7. We shall still prove that the commutativity \((f * g) * h = f * (g * h)\) holds also, if one of distributions \(f, g, h\) is of bounded carrier, provided the convolution of two remaining ones exists. We have namely:

**Theorem 7.** If the convolution \(f * g\) exists and \(h\) is a distribution of bounded carrier, then, in the equality \((f * g) * h = f * (g * h)\), all the convolutions exist and the equality holds.

**Proof.** The existence of convolutions \((f * g) * h\) and \(g * h\) follows, as a particular case, from Theorem 5. It remains to prove the existence of \(f * (g * h)\) and the equality. Let \(h_n = (f * \delta_n) * (g * \delta_n)\). Then \(h_n \to f * g\) by hypothesis that \(f * g\) exists. Now, by Theorem 5 we have \(h_n \to (f * g) * h\). Assume that \(h\) is a continuous function. Then

\[
\begin{align*}
  h_n * h &= [(f * \delta_n) * (g * \delta_n)] * h = (f * \delta_n) * [(g * \delta_n) * h] = (f * \delta_n) * [(g * h) * \delta_n],
\end{align*}
\]

here, the second equality follows from the fact that \(f * \delta_n, g * \delta_n, h\) are functions for which the convolutions \((f * \delta_n) * (g * \delta_n) * h\) and \((g * \delta_n) * h\) exist, and the last equality follows from the remark at the end of section 6. Since \(h_n * h\) converges, the convolution \(f * (g * h)\) exists, by definition, and the equality \((f * g) * h = f * (g * h)\) holds. If \(h\) is not a continuous function, then the assertion follows by Theorem 3.

**References**


Reçu par la Redaction le 3. 2. 1967

---

**On the uniqueness of the ideals of compact and strictly singular operators**

by

RICHARD H. HERMAN (Rochester)

The purpose of this note is to extend a result of [3]. In particular, it is shown there that the ideal of compact operators is unique in \(X\), the bounded linear operators from \(X\) to \(X\), where \(X = l_p, 1 \leq p < \infty\) and \(q\). An obvious question is do there exist other spaces for which this is true? We obtain a partial result in this direction by requiring our space to have two properties which \(l_p\) and \(q\) enjoy. In addition, using one of these properties, we show that the ideal of compact and strictly singular operators agree. The phrase “partial result” is used since we cannot at this time exhibit a space with the above-mentioned properties other than \(l_p\) or \(q\). However, the proofs given here have the advantage of treating all cases simultaneously, as opposed to what is done in [3].

We will assume that the reader is somewhat familiar with the theory of Schauder bases in Banach spaces. Results used from this theory may be found in [2].

1. **Definition.** \(\{a\}\) is a *Schauder basis* for \(X\) if for each \(x \in X\),

\[ x = \sum a_i x_i \text{ uniquely. In this case } a_i = g_i(x), g_i \subseteq X^*. \]

2. **Definition.** \(\{a\}\) is said to be a *block basis* if

\[ a = \sum a_i b_i x_i, \text{ } a_i < a_i < \ldots \]

If \(\{a\}\) is a Schauder basis for \(X\), then \(\{a\}\) is a Schauder basis for \(\overline{\text{B}}(\{a\})\), [1].

3. **Definition.** We will say that a Banach space \(X\) has a basis \(\{a\}\) if given a block basis \(\{a\}\), there exists \(P: X \to \overline{\text{B}}(\{a\})\) a projection.

* This work was done while the author was an NDEA Fellow at The University of Maryland.

11 — Studia Mathematica
4. **Lemma.** If \( \Omega \) is a non-zero, two-sided ideal in \([X] \), \( X \) a Banach space, then \( \Omega \) contains all operators with finite-dimensional range.

The following is due to Maddaus [6].

5. **Lemma.** Let \( X \) be a Banach space with a Schauder basis. Every compact operator in \( X \) is the uniform limit of finite-dimensional operators.

6. **Lemma.** Let \( A \in [X] \), \( X \) a Banach space with a Schauder basis \( \{e_i\} \). A sufficient condition for \( A \) to be compact is that

\[
\sum_{i=1}^{\infty} ||Ae_i|| < \infty.
\]

Proof. Since \( X \) is separable, we may apply the condition for compactness given by Gelfand [4]. Let \( (f_j) \subset X^* \) be such that \( f_j \to 0 \) in the weak* topology. We must show \( f_j \to 0 \) uniformly on the image, under \( A \), of the unit ball \( S \) in \( X \). Let \( \varepsilon > 0 \). Then

\[
x = \sum_{i=1}^{\infty} a_i e_i \quad \text{and} \quad A \varepsilon = \sum_{i=1}^{\infty} a_i Ae_i.
\]

Let \( g_j \) be the coordinate functionals. There exists \( M \) such that \( ||g_j|| \leq M \), hence, \( ||a_i|| = ||g_i(x)|| \leq M \). Let \( \varepsilon > 0 \) be given, and pick \( N(e) \) such that

\[
\sum_{i>N} ||Ae_i|| < \frac{\varepsilon}{2M}.
\]

Pick \( J(e) \) large enough so that \( ||f_j(Ae_i)|| < \varepsilon/2NM \) for \( j > J \) and \( i = 1, \ldots, N \). Then

\[
||f_j(Ae)|| < ||f_j\left(\sum_{i=1}^{N} a_i Ae_i\right)|| + ||f_j\left(\sum_{i>N} a_i Ae_i\right)|| < \varepsilon.
\]

Since \( x \) was arbitrary, we are done.

