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On differentiability of vector-valued functions of a real variable

by

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I. Introduction. Let $F(x)$ be a function defined on $I = [0, 1]$ and taking on values in a real or complex Banach space Y. One says that F is *strongly differentiable* at x in $(0, 1)$ and has *strong derivative* $F'(x)$, if

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = F'(x).$$

One says that F is *weakly differentiable* at x in $(0, 1)$ if there is a vector $wF'(x)$, which one calls the *weak derivative* of F at x, such that

$$\lim_{h\to 0}\frac{y^*\big(F(x+h)\big)-y^*\big(F(x)\big)}{h} = y^*\big(wF'(x)\big)$$

for all $y^* \in Y^*$, where Y^* is the Banach space of continuous linear functionals on Y. If there is a vector-valued function $pF'(x)$ defined on a measurable $E \subseteq (0, 1)$ such that for each $y^* \in Y^*$,

$$\lim_{h\to 0}\frac{y^*\big(F(x+h)\big)-y^*\big(F(x)\big)}{h} = y^*\big(pF'(x)\big)$$

for almost every $x \in E$, one says ([6], p. 300) that F has a *pseudo-derivative* on E (is *pseudo-differentiable* on E), and calls $pF'(x)$ a *pseudo-derivative* of F on E.

Clearly $F'(x)$ and $wF'(x)$ are unique. Also, if F is strongly differentiable a.e. on $E \subseteq (0, 1)$, then F is weakly differentiable a.e. on E, and if F is weakly differentiable a.e. on E, then F is pseudo-differentiable on E.

In general, two pseudo-derivatives of F need not be a.e. equal (see below). However, if Y has a countable determining set, i.e., countable set $\Lambda \subseteq Y^*$ such that

$$\|y\| = \sup_{y^*\in\Lambda}|y^*(y)|$$

for all $y \in Y$, then we shall show that any two pseudo-derivatives of F must be a.e. equal. One can show as a result of this theorem that, in particular, the space B of bounded, real-valued functions on I has no countable determining set (see below). We recall ([5], p. 34) that if Y is separable, then Y and Y^* have countable determining sets. Thus, in particular, $L_\infty(I)$ and $BV(I)$ ([3], p. 289 and 265) are Y's in which two pseudo-derivatives of F can differ on at most a set of measure zero.

One says that F is AC^* on I if, given $\varepsilon > 0$, there is an $\eta = \eta(\varepsilon) > 0$ such that if $\{I_i = [a_i, b_i]\}$ is a finite, non-overlapping sequence of subintervals such that $\sum (b_i - a_i) < \eta$, then $\sum \|F(b_i) - F(a_i)\| < \varepsilon$. Even if F is AC^* (in fact, Lipschitzian) and Y^* is separable, we have no guarantee that $pF'(x)$ exists on I (see below) although, as Clarkson has shown [2], $F'(x)$ exists a.e. if Y is uniformly convex. We shall show that an arbitrary F taking on values in a Y with countable determining set and satisfying a certain local pseudo-differentiability condition must have a pseudo-derivative on I.

II. Uniqueness and existence of derivatives. In general, two pseudo-derivatives of F need not agree a.e., and a pseudo-derivative of F need not be measurable. For example, let $Y = B$, $E \subseteq I$ be non-measurable and nowhere dense. Define

$$f(x) = \begin{cases} \chi_{\{x\}} & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Then $f(x)$ is integrable in the Graves sense [4] to zero. Since a Graves integral is a Birkhoff integral ([1], p. 375), and hence a Pettis integral ([6], p. 281), $f(x)$ is a pseudo-derivative of the function which is identically zero ([6], p. 300). But so is $g(x) \equiv 0$. Moreover $f(x)$ and $g(x)$ are clearly not equal a.e., and $f(x)$ is not measurable.

However, we have

1. Theorem. *If Y has a countable determining set, then two pseudo-derivatives of F can differ on at most a set of measure zero.*

Proof. Let $\{y_i^*\} \subseteq Y^*$ be a countable determining set, S a set on which $pF'(x)$ exists, and

$$H_i = \{x\colon x \in S \text{ and } [y_i^*(F)]'(x) \neq y_i^*(pF'(x))\}.$$

Let $H = \bigcup H_i$. Then $|H| = 0$. Suppose $g(x)$ is also a pseudo-derivative of F and

$$K_i = \{x\colon x \in S \text{ and } [y_i^*(F)]'(x) \neq y_i^*(g(x))\}.$$

Let $K = \bigcup K_i$. Then $|K| = 0$. Let $E = H \cup K$ and $x \in S - E$. Then

$$y_i^*(pF'(x)) = y_i^*(g(x))$$

for all i. Thus

$$\|pF'(x) - g(x)\| = \sup_i |y_i^*(pF'(x) - g(x))| = 0.$$

Thus we have a necessary condition for Y to have a countable determining set. In particular, B does not have a countable determining set.

Even if Y^* is separable and F is Lipschitzian, we have no guarantee that $pF'(x)$ exists on I; in fact, $pF'(x)$ need exist on no subset of I of positive measure.

