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On a condition for almost everywhere Bochner-Riesz summability of multiple Fourier series

by

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1. Let $f(x)$ be a periodic function with period 2π. We say that f satisfies the condition I_p $(p \geqslant 1)$ if and only if

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(x+t)-f(x-t)|^p}{|t|}\, dt\, dx < \infty. \tag{1}$$

The following theorem is due to J. Marcinkiewicz [2]:

THEOREM A. *If $f(x) \in L^p[-\pi, \pi]$, f is periodic and satisfies the condition I_p $(1 \leqslant p \leqslant 2)$, then its Fourier series converges almost everywhere.*

The purpose of this paper is to prove the k-dimensional $(k \geqslant 2)$ version of the above theorem.

We now introduce notations and definitions in connection with multiple Fourier series. E_k will denote the k-dimensional Euclidean space. A single letter such as $x, y, u, t, \ldots$ will usually denote point in E_k. A point $x = (x_1, \ldots, x_k)$ is called a *lattice point* if its coordinates $x_1, \ldots, x_k$ are integers. The letter n will always denote lattice point unless otherwise stated. For any two points $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$ we define a scalar product $x \cdot y = x_1 y_1 + \ldots + x_k y_k$. The usual Euclidean norm in E_k is then given by $|x| = (x \cdot x)^{1/2}$. $dx = dx_1 \ldots dx_k$ will denote the k-dimensional Lebesgue measure in E_k while $d\sigma$ will denote the $(k-1)$-dimensional Lebesgue measure in E_k.

The term *function* throughout this paper is understood to be complex-valued function unless it is stated to the contrary. A function $f(x) = f(x_1, \ldots, x_k)$ is said to be *periodic* if f is periodic with period 2π in each of its variables $x_1, \ldots, x_k$. Q_k will denote the fundamental cube in E_k consisting of all points $x = (x_1, \ldots, x_k)$ satisfying the inequalities

$$-\pi \leqslant x_j \leqslant \pi \quad (j = 1, 2, \ldots, k).$$

If $f(x) \in L(Q_k)$, f is periodic, we define the numbers

$$a_n = (2\pi)^{-k} \int_{Q_k} f(x) e^{-in\cdot x} dx \quad (n \text{ being lattice points})$$

and call them the *Fourier coefficients* of $f(x)$. The formal series

$$\sum a_n e^{in\cdot x},$$

where n ranges over all lattice points, is called the *multiple Fourier series* of $f(x)$, and will be denoted by $S[f] = S[f(x)]$. Following the notation of the one-variable case this relationship is indicated by $f(x) \sim \sum a_n e^{in\cdot x}$.

For an $R > 0$ and a complex number δ we define

$$S_R^\delta(x) = S_R^\delta(x, f) = \sum_{|n|<R} (1-|n|^2/R^2)^\delta a_n e^{in\cdot x}$$

and call it the *Bochner-Riesz means of order* δ of the Fourier series $\sum a_n e^{in\cdot x}$. We say that the series is *Bochner-Riesz summable of order* δ, in symbol summable (B-R, δ), to a finite complex number s if

$$\operatorname*{Lim}_{R\to\infty} S_R^\delta(x, f) = s.$$

Bochner-Riesz summability for multiple Fourier series of k variables is a generalization of Cesàro summability for Fourier series of one variable. In fact, when $k = 1$, summability (B-R, δ) is equivalent to the classical Cesàro summability (C, δ) of order δ. Bochner-Riesz summability of the special order $\delta = (k-1)/2$ is particularly important and throughout this paper we shall consider only summability $\big($B-R, $(k-1)/2\big)$ for multiple Fourier series of k variables. This method of summability is, in fact, the analog of the ordinary convergence of Fourier series of one variable because it has been found that some theorems on the convergence of Fourier series of one variable have their analogs valid for multiple Fourier series when ordinary convergence is replaced by summability $\big($B-R, $(k-1)/2\big)$ in multiple Fourier series of k variables.

$L\log^+L(Q_k)$ will denote the class of all measurable functions $f(x)$ defined on Q_k such that $\int_{Q_k} |f(x)|\log^+|f(x)|\,dx < \infty$.