7. **Definition.** A Banach space \( X \) with a Schauder basis \( \{e_i\} \) is said to be block homogeneous if given a block basis \( \{a_i\} \), \( ||a_i|| = 1 \), then \( \sum a_i e_i \) converges if, and only if, \( \sum b_i e_i \) converges. In this case, \( A(\sum b_i e_i) = \sum a_i Ae_i \) is an isomorphism between \( sp(a_i) \) and \( X \).

The following lemma is stated in [3] somewhat differently:

8. **Lemma.** Let \( X \) be a block homogeneous Banach space, with \( \alpha \) and a Schauder basis \( \{e_i\} \). No proper two-sided ideal \( \Omega \subset [X] \) can contain an operator \( C \) such that \( inf ||C e_i|| = \delta > 0 \) and \( lim_{k \to \infty} g_k(C e_i) = 0 \) for each coordinate functional \( g_k \).

Proof. Suppose such an operator was in \( \Omega \). By theorem 3 of [1] it follows that some subsequence \( \{C e_i\} \) is a basic sequence equivalent to \( \{e_i\} \), a block basis. By hypothesis, \( \{e_i\} \) is equivalent to \( \{e_i\} \). Thus there exist \( A, B \in [X] \) such that \( B e_i = C e_i \) and \( A e_i = \gamma_i \). Then \( ||(A - CB)e_i|| = ||e_i - C e_i|| \). An examination of the above-mentioned theorem indicates that we may choose \( ||e_i - C e_i|| < \delta \) and \( ||e_i|| > \delta/2 \), with \( \sum \delta < \infty \). Hence, by Lemma 6, \( A - CB \) is compact. By assumption, \( X \) has \( \gamma \). Therefore, there exists \( P : X \to sp(e_i) \).

By Lemma 4 and \( \gamma \), \( A - CB + \Omega \). Hence, \( A - P \cdot (A - CB) = I - A - P \cdot C - B \). If \( C \in \Omega \), then \( I \in \Omega \) which is a contradiction.

9. **Theorem.** Let \( X \) be as in Lemma 8. If \( \Omega \) is a non-trivial ideal in \([X] \), then \( \Omega \) is the ideal of compact operators.

Proof. Let \( A \) be a non-compact operator and \( A \in \Omega \). Then there exists a sequence \( \{a_i\} \), \( ||a_i|| = 1 \), such that \( \{a_i\} \) has no convergent subsequence. If the \( g_j \) are the coordinate functionals, we may, by the Cantor process, extract a subsequence \( \{a_{i_n}\} \) such that \( g_j(a_{i_n}) \) converges for each \( j \). \( \{a_{i_n}\} \) has no convergent subsequence; therefore, choose a further subsequence \( \{e_i\} \) such that \( ||e_j - e_{i_n}|| > \delta > 0 \). Let \( s_1 = e_1 - e_{i_n} \), then

\[
\begin{align*}
& I \in \Omega \quad g_j(a_{i_n}) \to 0 \quad \text{and} \quad inf ||Ae_i|| > \delta > 0. \\
& \text{Pick a subsequence of the } s_i \text{ such that } g_j(a_{i_n}) \text{ converges for each } j. \text{ Now } \{a_{i_n}\} \text{ has no convergent subsequence for, if it did, say } A e_i \to x, \text{ then } g_j(a_1) = 0 \text{ for all } j \text{ by the continuity of the } g_i, \text{ hence } x = 0. \text{ But } inf ||Ae_i|| > \delta > 0, \text{ thus } x = 0, \text{ a contradiction. Therefore, there exists } b > 0 \text{ and a further subsequence such that } \\
& inf ||Ae_i|| > \delta > 0. \text{ Let } s_2 = e_2 - e_{i_n}, \text{ and } g_j(a_{i_n}) = 0 \text{ for each coordinate functional } g_i. \text{ Let } B e_i = s_2 \text{ and } AB = \Omega; \text{ then } \gamma_i(g_j(C e_i)) = 0, \text{ and } inf ||C e_i|| = inf ||A e_i|| > 0. \text{ According to Lemma 8, this is impossible, q.e.d.}
\end{align*}
\]

10. **Lemma.** The spaces \( l_p(1 < p < \infty) \) and \( c_0 \) are block homogeneous.

