ist, so besitzt die verallgemeinerte quasilineare elliptische Differentialgleichung

$$F_u(\eta) = 0 \quad \text{für alle } \eta \in H_B$$

mindestens eine Lösung $u \in H_B$, $\|u\|_{H_B} < e$.

Dabei haben a_r, b_r *und* p_r *die in den Voraussetzungen und im Hilfssatz 4 angegebene Bedeutung, während* K_p^k *die Einbettungskonstanten aus Hilfssatz 1 sind.*

FOLGERUNG 1. *Aus Formel* (18) *folgt, daß das Problem mindestens eine Lösung besitzt, wenn die Konstanten* a_r *hinreichend klein sind.*

FOLGERUNG 2. *Die explizite Angabe der Einbettungskonstanten, Formel* (8) *und Formel* (13) *zeigen, daß die Aufgabe bei gegebenen Konstanten* a_r *und* b_r *mindestens eine Lösung besitzt, wenn* $|\Omega|$ *hinreichend klein ist.*

FOLGERUNG 3. *Aus der Herleitung des Existenzsatzes folgt auch der bekannte Sachverhalt, daß das Problem mindestens eine Lösung besitzt, wenn die Funktionen* $B^\lambda(x, \zeta^{(\varrho)})$ *einer Wachstumsbeschränkung der Form*

$$|B^\lambda(x), \zeta^{(\varrho)})| \leqslant c \sum_{\substack{\varrho \\ |\varrho|=0,\ldots,m-1}} |\zeta^{(\varrho)}|^\gamma + b(x),$$

$b(x) \geqslant 0$, $b(x) \in L_1(\Omega)$, $\gamma < 1$, *genügen. In diesem Fall kann man bekanntlich eine Apriori-Abschätzung für eventuelle Lösungen herleiten.*

Die Überlegungen dieser Arbeit kann man auf unbeschränkte Gebiete übertragen, die kein endliches Maß besitzen. Ferner ist eine Ausdehnung der Betrachtungen auf Systeme von Differentialgleichungen möglich, [7].

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STUDIA MATHEMATICA, T. XXIX. (1968)

Characterizations of solutions of $\Delta f = cf$

by

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1. In [1] the author gives a characterization of families of analytic mappings between infinite-dimensional real Banach spaces, making no requirement of differentiability; employing Lipschitz type conditions, and such algebraic and geometric properties as closure under addition and closure under linear translation. Differentiability and power series expansions are obtained directly by functional analysis arguments. In this paper we restrict our attention to functions on subsets of a Euclidean space E into the reals R.

If Δ is the elementary Laplacian operator and c an arbitrary constant, then the families of functions in question are solution spaces of equations of the form

$$\Delta f = cf.$$

THEOREM 1. *Let* f_0 *be an element of the collection* $\mathfrak{F}$ *of continuous functions on open subsets of the Euclidean space* E *into* R. *Then the following statements are equivalent:*

(1) (basic characterization) f_0 *is an element of a family* F *of functions of* $\mathfrak{F}$ *such that*:

(a) *For* $f, g \in F$, *the sum* $f+g$ *defined on the intersection of the domains of* f *and* g, *lies in* f.

(Note. we allow the notion of the „null" function whose domain is the empty set.)

(b) $rf \in F$ *for* $r \in R$, $f \in F$.

(c) *For* $f \in F$, *and* S *an open set in* E, *the restriction* $f|S$ *of* f *to* S *lies in* F.

(d) *For* $f \in F$, $x \in E$, *the translate* f_x *lies in* F, *where* $f_x(y) = f(y-x)$ *for all* $y \in E$ *such that* $y-x \in \operatorname{dom} f$.

(e) *For* $f \in F$, *and* g *a rotation of* E *into itself, the composition of* f *and* g, fg, *lies in* F.

(f) *If* $f_1, f_2, \ldots$ *is a sequence of elements of* F *with common domain* S, *which converges uniformly on* S *to a limit function* f, *then* $f \in F$.