2. In analogy with (1) we now define the condition I_p for functions of k variables. Let $f(x) = f(x_1, \ldots, x_k)$ be periodic with period 2π in each of its variables $x_1, \ldots, x_n$. We say that f *satisfies the condition* I_p $(p \geqslant 1)$ if and only if

$$J(p, f) = \int_{Q_k}\int_{Q_k} \frac{|f(x+t)-f(x-t)|^p}{|t|^k}\,(dt\,dx) < \infty \tag{2}$$

where the integral is interpreted as a $2k$-dimensional integral taken over the cartesian product $Q_k \times Q_k$ (i.e. Q_{2k}) in E_{2k}. We now state below the theorem to be proved in this paper.

THEOREM 1. *If* $f(x) = f(x_1, \ldots, x_k)$, $k \geqslant 2$, *is periodic*, $f(x) \in L\log^+L(Q_k)$, $f(x) \in L^p(Q_k)$ *and satisfies the condition* I_p, *where* $1 \leqslant p \leqslant 2$, *then its Fourier series is summable* $\big($B-R, $(k-1)/2\big)$ *at almost every point.*

We remark that the assumption $f(x) \in L\log^+L(Q_k)$ is essential only when $p = 1$. When $p > 1$ it is implied by $f(x) \in L^p(Q_k)$.

A review of the proof of Theorem A reveals that the following three theorems (Theorems B, C and D below) on Fourier series of one variable have been used.

THEOREM B (DINI'S TEST FOR CONVERGENCE). *Let* $f(x) \in L[-\pi, \pi]$, *f be periodic. For a point* x_0 *and a real number* $r > 0$ *we define*

$$f(x_0; r) = \tfrac{1}{2}\{f(x_0+r) + f(x_0-r)\}.$$

Then at a point x_0 *where* $f(x_0)$ *is finite-valued, the condition*

$$\int_0^\delta \frac{|f(x_0; r) - f(x_0)|}{r}\,dr < \infty \quad \textit{for some } \delta > 0$$

implies that the Fourier series $S[f]$ *of* f *converges to* $f(x_0)$ *at* $x = x_0$.

THEOREM C. *Suppose* $f(x) \in L[-\pi, \pi]$, *f is periodic*, $f(x_0)$ *is finite-valued. Then the condition.*

$$\int_{|t|\leqslant\delta} \frac{|f(x_0+t) - f(x_0)|}{|t|}\,dt < \infty \quad \textit{for some } \delta > 0$$

implies that the Fourier series $S[f]$ *of* f *converges to* $f(x_0)$ *at* $x = x_0$.

THEOREM D (PLESSNER'S THEOREM). *Suppose* $f(x) \in [-\pi, \pi]$, *f is periodic. Then the condition*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(x+t) - f(x-t)|^2}{|t|}\,(dt\,dx) < \infty$$

implies that the Fourier series $S[f]$ *of* f *converges at almost every point* x.

Theorem B is the well-known Dini's test for convergence, from which Theorem C follows easily as a corollary. Theorem D is due to A. Plessner [3].

The question arises naturally whether or not Theorems B, C and D above have their analogs valid for multiple Fourier series of k variables $(k \geqslant 2)$. The answer is in the affirmative and we state below Theorems 2, 3 and 4 being the k-dimensional analogs of Theorems B, C and D above respectively.

THEOREM 2 (DINI'S TEST FOR BOCHNER-RIESZ SUMMABILITY). *Suppose $f(x) \in L\log^+ L(Q_k)$ $(k \geqslant 2)$, f is periodic. For a point x_0 and a real number $r > 0$ we define*

$$f(x_0; r) = \frac{1}{\sigma_k r^{k-1}} \int\limits_{|x-x_0|=r} f(x)\,d\sigma, \quad \text{where} \quad \sigma_k = 2\pi^{k/2}/\Gamma\left(\frac{k}{2}\right).$$

(Thus $f(x_0; r)$ is the mean value of f taken over the surface of the sphere $|x-x_0| = r$.) Then at a point x_0 where $f(x_0)$ is finite-valued, the condition

$$\int\limits_0^\delta \frac{|f(x_0; r)-f(x_0)|}{r}\,dr < \infty \quad \text{for some } \delta > 0 \tag{3}$$

implies that the Fourier series $S[f]$ of f is summable (B-R, $(k-1)/2$) to $f(x_0)$ at $x = x_0$.