For example, let c_0 = the space of real null sequences. Then it is well known that l_1 = the space of sequences $\{a_i\}$ such that $\sum |a_i| < \infty$ is c_0^*. It is well known that l_1 is separable. We consider the following function (Clarkson [2] has remarked that the function we shall construct fails to have strong or weak derivative on a set of positive measure):

We define a sequence $\{\varphi_n(x)\}$ of functions on I as follows:

$$\varphi_1(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant \tfrac{1}{2}, \\ 2(1-x) & \text{if } \tfrac{1}{2} \leqslant x \leqslant 1; \end{cases}$$

extend φ_1 by periodicity to all of $(-\infty, +\infty)$;

$$\varphi_n(x) = \frac{\varphi(2^{n-1}x)}{2^{n-1}}$$

for $n = 2, 3, \ldots$ Define

$$f(x) = \{\varphi_i(x)\}.$$

Clearly $f(x) \in c_0$ for all $x \in I$. Moreover, $f(x)$ is clearly Lipschitzian on I. Suppose that f is pseudo-derivable on $E \subseteq I$. Consider the following family of members of Y^*: define

$$y_i^*(a_1, \ldots, a_i, a_{i+1}, \ldots) = a_i.$$

(Clearly $y_i^* \in Y^*$ for all i.) Then $[y_i^*(f)]'(x) = \varphi_i'(x)$ a.e. on E. But, then, $pf'(x) = \{2\varepsilon_i(x)\}$ a.e. on E, where $\varepsilon_i(x) = \pm 1$. Since $2\varepsilon_i(x) \nrightarrow 0$, this sequence is not in c_0. Thus $pF'(x)$ does not exist on E.

However, a local pseudo-differentiability condition is sufficient to insure pseudo-differentiability on I. By a *portion* of a set E we mean a set of the form $I' \cap E$, where I' is an open interval.

Definition. We say that F is *restrictedly pseudo-differentiable* (*rpd*) on I if, given any closed set $E \subseteq I$, there is a portion P of E such that F is pseudo-differentiable on P.

2. Theorem. *If Y has a countable determining set, and if F is rpd on I, then $pF'(x)$ exists on I.*

Proof. Let $\mathcal{F}$ be the family of open subintervals of I on which $pF'(x)$ exists. Then clearly $\mathcal{F} \neq \emptyset$. Suppose $\mathcal{F}$ does not cover I. Then we can assume $\mathscr{F} = \{I_n\}$ without affecting the set of points covered by $\mathscr{F}$. Now, if $E = (\bigcup I_n)^c$, then E is closed. Moreover $E \cap I \neq \emptyset$.

Suppose $E \cap I$ has an isolated point x_0. Then there is an open interval $I' \subseteq I$ such that $x_0 \in I'$ and $I' \cap E - \{x_0\} = \emptyset$. Let $G_n(x) = pF'(x)$ on I_n, and

$$E_{mn} = \{x\colon x \in I_m \cap I_n \text{ and } G_n(x) \neq G_m(x)\}.$$

Then, by Theorem 1, $|E_{mn}| = 0$ for all m and n. Define

$$G(x) = \begin{cases} G_n(x) & \text{if} \quad x \in I_n - \bigcup E_{mn}, \\ 0 & \text{if} \quad x \in (\bigcup I_n)^c \cup (\bigcup E_{mn}). \end{cases}$$

Then, clearly $G(x) = pF'(x)$ on I'. But $I' \cap E \neq \emptyset$, a contradiction.

Now suppose $E \cap I$ has no isolated point. Then there is an open $I' \subseteq I$ such that $pF'(x)$ exists on $P = I' \cap E \neq \emptyset$. Let $G_P(x) = pF'(x)$ on P. Let $H = E \cup (\bigcup E_{mn})$. Define

$$G(x) = \begin{cases} G_n(x) & \text{if} \quad x \in I_n - H, \\ G_P(x) & \text{if} \quad x \in P, \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly $G(x) = pF'(x)$ on I'. Hence $I' \in \mathscr{F}$. But $I' \cap E \neq \emptyset$, a contradiction. Thus $\mathscr{F}$ covers I. Define

$$f(x) = \begin{cases} G_n(x) & \text{if} \quad x \in I_n - (\bigcup E_{mn}), \\ 0 & \text{otherwise.} \end{cases}$$

Then f is well-defined and $pF'(x) = f(x)$ on I.

References

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The Stone-Čech operator and its associated functionals

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1.1. Introduction. The object of this work is to provide a realization of a certain Hilbert space of vector-valued sequences and to show how the structure obtained applies to a class of functionals on the space $\mathscr{L}(H)$. We use the symbol H to denote a separable Hilbert space, $\mathscr{L}(H)$ to denote the space of bounded linear transformations thereon, and m to denote the space of bounded complex-valued sequences.

1.1.1. Definition. A *generalized limit* is a bounded linear functional L on m which preserves the ordinary notion of convergence. That is, if $\lim(a_n) = a$, then $L((a_n)) = a$.

Generalized limits may be characterized as those continuous functionals which satisfy

1) $a_n \geqslant 0$ for all n implies $L((a_n)) \geqslant 0$.
2) $L((1)) = 1$, where $(1) = (1, 1, 1, \ldots)$.
3) If $a_n = b_n$ for $n \geqslant K$, then $L((a_n)) = L((b_n))$.

A stronger requirement than 3) is the *translation invariant* property:

4) $L((a_{n+1})) = L((a_n))$,

which we will assume only in special cases. The existence of generalized limits satisfying 1)-4) was proved by Banach [1].

1.2. Extensions and measures. It is well known that each completely regular topological space X possesses a Stone-Čech compactification βX with the property that X is densely embeddable in βX and every continuous function f mapping X into a compact space S possesses a continuous extension $f^\beta\colon \beta X \to S$. In particular, each bounded continuous complex-valued function has such an extension, and the correspondence $f \to f^\beta$ is an isometric isomorphism between $C_b(X)$ and $C(\beta X)$. Applying this to m (where the integers N are given the discrete topology), we see that m is isomorphic to $C(\beta N)$, that each sequence $(a_n) \in m$ has a continuous extension a^β defined in βN, and that

$$\sup_{n \in N} |a_n| = \sup_{t \in \beta N} |a^\beta(t)|.$$