(g) *If h is a radial function of F, that is a function defined on a spherical neighborhood of zero such that $\|x\| = \|y\|$ implies $h(x) = h(y)$ for $x, y \in \operatorname{dom} f$, then $h(0) = 0$, implies that h vanishes identically.*

(2) *f_0 is a volume mean function, i.e. for all $\delta > 0$, there exists $N(\delta) \in R$ such that for all $x \in \operatorname{dom} f_0$, such that the closure $\overline{U}_x(\delta)$ of the spherical region $U_x(\delta) = \{y \in E;\ \|y - x\| < \delta\}$ lies in $\operatorname{dom} f_0$, we have*

$$N(\delta)\cdot f_0(x) = \delta^{-p} \int_{U_x(\delta)} f_0\, dm,$$

where p is the dimension of E, and m is Lebesgue measure on E normalized so that $m[U_0(1)] = 1$.

(3) *f_0 is a surface mean function, i.e. for all $\delta > 0$, there exists $M(\delta) \in R$ such that for all $x \in \operatorname{dom} f_0$, $\overline{U}_x(\delta) \subseteq \operatorname{dom} f_0$, implies*

$$M(\delta)\cdot f_0(x) = \int_{B_x(\delta)} f_0\, d\mu$$

where μ is normalized surface measure on the boundary $B_x(\delta)$ of $U_x(\delta)$.

(4) *f_0 is twice continuously differentiable and there exists $c \in R$, such that $\Delta f_0(x) = c f_0(x)$ for all $x \in \operatorname{dom} f_0$.*

Moreover f_0 satisfies the basic characterization (1), *if f_0 is a member of a TG family F of $\mathfrak{F}$, that is a family F of $\mathfrak{F}$ satisfying conditions* (a)-(f) *of* (1), *such that:*

(5) *For $h \in F$, h satisfies the maximum modulus theorem, i.e. $\delta > 0$, $x \in \operatorname{dom} h$, $\overline{U}_x(\delta) \subseteq \operatorname{dom} h$, implies $|h(z)| \leqslant \sup\{|h(t)|;\ t \in B_x(\delta)\}$ for all $z \in U_x(\delta)$; or*

(6) *For $h \in F$, h satisfies the minimum modulus theorem, i.e. $\delta > 0$ $x \in \operatorname{dom} h$, $\overline{U}_x(\delta) \subseteq \operatorname{dom} h$, implies $|h(z)| \geqslant \inf\{|h(t)|;\ t \in B_x(\delta)\}$ for all $z \in U_x(\delta)$.*

If in the basic characterization (1) we adjoin the condition (h) that F contains the constant function $\bar{1}$, then we have $c = c\bar{1}(0) = \Delta\bar{1}(0) = 0$, and thus $\Delta f_0 \equiv 0$ and f_0 is a harmonic function.

In Theorem 2 we restrict our attention to *TG* families whose elements satisfy the maximum modulus theorem.

Definition 1. Let $\delta > 0$, and let f be a continuous function on $U(\delta) = U_0(\delta)$ into R. Then we set

$$\tilde{f} = \int_{B(\|x\|)} f\, dm$$

for all $x \in U(\delta)$, where $B(\|x\|) = B_0(\delta)$.

Thus $\tilde{f}$ is the radial function obtained by averaging f over concentric spheres about zero. If f lies in a *TG* family F, then $\tilde{f} \in F$. This follows from the fact that we can write

$$\tilde{f} = \int_G fg\, d\mu(g),$$

where G is the rotation group (group of unitary linear transformations) of E onto itself, and μ is normalized invariant (Haar) measure on G. The latter expression shall be taken as our working definition.

Also in the spirit of the paper for our working definition of the Laplacian[1] we shall write

$$\Delta f(x) = \int_B \partial^2 f/\partial n^2\, d\mu(n),$$

where for $n \in B = B(1)$, $\partial^2 f/\partial n^2$ is the second directional derivative of f at x in the direction given by n.