THEOREM 3. *Suppose $f(x) \in L\log^+ L(Q_k)$ $(k \geqslant 2)$, f is periodic, $f(x_0)$ is finite-valued. Then the condition*

$$\int\limits_{|t|\leqslant\delta} \frac{|f(x_0+t)-f(x_0)|}{|t|^k}\,dt < \infty \quad \text{for some } \delta > 0 \tag{4}$$

implies that the Fourier series $S[f]$ of f is summable (B-R, $(k-1)/2$) to $f(x_0)$ at $x = x_0$.

THEOREM 4. *Suppose $f(x) \in L(Q_k)$ $(k \geqslant 2)$, f is periodic. Then the condition*

$$\int\limits_{Q_k}\int\limits_{Q_k} \frac{|f(x+t)-f(x-t)|^2}{|t|^k}\,(dt\,dx) < \infty \tag{5}$$

implies that the Fourier series $S[f]$ of f is summable (B-R, $(k-1)/2$) at almost every point x in E_k.

Theorem 2 is due to E. M. Stein (see [4], p. 107). Theorem 3 follows from Theorem 2 because Dini's condition (3) of Theorem 2 is implied by condition (4) of Theorem 3 (see [1], p. 30-31 for a proof). For Theorem 4, reference is made to [1], p. 30.

Once we have got Theorems 2, 3 and 4 the proof of Theorem 1 is similar to that of the one variable case in [2]. We give details of the proof in the following section.

3. Proof of Theorem 1. Let $f(x) = g(x)+ih(x)$ where $g(x)$, $h(x)$ are the real and imaginary parts of $f(x)$. Clearly $f(x) \in L^p(Q_k)$ if and only if both $g(x)$ and $h(x)$ belong to $L^p(Q_k)$ $(1 \leqslant p \leqslant 2)$ and the same remark holds for the class $L\log^+ L(Q_k)$. Setting $a = g(x+t)-g(x-t)$, $b = h(x+t)-h(x-t)$ in the inequalities

$$|a|^p \leqslant |a+bi|^p, \quad |b|^p \leqslant |a+bi|^p,$$
$$|a+bi|^p \leqslant 2^p\{|a|^p+|b|^p\},$$

we have

$$|g(x+t)-g(x-t)|^p \leqslant |f(x+t)-f(x-t)|^p,$$
$$|h(x+t)-h(x-t)|^p \leqslant |f(x+t)-f(x-t)|^p,$$
$$|f(x+t)-f(x-t)|^p \leqslant 2^p\{|g(x+t)-g(x-t)|^p+|h(x+t)-h(x-t)|^p\}.$$

These show that f satisfies the I_p condition if and only if both g and h satisfy the I_p condition. If $S[g]$ and $S[h]$ are summable (B-R, $(k-1)/2$) to $g(x)$ and $h(x)$ at almost every point x respectively, it follows that $S[f] = S[g+ih] = S[g]+iS[h]$ is summable (B-R, $(k-1)/2$) to $g(x)+ih(x) = f(x)$ at almost every point x. Hence it is enough to prove the theorem for real-valued function.

Let $f(x)$ be a real-valued function satisfying the condition I_p. Clearly both its positive part $f^+(x)$ and negative part $f^-(x)$ also satisfy the condition I_p. Hence without loss of generality it is enough to prove the theorem for non-negative real-valued function $f(x)$.

Consider first the two extreme cases of $p = 1$ and $p = 2$. When $p = 2$, Theorem 1 is precisely Theorem 4 above. When $p = 1$, we shall show that the function f satisfies the condition

$$\int\limits_{2Q_k} \frac{|f(x_0+t)-f(x_0)|}{|t|^k}\,dt < \infty$$

for almost every point x_0 and hence, by Theorem 3, the Fourier series $S[f]$ of f is summable (B-R, $(k-1)/2$) to $f(x_0)$ at almost every point x_0.

