Posons

(40) $$F_{i,\varepsilon}(t_1, \ldots, t_n) = \int \underset{R_n}{\ldots} \int K(t_1-s_1, \ldots, t_n-s_n) F_i(s_1, \ldots, s_n) ds_1 ds_2 \ldots ds_n.$$

On tire de 1°, 2° et 3°

(41) $$\delta_\varepsilon = \sup_{t_k} |F_{i,\varepsilon}(t_1, \ldots, t_n) - F_i(t_1, \ldots, t_n)| \to 0 \quad (\varepsilon \to 0).$$

En appliquant à (38) et (39) l'opération K_ε, définie par (40), on vérifie que les fonctions $F_{i,\varepsilon}$ satisfont aussi à (38) et (39); par conséquent, les $F_{i,\varepsilon}$ étant de classe C_1, elles satisfont à (26) et (27). Définissons $\Psi_\varepsilon(u, h)$ comme la fonctionnelle qui résulte de $\Psi(u, h)$, si l'on remplace dans la formule (22) les fonctions F_i par $F_{i,\varepsilon}$; définissons ensuite l'opération F_ε par l'identité $(F_\varepsilon(u), h)_1 = \Psi_\varepsilon(u, h)$, pour $\varepsilon > 0$. En vertu de (41) on a

(42) $$|\Psi(u, h) - \Psi_\varepsilon(u, h)| \leqslant \delta_\varepsilon \|h\|_1;$$

par conséquent, pour $u \in M$,

(43) $$\|F(u) - F_\varepsilon(u)\|_1 \leqslant \delta_\varepsilon \to 0 \quad (\varepsilon \to 0)$$

vu que $(F_\varepsilon(u), h) = \Psi_\varepsilon(u, h)$, $(F(u), h) = \Psi(u, h)$. Or, les fonctions $F_{i,\varepsilon}$ satisfaisant aux hypothèses de 3.1, l'opération F_ε satisfait aux inégalités (4), (5) et (6) du moins pour $x, y \in M$; en vertu de (43) il en est de même de F pour $x, y \in M$, d'où l'on conclut, en suivant la démonstration du théorème 1, qu'il existe un seul $\bar{u} \in H_1$ tel que $F(\bar{u}) = 0$. Un remarque analogue sera vraie aussi pour 3.2.

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Reçu par la Rédaction le 12. 11. 1966

STUDIA MATHEMATICA, T. XXIX. (1968)

On the spectrum of finitely-generated locally m-convex algebras *

by

R. M. BROOKS (Minneapolis)

If A is a commutative complete locally m-convex algebra with identity, then there is a natural map $m \to (\hat{a}_1(m), \ldots, \hat{a}_N(m))$ of the spectrum M of A onto the joint spectrum of any generating family $\{a_1, \ldots, a_N\}$ for A. If A is a Banach algebra, then the mapping is topological. We shall show that for non-Banach algebras one can make only the obvious statement that the map is a continuous injection, even in the simplest case (an F-algebra with one generator). We demonstrate this with a series of examples and show (i) it may occur that one generating family yields a topological map, while a second fails to, (ii) it may be that no generating family induces a homeomorphism, (iii) the joint spectrum of a generating family need not be polynomially convex (contrary to the Banach algebra results).

We show that while $\sigma(a_1, \ldots, a_N)$ need not be polynomially convex, it is polynomially convex with respect to a certain family of compact subsets determined by the algebra, and we give conditions in terms of the family $\{a_1, \ldots, a_N\}$ and its action on the equicontinuous subsets of M in order that the natural map be topological. These conditions are necessary and sufficient in case A is an F-algebra, sufficient but not necessary for more general algebras.

If S is a compact subspace of C^N, then there exists an N-generated Banach algebra A such that S is the spectrum of A if, and only if, S is polynomially convex. We consider this question for locally m-convex algebras and show (i) if S is a subspace of C^N, then S is the spectrum of an N-generated F-algebra if, and only if, S is hemi-compact and polynomially convex, (ii) every subspace of C^N is the spectrum of an N-generated locally m-convex algebra.