Proof of Theorem 1. We first show that the basic characterization (1) implies (2), (3) and (4). Let H be the set of all $h \in F$ such that $h(0) = 1$ and $\operatorname{dom} h = U(\delta)$ for some $\delta > 0$. We consider the non-trivial case when H is non-empty. For $g, h \in H$, $\tilde{g}(x) = \tilde{h}(x)$ for all $x \in \operatorname{dom} g \cap \operatorname{dom} h$. Hence there exists $\delta_0 > 0$, and a radial function w of $\mathfrak{F}$ with domain $U(\delta_0)$, such that for all $h \in H$, we have $\operatorname{dom} h \subseteq U(\delta_0)$ and $w(x) = \tilde{h}(x)$ for all $x \in \operatorname{dom} h$. For $0 < \delta < \delta_0$, set

$$N(\delta) = \delta^{-p} \int_{U(\delta)} w\, dm \quad \text{and} \quad M(\delta) = \int_{B(\delta)} w\, d\mu$$

and set $M(\delta) = N(\delta) = 1$ for $\delta \geqslant \delta_0$.

Let $\delta > 0$ and $g \in F$ with domain $U(\delta)$. Now if $0 < \delta \leqslant \delta_0$, $\tilde{g}(x) = g(0)w(x)$ for all $x \in U(\delta)$, and thus for $0 < r < \delta$,

$$\int_{U(r)} g\, dm = \int_{U(r)} \tilde{g}\, dm = \int_{U(r)} g(0)\cdot w\, dm = g(0) \int_{U(r)} w\, dm = g(0) N(r) r^p.$$

If $\delta > \delta_0$, then $g(0) = 0$, and $\tilde{g} \equiv 0$, and thus

$$N(r) g(0) = 1\cdot 0 = 0 = r^{-p} \int_{U(r)} \bar{0}\, dm = r^{-p} \int_{U(r)} \tilde{g}\, dm = r^{-p} \int_{U(r)} g\, dm,$$

for $0 < r < \delta$. Thus the elements of F are volume mean functions. Similarly the elements of F are surface mean functions.

We shall now show that the elements of F satisfy a uniform Lipschitz condition. This plus the fact that F satisfies conditions (a)-(d) of the basic characterization is sufficient [1, 2] to insure that the elements of F are twice differentiable, and indeed analytic.

Now

$$\lim_{r\to 0} N(r) = \lim_{r\to 0} r^{-p} \int_{U(r)} w\, dm = 1,$$

[1] To obtain the elementary Laplacian from the working Laplacian multiply by a factor of p.

and thus there exists $r_0 > 0$, $\varepsilon > 0$, such that for $r \in (0, r_0)$, $N(r) \geqslant \varepsilon$. Let $f \in F$, $0 < s < r_0$, such that $\overline{U}(s) \subseteq \operatorname{dom} f$, and set $M = \sup\{|f(t)|; t \in \overline{U}(s)\}$. Then setting $r = s/2$, for $x \in U(r)$,

$$|f(x)-f(0)| = [N(r)r^p]^{-1}\Big|\int_{U_x(r)} f\,dm - \int_{U(r)} f\,dm\Big|$$
$$\leqslant [N(r)r^p]^{-1}\cdot 2[\|x\|Cr^{p-1}]M$$
$$= [2C/N(r)]M\|x\|r^{-1},$$

where $\|x\|Cr^{p-1}$, $C > 0$, is the volume of a „cylinder" with base a sphere of radius r contained in a hyperplane of dimension $p-1$ passing through 0, and with altitude $\|x\|$. Setting $N = \max\{2, 2C/\varepsilon\}$, we obtain for $x \in U(s)$,

$$|f(x)-f(0)| \leqslant NM\|x\|r^{-1}.$$

Let $f \in F$ such that $0 \in \operatorname{dom} f$. Let $0 < \delta < \delta_0$ such that $U(\delta) \subseteq \operatorname{dom} f$, and set $h = f | U(\delta)$. Now $\Delta h(0) = \Delta(hg)(0)$ for all $g \in G$, and thus

$$\Delta f(0) = \Delta h(0) = \int_G \Delta hg(0)\,d\mu(g) = \Delta\Big[\int_G hg\,d\mu(g)\Big](0)$$
$$= \Delta\tilde{h}(0) = \Delta[f(0)w](0) = cf(0),$$

where $c = \Delta w(0)$.