Applying Fubini's theorem twice and using the change of variables $t = \frac{1}{2}t'$ we write the integral $J(p,f)$ in (2) as

$$J(p,f) = \int\limits_{Q_k}\left[\int\limits_{Q_k} \frac{|f(x+t)-f(x-t)|}{|t|^k}\,dt\right]dx$$
$$= \int\limits_{Q_k}\left[\int\limits_{2Q_k} \frac{|f(x+\frac{1}{2}t')-f(x-\frac{1}{2}t')|}{|t'|^k}\,dt'\right]dx$$
$$= \int\limits_{2Q_k}\left[\int\limits_{Q_k} |f(x+\tfrac{1}{2}t')-f(x-\tfrac{1}{2}t')|\,dx\right]\frac{1}{|t'|^k}\,dt',$$

where $2Q_k$ denotes the cube consisting of all points $(x_1, \ldots, x_k)$ satisfying $-2\pi \leqslant x_j \leqslant 2\pi$ $(j = 1, 2, \ldots, k)$.

In the inner integral on the right-band side above, viz.

$$\int\limits_{Q_k} |f(x+\tfrac{1}{2}t')-f(x-\tfrac{1}{2}t')|\,dx \tag{6}$$

the integrand $|f(x+\frac{1}{2}t')-f(x-\frac{1}{2}t')|$, with t' kept fixed, is periodic in the variable $x = (x_1, \ldots, x_k)$. Hence the value of the integral (6) remains unchanged if the region of integration Q_k is replaced by $Q_k+\frac{1}{2}t'$. By means of the translation $x = x'+\frac{1}{2}t'$ the integral (6) is equal to

$$\int_{Q_k+t'/2} |f(x+\tfrac{1}{2}t')-f(x-\tfrac{1}{2}t')|\,dx = \int_{Q_k} |f(x+t')-f(x')|\,dx'.$$

Hence

$$J(p,f) = \int_{2Q_k}\Big[\int_{Q_k} |f(x'+t')-f(x')|\,dx'\Big]\frac{1}{|t'|^k}\,dt'.$$
$$= \int_{Q_k}\Big[\int_{2Q_k} \frac{|f(x+t)-f(x)|}{|t|^k}\,dt\Big]\,dx.$$

In the last step above we have dropped the dashes in x' and t' and reverse the order of integration. Since $J(p,f) < \infty$ by the I_p condition on f, it follows that the inner integral above is $< \infty$ for almost every x in Q_k. This completes the proof for the case $p = 1$.

Reverting to the general case we can now suppose $1 < p < 2$ and $f(x) \geqslant 0$. For each positive integer n let A_n, B_n, C_n denote the sets of those points in E_k where

$$f(x) \leqslant n, \qquad n < f(x) \leqslant n+1, \qquad n+1 < f(x),$$

respectively. For a positive integer n and a point $x \in E_k$ we define a set

$$C_x = \{t|\, t\in Q_k,\, x+t\in C_n\} = \{t|\, t\in Q_k, f(x+t) > n+1\}.$$

Define two functions $\varphi(x)$, $\psi(x)$ $(-\infty < x < \infty)$ by

$$\varphi(x) = f(x) \text{ if } x\in A_n\cup B_n, \qquad \varphi(x) = n+1 \text{ if } x\in C_n, \qquad \psi(x) = f(x)-\varphi(x)$$

(observe that the functions $\varphi(x)$, $\psi(x)$ and the set C_x defined above depend on n but this dependence is not exhibited since we shall work on a fixed n in the discussions that follow). From the above definitions it is clear that

$$\left.\begin{aligned}\varphi(x) &= n+1,\\ \psi(x) &= f(x)-(n+1)\end{aligned}\right\}\text{at those points } x \text{ in } E_k \text{ where } f(x) > n+1,$$

$\varphi(x) = f(x)$ and $\psi(x) = 0$ at those points x in E_k where $f(x) \leqslant n+1$.

Also $\varphi(x) \geqslant 0$ for all x in E_k and $\varphi(x)$, $\psi(x)$ are periodic.

