_i\Big)\Big| < \varepsilon.$$

Since x was arbitrary, we are done.

7. Definition. A Banach space X with a Schauder basis $\{e_i\}$ is said to be *block homogeneous* if given a block basis $\{z_k\}$, $\|z_k\| = 1$, then $\sum t_k z_k$ converges if, and only if, $\sum t_k e_k$ converges. In this case, $A(\sum t_k z_k) = \sum t_k e_k$ is an isomorphism between $\overline{\mathrm{sp}}\{z_k\}$ and X.

The following lemma is stated in [3] somewhat differently:

8. Lemma. *Let X be a block homogeneous Banach space, with $(+)$ and a Schauder basis $\{e_k\}$. No proper two-sided ideal $\Omega \subset [X]$ can contain an operator C such that $\inf_k \|Ce_k\| = \delta > 0$ and $\lim_k g_j(Ce_k) = 0$ for each coordinate functional g_j.*

Proof. Suppose such an operator was in Ω. By theorem 3 of [1] it follows that some subsequence $\{Ce_{k_j}\}$ is a basic sequence equivalent to $\{z_j\}$, a block basis. By hypothesis, $\{z_j\}$ is equivalent to $\{e_j\}$. Thus there

exist $A, B \in [X]$ such that $Be_j = Ce_{k_j}$ and $Ae_j = z_j$. Then $\|(A - CB)e_j\| = \|z_j - Ce_{k_j}\|$. An examination of the above-mentioned theorem indicates we may choose $\|z_j - Ce_{k_j}\| < \varepsilon_j$ and $\|z_j\| > \delta/2$, with $\sum_1^\infty \varepsilon_j < \infty$. Hence, by Lemma 6, $A - CB$ is compact. By assumption, X has $(+)$. Therefore, there exists $P: X \to \overline{\mathrm{sp}}\{z_k\}$.

By lemmas 4 and 5, $A - CB \in \Omega$. Hence, $A^{-1}\cdot P\cdot(A - CB) = I - A^{-1}\cdot P\cdot C\cdot B$. If $C \in \Omega$, then $I \in \Omega$ which is a contradiction.

9. Theorem. *Let X be as in Lemma 8. If Ω is a non-trivial ideal in $[X]$, then Ω is the ideal of compact operators.*

Proof. Let A be a non-compact operator and $A \in \Omega$. Then there exists a sequence $\{x_n\}$, $\|x_n\| = 1$, such that $\{Ax_n\}$ has no convergent subsequence. If the g_j are the coordinate functionals, we may, by the Cantor process, extract a subsequence $\{Ax_{n_k}\}$ such that $g_j(Ax_{n_k})$ converges for each j. $\{Ax_{n_k}\}$ has no convergent subsequence; therefore, choose a further subsequence $\{\bar{x}_l\}$ such that $\|A\bar{x}_l - A\bar{x}_{l+1}\| \geqslant \varepsilon > 0$. Let $z_l = \bar{x}_l - \bar{x}_{l+1}$; then

(I) $g_j(Az_l) \to 0$ and $\inf_l \|Az_l\| \geqslant \varepsilon > 0$. Pick a subsequence of the z_l such that $g_j(z_{l_k})$ converges for each j. Now $\{Az_{l_k}\}$ has no convergent subsequence for, if it did, say $Az_r \to x$, then $g_j(x) = 0$ for all j, by the continuity of the g_j, hence $x = 0$. But $\inf_l \|Az_l\| \geqslant \varepsilon > 0$, thus x cannot $= 0$, a contradiction. Therefore, there exists $b > 0$ and a further subsequence such that

(II) $\inf \|A(z_r - z_{r+1})\| \geqslant b > 0$. Let $y_r = z_r - z_{r+1}$. Now $g_j(y_r) \to 0$ and we must have, by (II) that $\exists\, \delta > 0$ such that $\inf_r \|y_r\| \geqslant \delta$. By theorem 3 of [1] and our hypothesis on X, some subsequence $\{y_{r_k}\}$ is equivalent to $\{e_k\}$. Let $Be_k = y_{r_k}$ and $AB = C$; then $g_j(Ce_k) = g_j(Ay_{r_k}) \underset{k}{\to} 0$ by (I), and $\inf_k \|Ce_k\| = \inf_k \|Ay_{r_k}\| > 0$ by (II). According to Lemma 8, this is impossible, q.e.d.

10. Lemma. *The spaces $l_p (1 < p < \infty)$ and c_0 are block homogeneous.*

Proof. Verify.