1. The natural maps of the spectrum. In this paper we shall consider only commutative complete locally m-convex algebras with identity and shall write "locally m-convex algebra" rather than the longer, more complete, description. A locally m-convex algebra is a locally convex (Haus-

* The research for this paper was supported in part by NSF Grant 5707.

dorff) topological algebra A whose topology is given by a directed family $\{\|\cdot\|_n : n \in D\}$ of pseudonorms (submultiplicative convex symmetric functionals), where $(D, \leqslant)$ is a directed set and $n \leqslant m$ if, and only if, $\|a\|_n \leqslant \|a\|_m$ for each $a \in A$. Each $\|\cdot\|_n$ determines a Banach algebra A_n, the completion of $A/\{x: \|x\|_n = 0\}$ with respect to the norm $\|\pi_n x\| = \|x\|_n$ (π_n being the natural homomorphism). If $n \leqslant m$, then π_n and π_m induce a norm-decreasing homomorphism π_{nm} of A_m onto a dense subalgebra of A_n. The resulting family of Banach algebras and homomorphisms is an inverse limit system and $\operatorname{liminv}\{A_n\}$ is (topologically and algebraically) the completion of A. For details of this construction and the basic properties of these algebras the reader is referred to [6].

The *spectrum* of A is the space M of all non-zero continuous complex-valued homomorphisms on A endowed with the Gelfand (relative w^*-) topology. For each $a \in A$, the function $\hat{a}: M \to \boldsymbol{C}$ defined by $\hat{a}(m) = m(a)$ is continuous and the map $a \to \hat{a}$ is a homomorphism of A onto $A^\wedge \subseteq C(M)$. For each $n \in D$ the map π_n induces a homeomorphism $\pi_n^* | M(A_n)$ of $M(A_n)$ onto a compact subset M_n of M (π_n^* is the dual map of A_n^* to A^*), and $M = \bigcup\{M_n : n \in D\}$, where $n \leqslant m$ implies $M_n \subseteq M_m$. The family $\{M_n : n \in D\}$ covers every equicontinuous subset of M (cf. [1], Lemma 5.1.1), where by "covers" we mean that if E is equicontinuous, then there exists $n \in D$ such that $E \subseteq M_n$.

An F-algebra is a complete locally m-convex algebra whose topology is given by a countable (ascending) sequence of pseudonorms. In this case, M is σ-compact and every compact subset of M is equicontinuous (cf. [6], Proposition 4.2). Thus, $\{M_n : n = 1, 2, \ldots\}$ is a *k-covering sequence* for M ("k-covering" means that the family covers every compact subset). If a topological space has a k-covering sequence of compact subsets, we shall say that the space is *hemicompact*.

If $\{a_1, \ldots, a_N\} \subseteq A$, then the joint spectrum $\sigma(a_1, \ldots, a_N)$ of this family is $\{(\hat{a}_1(m), \ldots, \hat{a}_n(m)) : m \in M\}$ ($\sigma_A(a_1, \ldots, a_N)$ if we wish to specify the algebra with respect to which we consider the spectrum); and for each $n \in D$,

$$\sigma_n(a_1, \ldots, a_N) = \{(\hat{a}_1(m), \ldots, \hat{a}_N(m)) : m \in M_n\} = \sigma_{A_n}(\pi_n a_1, \ldots, \pi_n a_N).$$

If $\{a_1, \ldots, a_N\} \subseteq A$, then $P(a_1, \ldots, a_N)$ is the closure in A of $\{p(a_1, \ldots, a_N) : p \in P^N\}$, where P^N is the family of all polynomials in N variables with complex coefficients. In order that a family $\{a_1, \ldots, a_N\}$ generate A it is necessary and sufficient that for each $n \in D$ the family $\{\pi_n a_1, \ldots, \pi_n a_N\}$ generate A_n. This last statement is easily verified; see, for example, ([4], p. 334-340) for the case $N = 1$.