We first show that

$$(7)\qquad J_p(x) = \int_{2Q_k} \frac{|f(x+t)-f(x)|^p}{|t|^k}\,dt < \infty \quad \text{for almost every } x \text{ in } E_k.$$

To see this we apply the change of variables $t = 2t'$ to the above integral and thus obtain

$$J_p(x) = \int_{2Q_k} \frac{|f(x+2t')-f(x)|^p}{|t'|^k}\,dt' = \int_{Q_k} \frac{|f(x+2t)-f(x)|^p}{|t|^k}\,dt.$$

Therefore, by Fubini's theorem,

$$(8)\qquad \int_{Q_k} J_p(x)\,dx = \int_{Q_k}\Big[\int_{Q_k} |f(x+2t)-f(x)|^p\,dx\Big]\frac{1}{|t|^k}\,dt.$$

In the inner integral on the right-hand side above the integrand $|f(x+2t)-f(x)|^p$, with t kept fixed, is periodic in the variable $x = (x_1, \ldots, x_k)$. Hence the value of the inner integral remains unchanged if the region of integration Q_k is replaced by Q_k-t. Hence

$$(9)\qquad \int_{Q_k} |f(x+2t)-f(x)|^p\,dx = \int_{Q_k-t} |f(x+2t)-f(x)|^p\,dx$$
$$= \int_{Q_k} |f(x+t)-f(x-t)|^p\,dx$$

where a translation of the variable x reduces the middle integral above to the integral on the right-hand side. Substituting (9) into (8) we obtain

$$\int_{Q_k} J_p(x)\,dx = \int_{Q_k}\Big[\int_{Q_k} |f(x+t)-f(x-t)|^p\,dx\Big]\frac{1}{|t|^k}\,dt$$
$$= \int_{Q_k}\Big[\int_{Q_k} \frac{|f(x+t)-f(x-t)|^p}{|t|^k}\,dt\Big]\,dx.$$

By hypothesis the repeated integral on the right-hand side above is $< \infty$ and hence $J_p(x) < \infty$ for almost every x in Q_k. Now the function $J_p(x)$ defined by (7) is clearly periodic in the variable $x = (x_1, \ldots, x_k)$ and hence $J_p(x) < \infty$ for almost every point x in E_k.

Fixing the positive integer n we now establish the following inequality

$$(10)\qquad \int_{Q_k} \frac{|\psi(x+t)-\psi(x)|}{|t|^k}\,dt < \infty \quad \text{for almost every point } x \text{ in } A_n.$$

The proof of the above inequality is divided into the following three steps:

(11)
$$x \in A_n \text{ and } J_p(x) = \int\limits_{2Q_k} \frac{|f(x+t)-f(x)|^p}{|t|^k}\,dt \Rightarrow \int\limits_{C_x} \frac{|f(x+t)-f(x)|}{|t|^k}\,dt < \infty.$$

(12) $$x \in A_n \text{ and } \int\limits_{C_x} \frac{|f(x+t)-f(x)|}{|t|^k}\,dt < \infty \Rightarrow \int\limits_{C_x} \frac{|\psi(x+t)-\psi(x)|}{|t|^k}\,dt < \infty.$$

(13) For every $x \in A_n$, $$\int\limits_{C_x} \frac{|\psi(x+t)-\psi(x)|}{|t|^k}\,dt = \int\limits_{Q_k} \frac{|\psi(x+t)-\psi(x)|}{|t|^k}\,dt.$$

Clearly the above three steps together imply (10) since $J_p(x) < \infty$ for almost every x in Q_k.

Fix on a point $x \in A_n$. We have clearly

$$f(x+t)-f(x) > (n+1)-n = 1 \quad \text{for all } t \in C_x$$

since $f(x) \leqslant n$ and $f(x+t) > n+1$ for all points $t \in C_x$ by the definition of C_x. Therefore for every point $t \in C_x$ we have

$$\frac{|f(x+t)-f(x)|}{|t|^k} < \frac{|f(x+t)-f(x)|^p}{|t|^k}.$$

Integrating both sides of the above inequality over C_x we have

$$\int\limits_{C_x} \frac{|f(x+t)-f(x)|}{|t|^k}\,dt < \int\limits_{C_x} \frac{|f(x+t)-f(x)|^p}{|t|^k}\,dt.$$

Now

$$J_p(x) = \int\limits_{2Q_k} \frac{|f(x+t)-f(x)|^p}{|t|^k}\,dt < \infty$$

implies that the integral on the right-hand side above is $< \infty$ since $C_x \subset 2Q_k$ and the integrand is non-negative. This implies that the integral on the left-hand side above is $< \infty$, thus proving (11).