11. Lemma. *The spaces $l_{p'} (1 \leqslant p < \infty)$ and c_0 have $(+)$.*

Proof. Let $\{z_k\}$ be a block basis in c_0. There exists $h_k \in [z_k]^*$ such that $\{h_k\}$ is a biorthogonal sequence for the $\{z_k\}$, i.e. $h_k(z_j) = \delta_{kj}$ and $\|h_k\| \leqslant 2$ for all k, [2]. By the Hahn-Banach theorem we extend each h_k to $h'_{k'}$ a continuous linear functional on c_0. Hence, h'_k belongs to l_1 for all k, [1]. Suppose

$$z_k = \sum_{a_k+1}^{a_{k+1}} t_1^{(k)} e_i,$$

where e_i is the standard basis for c_0. Each

$$h'_k = \sum_1^{\infty} a_1^{(k)} f_i$$

and $\|h'_k\| \leqslant 2$ (f_i is the standard basis in l_1). By the form of the norm in l_1 we may take

$$h''_k = \sum_{a_k+1}^{a_{k+1}} a_1^{(k)} f_i$$

and still have $\|h''_k\| \leqslant 2$ and $h''_k(z_j) = \delta_{kj}$. Let $x = \sum_1^{\infty} t_i e_i$. We write

$$x = \sum x_k \quad \text{where} \quad x_k = \sum_{a_k+1}^{a_{k+1}} t_i e_i.$$

The desired projection is then

$$Px = \sum_1^{\infty} h''_k(x_k) z_k.$$

P is clearly the identity on $[z_k]$, therefore, it suffices to show that P is well defined or, using the fact that c_0 is block homogeneous, that the sequence $\{h''_k(x_k)\} \epsilon c_0$. But $x = \sum_1^{\infty} x_k$ hence, $\|x_k\| \to 0$. Using the fact that $\|h''_k\| \leqslant 2$ for all k, the result is obvious.

The proof for the case of l_p, $1 \leqslant p < \infty$, is similar. We may apply the same method arriving at functionals h''_k and letting $x = \sum_1^{\infty} x_k$ we again define $Px = \sum h''_k(x_k) z_k$. However, we must now show $\{h''_k(x_k)\}$ to belong to the appropriate l_p. But

$$\sum |h''_k(x_k)|^p \leqslant \sum \|h''_k\|^p \|x_k\|^p \leqslant 2^p \sum \sum_{a_k+1}^{a_{k+1}} |t_i|^p = 2^p \sum_1^{\infty} |t_i|^p < \infty.$$

12. Corollary [3]. *The ideal of compact operators is unique in l_p, $1 \leqslant p < \infty$, and c_0.*

Proof. Lemmas 10 and 11.

13. Definition. An operator $T \epsilon [X]$ is said to be *strictly singular* if, whenever T has a bounded inverse on a subspace $M \subset X$, then M is finite-dimensional [5].

14. Corollary. *If X has a block homogeneous basis, then the ideal of strictly singular operators $(K(X))$ equals the ideal compact operators $(T(X))$.*

Proof. Suppose not, i.e., $A \epsilon K(X)$ and $A \notin T(X)$. As in the proof of Theorem 9 we obtain a sequence (after relabeling) $\{y_m\}$ such that

(i) $\inf_m \|y_m\| \geqslant \delta > 0$,

(ii) $\inf_m \|Ay_m\| \geqslant b > 0$,

(iii) $g_j(y_m) \underset{m}{\to} 0$,

(iv) $g_j(Ay_m) \underset{m}{\to} 0$,

(v) Ay_m is a basic sequence equivalent to $\{e_m\}$, the basis of X.

By applying Theorem 3 of [1] to $\{y_m\}$ we may extract $\{y_{m_k}\}$, a basic sequence equivalent to $\{e_k\}$, since the space is block homogeneous. If we now take the corresponding Ay_{m_k} and use the block homogeneity again, we get $\{Ay_{m_k}\}$ basic and equivalent to $\{e_k\}$. Let $W = \overline{\mathrm{sp}}\{y_{m_k}\}$. Since $\{y_{m_k}\}$ is a basic sequence, all the y_{m_k} are topologically free, i.e., no one is in the closed linear span of the other. Hence, W is infinite-dimensional. By the above equivalence, we obtain the existence of isomorphism B and C such that

$$\begin{array}{ccc} y_{m_k} & \xrightarrow{B} & e_k \\ & \swarrow{\scriptstyle C} & \\ Ay_{m_k} & & \end{array}$$

i.e., $A = CB$ is an isomorphism between W and AW. This contradicts the fact that A is strictly singular.

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