2. We now show that the volume mean characterization implies the basic characterization. Let F be the family of all $h \in \mathfrak{F}$, such that for $\delta > 0$, $x \in \operatorname{dom} h$, $\overline{U}_x(\delta) \subseteq \operatorname{dom} h$, implies

$$N(\delta)h(x) = \delta^{-p}\int_{U(\delta)} h\,dm.$$

Clearly F is a TG family.

Let $\delta > 0$, and let h be a radial function of F with domain $U(\delta)$ such that $h(0) = 0$. Then for $0 < r < \delta$,

(1) $$0 = r^p N(r)\cdot h(0) = \int_{U(r)} h\,dm = p\int_0^r h(sn)s^{p-1}ds.$$

Differentiating both sides of (1) we obtain for $0 < r < \delta$, $0 = p\cdot f(rn)r^{p-1}$, and thus $h \equiv 0$. Thus F satisfies the basic characterization. The argument for the surface mean case is similar.

3. We now show that condition (4) implies the basic characterization. Let F be the family of all twice continuously differentiable functions f in $\mathfrak{F}$ which satisfy the equation

$$\Delta f = cf.$$

Let F_0 be the closure of F, that is the family of all functions h of $\mathfrak{F}$ which are uniform limits on $S = \operatorname{dom} h$ of sequences of elements of F with common domain S. Then F_0 is a TG family.

Let $\delta > 0$, and let h be a radial function of F_0 with domain $U(\delta)$ such that $h(0) = 0$. Then there exists a sequence $h_1, h_2, \ldots$ of elements of F with common domain $U(\delta)$ which converges uniformly on $U(\delta)$ to h. Without loss of generality we may take $h_i(0) = 0$ for $i = 1, 2, \ldots$ It is easy to show for $i = 1, 2, \ldots$ that $\tilde{h}_i$ is twice continuously differentiable, and thus since $\Delta(h_i g) = (\Delta h_i)g$ for $g \in G$,

$$\Delta\tilde{h}_i = \Delta\int_G h_i g\,d\mu(g) = \int_G (\Delta h_i)g\,d\mu(g) = \int_G ch_i g\,d\mu(g) = c\tilde{h}_i.$$

Thus for $i = 1, 2, \ldots$, $\tilde{h}_i \in F$ and $\tilde{h}_i(0) = 0$. If we can show that $\tilde{h}_i \equiv 0$ for $i = 1, 2, \ldots$ we will obtain taking the limit as $i \to \infty$, that h vanishes identically.

Let $i = 1, 2, \ldots$ and set $k = \tilde{h}_i$. Let $x_0 \in B$, and set $w(r) = k(rx_0)$ for $0 \leqslant r < \delta$. Then w is twice differentiable, and for $0 < r < \delta$, setting $r = r(x) = \|x\| = [x, x]^{1/2}$,

$$\partial^2 k(x)/\partial n^2 = w''(r)[\partial r/\partial n]^2 + w'(r)\partial^2 r/\partial n^2$$
$$= w''(r)\{[x, n]/r\}^2 + w'(r)\{r^{-1} - [x, n]^2/r^3\}$$

and

$$cw(r) = ck(rx_0) = k(rx_0) = \int_B \partial^2 k(x)/\partial n^2\,d\mu(n)$$
$$= \{w''(r)/r^2 - w'(r)/r^3\}\int_B [x, n]^2 d\mu(n) + w'(r)/r$$
$$= aw''(r) + (1-a)w'(r)/r,$$

where $\int_B [x, n]^2 d\mu(n) = a\|x\|^2$ and $0 < a < 1$. Thus for $0 < r < \delta$, $s = (1-a)/a$, and $c' = c/a$,

(1) $$w''(r) + sw'(r)/r + c'w(r) = 0.$$

It is well known from the elementary theory of differential equations that any function w satisfying (1) with $s \geqslant 0$, and vanishing at 0, is identically zero. Thus $k = \tilde{h}_i \equiv 0$.