We begin with a series of examples which show the nature and extent of the pathology involved. We note that anything that goes awry does so for singly-generated F-algebras.

EXAMPLE 1.1. We let $A = C[0, 2\pi)$ with the pointwise operations and the compact-open topology, and define a, b in A by $a(t) = t$ and $b(t) = \exp(it)$ for each $t \in [0, 2\pi)$. From Example 7.6 of [6] it follows that $M = [0, 2\pi) = \sigma(a)$, and it is clear that a generates A. An ascending sequence of pseudonorms for A is obtained by taking an ascending sequence $\{t_n : n = 1, 2, \ldots\}$ of positive numbers satisfying $\lim t_n = 2\pi$, and defining

$$\|a\|_n = \sup\{|a(t)| : 0 \leqslant t \leqslant t_n\}.$$

In this case $A_n = C[0, t_n]$ and π_n is the restriction mapping. Since $b | [0, t_n]$ generates A_n for each n, we see that b generates A. But $\sigma(b)$ is the unit circle. Hence, it may occur that one generator induces a topological mapping while another doesn't. Also, we see that the spectrum of a generator need not be polynomially convex. We shall define this below.

We give two examples to show that it is possible to have no homeomorphism-inducing generators; the first is extremely simple, while the second (for F-algebras) is somewhat more complicated.

EXAMPLE 1.2. Let $T = \boldsymbol{R}$ with the discrete topology and $A = C(T)$ with the compact-open topology (topology of pointwise convergence). Then A is a complete locally m-convex algebra and is generated by $a(t) = t$. However, if b is any generator of A, then $\sigma(b) = \hat{b}(M) = b(T)$ ($M = T$ as in Example 1.1) and $t \to b(t)$ cannot be topological, since T is an uncountable discrete space and cannot be embedded in $\boldsymbol{C}^1$.

Suppose (T, τ) is a completely regular Hausdorff space and $\mathscr{K} = \{K_n : n = 1, 2, \ldots\}$ is a sequence of compact subsets of T, nested upward, which covers T. Let k denote the weak topology generated by the family $\mathscr{K}$ and let T^* denote T with the k-topology. Then $\tau \leqslant k$ and the topology k is generated by the family $\mathscr{K}^*$ ($\mathscr{K}$ as family of subspaces of T^*) in the sense of Michael ($U \subseteq T^*$ is k-open if, and only if, $U \cap K_n$ is relatively k-open for each n). Thus, $C(T^*)$ with the topology of uniform convergence on members of $\mathscr{K}^*$ is an F-algebra and $M = T^*$. Moreover, every compact subset of T^* (i.e., of M) is equicontinuous and contained in some $K_n (= M_n)$ (cf. [6], Proposition 4.2, Lemma D.5, and Example 7.6, and [1], Lemma 5.1.1).

EXAMPLE 1.3. We fix a decreasing sequence $\{\theta_n : n = 1, 2, \ldots\}$ of positive numbers such that $\theta_1 < \pi$ and $\lim \theta_n = 0$, let L_n denote the segment $[0, \exp(i\theta_n)]$ in the plane and set $K_n = \bigcup\{L_i : i = 1, 2, \ldots, n\}$. We let $T = \bigcup\{K_n : n = 1, 2, \ldots\}$, let t_0 denote the common point (0) of the segments L_n, and let π denote the relative plane topology on T. The space (T, π) and the sequence $\mathscr{K} = \{K_n\}$ satisfy the requirements of our discussion above, and $A = C(T^*)$ with the topology of uniform convergence on members of $\mathscr{K}$ is an F-algebra, and every k-compact subset of T^* is contained in some K_n.

We now show that T^* is not metrizable. Suppose d is a metric on T which defines the topology k; then $d|(K_n \times K_n)$ must be equivalent to the usual metric of the plane restricted to $K_n \times K_n$, since k and π agree on each K_n. Therefore, we can choose for each n an element $t_n \in I_n \subseteq K_n$ such that $d(t_n, t_0) < 2^{-n}$. But then $\{t_n: n = 1, 2, \ldots\}$ converges to t_0 in the k-topology; hence, $K = \{t_n: n = 0, 1, \ldots\}$ is k-compact. But $K \nsubseteq K_n$, for each n, a contradiction.