Proof. Verify.

11. **Lemma.** The spaces \( l_p(1 \leq p < \infty) \) and \( c_0 \) have \( \gamma \).

Proof. Let \( \{a_i\} \) be a block basis in \( c_0 \). There exists \( a_i^\star \) such that \( (a_i^\star) \) is a biorthogonal sequence for the \( \{a_i\} \), i.e. \( a_i^\star(a_i) = a_i 
(\gamma) \leq 2k \). By the Hahn-Banach theorem we extend each \( a_i \) to \( h_k \) a continuous linear functional on \( c_0 \). Hence, \( h_k \) belongs to \( \gamma \), for all \( k, \{1\} \). Suppose

\[
z_k = \sum_{i=1}^{k} \beta_i e_i.
\]
where \(e_i\) is the standard basis for \(e_i\). Each
\[ h'_k = \sum_{i=1}^{\infty} a_i^p f_i \]
and \(\|h'_k\| \leq 2 \) (where \(f_i\) is the standard basis in \(l_i\)). By the form of the norm in \(l_i\) we may take
\[ h'_k = \sum_{\frac{a_k}{k} = 1}^{\infty} a_i^p f_i \]
and still have \(\|h'_k\| \leq 2\) and \(h'_k(a_i) = \delta_{ik}\). Let \(x = \sum_{i=1}^{\infty} \frac{x_i}{i} q_i\). We write
\[ x = \sum_{i=1}^{\infty} x_i q_i \quad \text{where} \quad x_i = \sum_{\frac{x_i}{i}}^{a_k} \frac{a_k}{k} q_i. \]
The desired projection is then
\[ Px = \sum_{i=1}^{\infty} h'_k(a_i) x_i. \]

P is clearly the identity on \([x_0]\), therefore, it suffices to show that \(P\) is well defined or, using the fact that \(e_i\) is block homogeneous, that the sequence \((h'_k(a_i)) e_i\). But \(x = \sum_{i=1}^{\infty} x_i q_i\) hence, \(\|x_0\| \to 0\). Using the fact that \(\|h'_k\| \leq 2\) for all \(k\), the result is obvious.

The proof for the case of \(l_p, 1 < p < \infty\), is similar. We may apply the same method arriving at functionals \(h''_k\) and letting \(x = \sum_{i=1}^{\infty} x_i q_i\) again define \(Px = \sum_{i=1}^{\infty} h''_k(a_i) x_i\). However, we must now show \((h''_k(a_i))\) to belong to the appropriate \(l_p\). But
\[ \sum_{i=1}^{\infty} |h''_k(a_i)|^p \leq \sum_{i=1}^{\infty} |h'_k|^p |x_i|^p \leq 2^p \sum_{i=1}^{\infty} |x_i|^p = 2^p \sum_{i=1}^{\infty} |x_i|^p < \infty. \]

12. Corollary [3]. The ideal of compact operators is unique in \(l_p, 1 < p < \infty\), and \(e_i\).

Proof. Lemmas 10 and 11.

13. Definition. An operator \(T \in X\) is said to be strictly singular if, whenever \(T\) has a bounded inverse on a subspace \(M \subseteq X\), then \(M\) is finite-dimensional [5].

14. Corollary. If \(X\) has a block homogeneous basis, then the ideal of strictly singular operators \((K(X))\) equals the ideal compact operators \((T(X))\).

Proof. Suppose not, i.e., \(A \in K(X)\) and \(A \notin T(X)\). As in the proof of Theorem 9 we obtain a sequence (after relabeling) \((y_m)\) such that

(i) \(\inf \|y_m\| \geq \delta > 0\),
(ii) \(\inf \|Ay_m\| \geq \beta > 0\),
(iii) \(g(y_m) \to 0\),
(iv) \(g(Ay_m) \to 0\),
(v) \(Ay_m\) is a basic sequence equivalent to \((e_i)\), the basis of \(X\).

By applying Theorem 3 of [1] to \((y_m)\) we may extract \((y_{m_0})\), a basic sequence equivalent to \((e_i)\), since the space is block homogeneous. If we now take the corresponding \(AY_{m_0}\) and use the block homogeneity again, we get \((AY_{m_0})\) basic and equivalent to \((e_i)\). Let \(W = \overline{\text{SP}}(y_{m_0})\).

Since \((y_{m_0})\) is a basic sequence, all the \(y_{m_0}\) are topologically free, i.e., no one is in the closed linear span of the other. Hence, \(W\) is infinite-dimensional. By the above equivalence, we obtain the existence of isomorphism \(B\) and \(C\) such that

\[ y_{m_0} \xrightarrow{B} e_k \]

\[ A \xrightarrow{C} \]

i.e., \(A = CB\) is an isomorphism between \(W\) and \(AW\). This contradicts the fact that \(A\) is strictly singular.

References