4. We now show that (5) and (6) imply the basic characterization. Let F be a TG family of functions satisfying the maximum modulus theorem and let h be a radial function of F such that $h(0) = 0$. Assume for some $x, y \in \operatorname{dom} h$, $\|x\| < \|y\|$, that $h(x) < 0$ and $h(y) > 0$, or $h(x) > 0$ and $h(y) < 0$. Then there exists $z \in \operatorname{dom} h$, such that $\|x\| < \|z\| < \|y\|$

and $h(z) = 0$. Thus $|h(x)| \leqslant \sup\{|h(t)|; t \in B(\|z\|)\} = |h(z)| = 0$, and $h(x) = 0$. Thus $h(x) \geqslant 0$ for all $x \in \operatorname{dom} h$ or $h(x) \leqslant 0$ for all $x \in \operatorname{dom} h$. Set $H = \{x \in \operatorname{dom} h;\ h(x) = 0\}$.

Assume h does not vanish identically. Then there must exist a radial function $u \in F$, such that $u(0) = 1$ and $u(x) > 0$ for $x \in \operatorname{dom} u$. Let $z \in H$, and let $\delta > 0$ such that $U_z(\delta) \subseteq \operatorname{dom} h$ and $S = U(\delta) \subseteq \operatorname{dom} u$. For $x \in S$, set $w(x) = h(x-z)$. Then $\tilde{w}(0) = 0$, and $w(x) \geqslant 0$ for all $x \in S$. Thus if $\tilde{w}(x) = 0$ for some $x \in S$, we must have since w is continuous, $w(t) = 0$ for all $t \in U(\|x\|)$, and $w(x) = 0$.

Let $x \in S$. Set $a = -\tilde{w}(x)/u(x)$ and set $\varrho(t) = \tilde{w}(t) + au(t)$ for $t \in S$. Then $\varrho(x) = 0$, and since ϱ satisfies the maximum modulus theorem $0 = \varrho(0) = \tilde{w}(0) + au(0) = 0 + a\cdot 1 = a$, and $\tilde{w}(x) = au(x) = 0\cdot u(x) = 0$. Thus $w(x) = 0$, and $w \equiv 0$, and consequently $U_z(\delta) \subseteq H$. Thus H is an open and closed subset of $\operatorname{dom} h$ containing 0, and thus $H = \operatorname{dom} h$, and $h \equiv 0$.

THEOREM 2. *Let $f_0 \in F$. Then the following statements are equivalent:*

(1) *There exists a TG family F of $\mathfrak{F}$ all elements of which satisfy the maximum modulus theorem, which contains f_0.*

(2) *f_0 is a subvolume mean function, i.e. for all $\delta > 0$, there exists $N(\delta) \geqslant 1$, such that $x \in \operatorname{dom} f$, $\overline{U}_x(\delta) \subseteq \operatorname{dom} f$, implies*

$$f_0(x) N(\delta) = \delta^{-p} \int_{U_x(\delta)} f_0\, dm.$$

(3) *f_0 is a subsurface mean function, i.e. for all $\delta > 0$, there exists $M(\delta) \geqslant 1$, such that $x \in \operatorname{dom} f$, $\overline{U}_x(\delta) \subseteq \operatorname{dom} f_0$, implies*

$$f_0(x) M(\delta) = \int_{B_x(\delta)} f_0\, d\mu.$$

(4) *f_0 is twice continuously differentiable and there exists $c \geqslant 0$ such that for all $x \in \operatorname{dom} f_0$,*

$$\Delta f_0(x) = c f_0(x).$$

We observe that functions satisfying the above conditions are special cases of subharmonic functions.