We complete the example by showing that A is singly-generated. From the non-metrizability of T^* it is clear that there is no homeomorphism-inducing generator. The function $a(t) = t$ generates A since $A_n = C(K_n)$ for each n and $a|K_n$ generates $C(K_n)$. This last fact follows from Mergelyan's theorem [5], since each K_n is a compact polynomially convex subset of C^1 with no interior. We note here that this also yields a simple example of a separable F-algebra with non-metrizable spectrum.

Definition 1.1. If K is a compact subset of C^N, the *polynomially convex hull* $H(K)$ of K is the set of all $\zeta \in C^N$ such that

$$|p(\zeta)| \leqslant \|p\|_K = \max\{|p(\xi)|: \xi \in K\}$$

for every $p \in P^N$ (polynomials in N variables). A compact set K is called *polynomially convex* provided $H(K) = K$.

Definition 1.2. If S is a subset of C^N and $\mathscr{K}$ is a family of compact subsets of S which covers S we define $H(S, \mathscr{K}) = \bigcup \{H(K): K \in \mathscr{K}\}$ and say that S is *$\mathscr{K}$-polynomially convex* if $H(S, \mathscr{K}) \subseteq S$ (hence, $H(S, \mathscr{K}) = S$). If S is $\mathscr{K}$-polynomially convex for $\mathscr{K}$ = the family of all compact subsets of S we shall say that S is *polynomially convex*.

We state, for completeness, the following theorem without proof:

THEOREM 1.1 *Let A be a commutative complete locally m-convex algebra with identity. Suppose A is finitely-generated and $\{a_1, \ldots, a_N\}$ is a generating family for A. Then the mapping $m \to (\hat{a}_1(m), \ldots, \hat{a}_N(m))$ of M onto $\sigma(a_1, \ldots, a_N)$ is a continuous injection.*

THEOREM 1.2. *Let A be a finitely-generated locally m-convex algebra and $\{a_1, \ldots, a_N\}$ a generating family for A. If $\{\|\cdot\|_n: n \in D\}$ is any directed family of pseudonorms for A, then* (i) *for each $n \in D$ the compact set $\sigma_n(a_1, \ldots, a_N)$ in C^N is polynomially convex*; *thus,* (ii) *$\sigma(a_1, \ldots, a_N)$ is $\{\sigma_n(a_1, \ldots, a_N)\}$-polynomially convex.*

Proof. (ii) is an immediate consequence of (i), and (i) follows from the fact that for each $n \in D$ the family $\{\pi_n(a_1), \ldots, \pi_n(a_N)\}$ generates the Banach algebra A_n and $\sigma_n(a_1, \ldots, a_N)$ is exactly the joint spectrum of this family.

LEMMA 1.3. *If X is a topological space, Y is a k-space ($F \subseteq Y$ is closed if, and only if, $F \cap K$ is compact for each compact subset K of Y), and $f: X \to Y$ is a continuous bijection, then f is a topological map if, and only if, the inverse image of each compact subset of Y is compact.*

Proof. It is clear that the condition is necessary (even for Y not a k-space). For the sufficiency we fix a closed subset S of X. Then $f(S) \subseteq Y$ and $f(S) \cap K = f[S \cap f^{-1}(K)]$ for each compact $K \subseteq Y$. But $f^{-1}(K)$ is compact. Therefore, $S \cap f^{-1}(K)$ and $f(S) \cap K$ are also compact. Thus, $f(S)$ is closed and f is a homeomorphism.