Proof. We first show that (2), (3) and (4) imply (1). Let S be a spherical region such that $\overline{S} \subseteq \operatorname{dom} f$. Assume f_0 is a subvolume mean function. Let F be the TG family of all $f \in F$, such that $x \in \operatorname{dom} f$, $\delta > 0$, $\overline{U}_x(\delta) \subseteq \operatorname{dom} f$ implies

$$f(x) N(\delta) = \int_{U_x(\delta)} f\, dm.$$

Let $f \in F$ and set $M = \sup\{|f(x)|; x \in \overline{S}\}$, and $H = \{x \in \overline{S}; f(x) = M\}$. Then H is closed in S. Let $x \in H$, $\delta > 0$, such that $\overline{U}_x(\delta) \subseteq S$. Then there exists $N(\delta) \geqslant 1$, such that

$$M = |f(x)| \leqslant N(\delta)|f(x)| \leqslant \delta^{-p} \int_{U_x(\delta)} |f|\, dm \leqslant \delta^{-p} \int_{U_x(\delta)} M\, dm = M.$$

Thus, since f is continuous, $|f(t)| = M$ for all $t \in U_x(\delta)$, and thus H is open in S. Since S is connected, H is empty or $H = S$, and thus f satisfies the maximum modulus theorem. The proof in the case that f_0 is a subsurface mean function is similar.

Assume that f satisfies (4) and that $c > 0$, and set $N = \sup\{f(t); t \in \overline{S}\}$ and $M = \inf\{f(t);\ t \in \overline{S}\}$. Assume $N > 0$ and there exists $x \in S$ such that $f(x) = N$. Let $\delta > 0$ such that $U_x(\delta) \subseteq S$. For $e \in B$ and $r \in (-\delta, +\delta)$, set $f_e(r) = f(x+re)$. Then for $e \in B$, f_e attains its absolute minimum at 0, $h_e'(0) = 0$, and $f''(0) \leqslant 0$. But then

$$0 < cf(x) = \Delta f(x) = \int_B \partial^2 f(x)/\partial e^2\, d\mu(e) = \int_B f_e''(0)\, d\mu(e) \leqslant 0.$$

Similarly, if $M < 0$, we have $f(x) > M$ for all $x \in S$. It now readily follows that f satisfies the maximum modulus theorem.

To handle the case when $c = 0$, we consider functions of the form $w(x) = f(x) \pm (\varepsilon/2)\|x\|^2$ for $x \in \operatorname{dom} f$, where ε is arbitrarily small, and $\Delta w = \pm\varepsilon$.

We now show that (1) implies (2), (3) and (4). Let H be the family of all radial functions of F such that $h(0) = 1$. Then for $h \in H$, $x \in \operatorname{dom} h$, we have

$$|h(x)| = \sup\{|h(t)|; t \in U(\|x\|)\} \geqslant |h(0)| = 1,$$

and $h(x) \geqslant 1$. From Theorem 1, F satisfies the basic characterization, and from the proof of Theorem 1, there exists a function N on $(0, \infty)$ into R such that f is a volume mean function with respect to N, and such that for $\delta > 0$, $N(\delta) = 1$, or there exists $h \in H$, such that $\overline{U}(\delta) \subseteq \operatorname{dom} h$, and

$$N(\delta) = N(\delta)\cdot h(0) = \delta^{-p} \int_{U(\delta)} h\, dm \geqslant \int_{U(\delta)} \bar{1}\, dm = 1.$$

Thus f is a subvolume mean function. The argument that f is a subsurface mean function is similar.