THEOREM 1.3. *Let A be a finitely-generated locally m-convex algebra with generating family $\{a_1, \ldots, a_N\}$ and let $\varphi: M \to \sigma(a_1, \ldots, a_N)$ be the natural mapping. A sufficient condition in order that φ be a topological map is that there exists a directed family $\{\|\cdot\|_n\}$ of pseudonorms for A such that the family $\{\sigma_n(a_1, \ldots, a_N)\}$ is a k-covering family for $\sigma(a_1, \ldots, a_N)$. If A is an F-algebra, then this condition is also necessary, where "directed family" is replaced by "ascending sequence."*

Proof. Let A, $\{a_1, \ldots, a_N\}$, and $\{\|\cdot\|_n: n \in D\}$ be as in the statement of the theorem. We fix a compact set K in $\sigma(a_1, \ldots, a_N)$. Then $K \subseteq \sigma_n(a_1, \ldots, a_N)$ for some $n \in D$ and $\varphi^{-1}(K) \subseteq M_n$. But $\varphi^{-1}(K)$ is closed and M_n is compact. Thus, $\varphi^{-1}(K)$ is compact and φ is topological by Lemma 1.3, since every metric space is a k-space (cf. [6], Example D.2).

If A is an F-algebra with generating family $\{a_1, \ldots, a_N\}$ and φ is a topological map, then $\varphi^{-1}(K)$ is compact for each compact K in $\sigma(a_1, \ldots, a_N)$ and $\varphi^{-1}(K) \subseteq M_n$ for some integer n. But then, $K \subseteq \sigma_n(a_1, \ldots, a_N)$.

We now give an example to show that the condition is not necessary for non-F-algebras.

EXAMPLE 1.4. We let I denote the unit interval $[0, 1]$, $\mathscr{K}$ the family of all compact and countable subsets of I, and A denotes $C(I)$ with the topology of uniform convergence on members of $\mathscr{K}$. Then A is a complete locally m-convex algebra and the family $\{\|\cdot\|_K = \max|\cdot| \text{ on } K: K \in \mathscr{K}\}$ defines the topology of A. Moreover, $M(A)$ is homeomorphic to I (cf. [6], Examples 3.8 and 7.6). If $K \in \mathscr{K}$, then $A_K = C(K)$, and π_K is the restriction map. The algebra A is generated by the function $a(t) = t$, since $a|K$ generates $C(K)$ for each $K \in \mathscr{K}$. Thus $M \to \sigma(a) = I$ is topological. But, $I \nsubseteq \sigma_K(a)$ for each $K \in \mathscr{K}$. It is then easily verified that no family of pseudonorms for A will yield a k-covering family for I.

Remarks. Admittedly, the condition in Theorem 1.3 is not a pleasing one because of the clumbsiness involved in stating it. However, the examples above indicate that any such condition must contain statements concerning the action of the generating family (i.e., of φ) on the building blocks M_n of M. Also, the last example shows that the general algebras do not behave nearly as nicely as the F-algebras. Theorem 2.1 (below) will indicate this fact even more strongly.

2. Spectra in C^N. In his notes on Banach algebras Hoffman [3] gives a succint description of those (necessarily) compact subsets of C^N which may be the spectrum of an N-generated Banach algebra with identity. The description depends on a proper interpretation of the word "be". For Banach algebras one says that a compact subset S of C^N is the spectrum of an N-generated Banach algebra A if, and only if, there exists a generating family $\{a_1, \ldots, a_N\}$ for A such that $S = \sigma(a_1, \ldots, a_N)$. The result is that a compact set $S \subseteq C^N$ is the spectrum of an N-generated Banach algebra if, and only if, S is polynomially convex.

Definition 2.1. If S is a subset of C^N, then we say that S is the spectrum of an N-generated locally m-convex algebra A, provided there exists a generating family $\{a_1, \ldots, a_N\}$ for A such that (i) $S = \sigma(a_1, \ldots, a_N)$ and (ii) $m \to (\hat{a}_1(m), \ldots, \hat{a}_N(m))$ is a homeomorphism.

We now show that general locally m-convex algebras differ radically from Banach algebras with respect to this problem.

LEMMA 2.1. *If $K \subseteq C^N$ is compact and countable, then $C(K) = P(K)$, the uniform closure on K of the algebra of all polynomials in N variables.*

Proof. We first show that if K is any countable subset of C^N, then there exists a linear polynomial q in P^N such that $q|K$ is one-to-one. In fact, enumerating $K = \{\zeta^1, \zeta^2, \ldots\}$ the problem is equivalent to finding $a \in C^N$ such that if $i \neq j$, then

$$\sum_{k=1}^{n} a_k(\zeta_k^i - \zeta_k^j) \neq 0.$$

A simple way of showing the existence of such an a is the following. Let $\lambda_{ij} = \zeta^i - \zeta^j$ for $i \neq j$. Then $\{\lambda_{ij}; i, j = 1, 2, \ldots, i \neq j\}$ is a sequence of vectors in C^N (considered as an N-dimensional Hilbert space with inner product $\langle , \rangle$), and we want a vector $a \in C^N$ such that $\langle \lambda_{ij}, a\rangle \neq 0$ for each pair (i, j), $i \neq j$. A category arguement shows that $\bigcup_{i \neq j} \{\zeta: \langle \lambda_{ij}, \zeta\rangle = 0\}$ is not C^N, since $\{\lambda_{ij}\}$ is countable. Choose an a not in the union and let $q(\zeta) = \sum_{k=1}^{N} \bar{a}_k \zeta_k$. Then $q|K$ is one-to-one.

If K is compact and countable and we choose q as above, then $q(K)$ is a compact and countable subset of C^1. Therefore $q(K)$ does not separate the plane and $C(q(K)) = P(q(K))$. Fix $f \in C(K)$ and $\varepsilon > 0$, and define $g: q(K) \to C$ by $g(q(\zeta)) = f(\zeta)$. Such a definition is possible since q is one-to-one, and g is continuous since f is continuous and q is closed. There exists $p \in P^1$ such that $\|g - p\|_{q(K)} < \varepsilon$. Then $p \circ q \in P^N$ and $\|f - p \circ q\|_K < \varepsilon$.

THEOREM 2.1. *If S is any subset of C^N, then there exists an N-generated locally m-convex algebra A such that S is the spectrum of A.*

Proof. Let $\mathscr{K}$ be the family of all compact and countable subsets of S, and let $A = C(S)$ with the topology of uniform convergence on members of $\mathscr{K}$. Then $\mathscr{K}$ generates the relative Euclidean topology of S and A is complete (cf. [6], Lemma D.5). Moreover, if for each $K \in \mathscr{K}$ we define $\|a\|_K = \sup\{|a(\zeta)|: \zeta \in K\}$, then $\{\|\cdot\|_K: K \in \mathscr{K}\}$ is a directed family of pseudeonorms defining the topology of A and $A_K = C(K)$ for each $K \in \mathscr{K}$. The N functions $z_1(\lambda) = \lambda_1, \ldots, z_N(\lambda) = \lambda_N$ generate A since for each $K \in \mathscr{K}$ the algebra generated by the family $\{z_i|K: i = 1, 2, \ldots, N\}$ is just $P(K)$ which we have shown to be $C(K)$. Moreover, $\sigma(z_1, \ldots, z_N)$ is exactly S which is homeomorphic to $M(A)$ by Example 7.6 of [6].

We consider now F-algebras. If $S \subseteq C^N$ is to be the spectrum of an F-algebra, then S must be hemi-compact, so we can restrict our attention to such subspaces of C^N.

THEOREM 2.2. *If S is a hemi-compact subspace of C^N, then there exists an N-generated F-algebra A such that S is the spectrum of A if, and only if, S is polynomially convex.*