Let $h \in H$. Then h attains its minimum at 0, and

$$c = ch(0) = \Delta h(0) \geqslant 0.$$

Remark. Let F be a TG family of functions satisfying the maximum modulus theorem, which contains the constant function $\bar{1}$ defined on all of E. For some $c \geqslant 0$, $\Delta f = cf$ for all $f \in F$. Now $0 = \Delta\bar{1}(0) = c\bar{1}(0) = c$, and thus the elements of F are harmonic functions. For $\delta > 0$, there exists $c = N(\delta)$ such that $f \in F$, $x \in \operatorname{dom} f$, $\bar{U}_x(\delta) \subseteq \operatorname{dom} f$ implies

$$cf(x) = \delta^{-p} \int_{U_x(\delta)} f dm$$

and hence

$$c = c\bar{1}(0) = \delta^{-p} \int_{U(\delta)} \bar{1} dm = 1.$$

Thus the elements of F are volume mean functions in the strong, i.e. classical, sense. Similarly they are surface mean functions in the strong sense.

Further developments along this line, including full radius of convergence of power series, the solution of the Dirichlet, Neumann, and Robin problems for the sphere, etc. may be found in [2, 3].

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STUDIA MATHEMATICA, T. XXIX. (1968)

Sur les solutions généralisées des équations quasi-linéaires

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Nous considérons dans ce travail les équations différentielles aux dérivées partielles quasi-linéaires elliptiques; nous y démontrons l'existence d'une solution faible, et donnons une méthode de solutions numérique. Soit

$$\Psi(u, h) = \int_{\Omega} f\left(\frac{\partial u}{\partial \xi_1}, \ldots, \frac{\partial u}{\partial \xi_n}, \frac{\partial h}{\partial \xi_1}, \ldots, \frac{\partial h}{\partial \xi_n}\right) d\Omega$$

une fonctionnelle de deux variables u et h, linéaire par rapport à h; supposons qu'une fonction fixée $\bar{u}(\xi_1, \ldots, \xi_n)$ de classe C_2 sur Ω satisfait à l'équation $\Psi(\bar{u}, h) = 0$ pour toutes les fonctions $h(\xi_1, \ldots, \xi_n)$ de classe C_2 sur Ω, remplissant la condition $\mathring{h}(\xi_1, \ldots, \xi_n) = 0$ sur le bord S de Ω. Une intégration par parties donne

$$\int_{\Omega} \Re(u) \cdot h d\Omega = 0$$

pour tout h de la classe mentionnée, où $\Re$ est une opération différentielle (en général non-linéaire) d'ordre 2. Par suite (en vertu d'un lemme classique du calcul des variations) on a $\Re(\bar{u}) = 0$.

Or, si la fonction $\bar{u}$ n'est pas de classe C_2, mais seulement de classe $L_{1,\Omega}$ (c'est-à-dire si les dérivées $\partial u / \partial \xi_i$ sont de carrés intégrables sur Ω, au sens de Lebesgue), et $\bar{u}(\xi_1, \ldots, \xi_n)$ satisfait à la relation $\Psi(u, h) = 0$ pour toutes les fonctions h (d'une classe assez large) nous appellerons la fonction $\bar{u}(\xi_1, \ldots, \xi_n)$ *solution généralisée* de l'équation $\Re(u) = 0$.

1. Soient H un espace réel de Hilbert, dont les éléments sont x, y, u, v, h, f etc., et le produit scalaire (x, y). Soient M un ensemble linéaire dense dans H, au sens de la norme $\|x\| = \sqrt{(x, x)}$, $(x, y)_1$ un autre produit scalaire sur M tel que

(1) $\quad \|x\|_1 = \sqrt{(x, x)_1} \geqslant \gamma \|x\|, \qquad (x, y)_1 \leqslant m(x) \|y\| \qquad$ pour $x, y \in M$.

Désignons par H_1 le complément de M en norme $\| \ \|_1$, par A l'opération linéaire, définie sur M par l'identité $(Ax, y) = (x, y)_1$; A étant,