Proof. Sufficiency. Let S be a polynomially convex hemi-compact subset of C^N and let $\{S_n: n = 1, 2, \ldots\}$ be an ascending k-covering sequence for S. Since, for each n, we have $H(S_n) \subseteq S$, we may assume that each S_n is polynomially convex. Since S is a first countable space, S is a k-space and $C(S)$ with the compact-open topology is an F-algebra. Moreover, the family $\{\|\cdot\|_n: n = 1, 2, \ldots\}$ of pseudonorms defines the topology of $C(S)$, where $\|a\|_n = \sup\{|a(\zeta)|: \zeta \in S_n\}$, since the family $\{S_n\}$ is k-covering. The functions $z_1(\lambda) = \lambda_1, \ldots, z_n(\lambda) = \lambda_N$ generate a closed subalgebra A of $C(S)$, which is necessarily an F-algebra. Also, $\sigma_C(z_1, \ldots, z_N) = S$ and $\sigma_n(z_1, \ldots, z_N) = S_n$ (spectrum with respect to $C(S)$) for each n. Since $\sigma_A(z_1, \ldots, z_N)$ is the $\{\sigma_n(z_1, \ldots, z_N)\}$-polynomically convex hull of $\sigma_C(z_1, \ldots, z_N) = S$ and S is polynomially convex, we have $\sigma_A(z_1, \ldots, z_N) = S$. In view of Theorem 1.3, we have only to show that $\{\sigma_{A,n}(z_1, \ldots, z_N)\}$ is a k-covering sequnce of S. But $\sigma_{A,n}(z_1, \ldots, z_N) = \sigma_{C,n}(z_1, \ldots, z_N) = S_n$ and the conclusion follows.

Necessity. Suppose S is a hemi-compact subset of C^N which is the spectrum of some N-generated F-algebra A. Let $\{a_1, \ldots, a_N\}$ be a generating family for A such that $S = \sigma(a_1, \ldots, a_N)$ and the natural map φ of M onto S is topological. By Theorem 1.3, $\{\sigma_n(a_1, \ldots, a_N)\}$ is a k-covering sequence for S, where $\{\sigma_n(a_1, \ldots, a_N)\}$ is determined by some sequence of pseudonorms for A. If K is any compact subset of S, then $K \subseteq \sigma_n(a_1, \ldots, a_N)$ for some n, and $H(K) \subseteq H(\sigma_n(a_1, \ldots, a_N)) = H(\sigma_{A_n}(\pi_n a_1, \ldots, \pi_n a_N)) = \sigma_{A_n}(\pi_n a_1, \ldots, \pi_n a_N) = \sigma_n(a_1, \ldots, a_N) \subseteq S$, since $\{\pi_n a_1, \ldots, \pi_n a_N\}$ generates the Banach algebra A_n. Thus, S is polynomially convex.

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UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA

Reçu par la Rédaction le 23. 11. 1966

STUDIA MATHEMATICA, T. XXIX. (1968)

Sequential theory of the convolution of distributions

by

J. MIKUSIŃSKI (Katowice)

1. In [3], a concept of *regular operations* has been introduced. Such operations extend automatically from functions to distributions and retain their properties. Operations which are not regular, are called *irregular*. Irregular operations can not be extended on arbitrary distributions. However, there exists a general method which permits, in cases where it is possible, to perform such an extension (see [2]).

Convolution is one of the most important irregular operations. Its extension on distributions was largely investigated by Laurent Schwartz [4] and other authors. The sequential approach which is the subject of the present paper makes use of the general method of defining irregular operations so that the definition of the convolution is nothing else but a particular case of it. It turns out that this definition embraces all cases in which the convolution was defined previously by other methods. This uniform approach can also be considered as more elementary, because it does not need any concepts of functional analysis or topology.

Beside the new approach to known facts, there is also a number of theorems which are stated, in this paper, for the first time.

In what follows we shall use the notation and the terminology of [3].

2. If φ is a smooth function of bounded carrier, then the convolutions

$$(1)\qquad f*\varphi = \int_{-\infty}^{\infty} f(x-t)\varphi(t)\,dt \quad\text{and}\quad \varphi*f = \int_{-\infty}^{\infty} \varphi(x-t)f(t)\,dt$$

are defined for every distribution f, as regular operations, performed on f. (It should be emphasized that the convolution is an irregular operation only if it is considered as an operation on two functions or distributions. Otherwise it is regular.) Such convolutions preserve, for any distribution f, their ordinary properties:

$$(2)\qquad f*\varphi = \varphi*f,$$