To prove (12) we observe that $f(x+t) > n+1$, $\varphi(x+t) = n+1$ at every point $t \in C_x$. Hence

$$\psi(x+t) = f(x+t)-\varphi(x+t) = f(x+t)-(n+1) < f(x+t)-n.$$

Now $f(x) \leqslant n$ because $x \in A_n$. It thus follows from the above inequality that

$$\psi(x+t) < f(x+t) - f(x).$$

Since $\psi(x) = 0$ and $\psi(x+t)$ is always non-negative, we actually have

$$|\psi(x+t)-\psi(x)| = \psi(x+t) < f(x+t)-f(x) = |f(x+t)-f(x)|, \quad t \in C_x.$$

Dividing the above inequality throughout by $|t|^k$ and integrating with respect to t over the set C_x we obtain (12).

We now come to the proof of (13). Let $\tilde{C}_x$ be the complement of

$$C_x = \{t | \, t \in Q_k,\ f(x+t) > n+1\}$$

relative to Q_k. For every point $t \in \tilde{C}_x$ we have $f(x+t) \leqslant n+1$ and so $\psi(x+t) = 0$ by the definition of ψ. Also $\psi(x) = 0$ since $x \in A_n$. Therefore in the integral

$$\int\limits_{\tilde{C}_x} \frac{|\psi(x+t)-\psi(x)|}{|t|^k}\,dt$$

the integrand vanishes identically for all points $t \in \tilde{C}_x$ and so the above integral is 0. We thus have

$$\int\limits_{Q_k} \frac{|\psi(x+t)-\psi(x)|}{|t|^k}\,dt = \int\limits_{\tilde{C}_x} + \int\limits_{C_x} = \int\limits_{C_x}.$$

This completes the proof of (13) and hence (10).

Lastly we now prove that

(14) $$\int\limits_{Q_k} \int\limits_{Q_k} \frac{|\varphi(x+t)-\varphi(x)|^2}{|t|^k}\,(dt\,dx) < \infty.$$

To see this we observe that the inequality

$$|\varphi(x+t)-\varphi(x-t)| \leqslant |f(x+t)-f(x-t)|$$

always holds for all x and all t from the definition of φ, and hence

$$|\varphi(x+t)-\varphi(x-t)|^p \leqslant |f(x+t)-f(x-t)|^p.$$

It thus follows from the condition I_p on f, namely

$$\int\limits_{Q_k} \int\limits_{Q_k} \frac{|f(x+t)-f(x-t)|^p}{|t|^k}\,(dt\,dx) < \infty$$

that

$$\int\limits_{Q_k} \int\limits_{Q_k} \frac{|\varphi(x+t)-\varphi(x-t)|^p}{|t|^k}\,(dt\,dx) < \infty.$$

Because $\varphi(x)$ is bounded the above inequality involving the p-th power of $|\varphi(x+t)-\varphi(x-t)|$ implies the same inequality with p replaced by 2, i.e. (14) (we omit details of proof here).

Inequality (10) and Theorem 3 show that the Fourier series $S[\psi]$ of ψ is summable (B-R, $(k-1)/2$) to $\psi(x)$ at almost every point x in A_n. Inequality (14) and Theorem 4 show that the Fourier series $S[\varphi]$ of φ is summable (B-R, $(k-1)/2$) to $\varphi(x)$ at almost every point x in E_k. Since $S[f] = S[\varphi+\psi] = S[\varphi]+S[\psi]$, it follows that $S[f]$ is summable (B-R, $(k-1)/2$) to $\varphi(x)+\psi(x) = f(x)$ at almost every point x in A_n.

The final part of the proof is now accomplished by passing to the limit $n\to\infty$ on the sets A_n. Since $S[f]$ is summable (B-R, $(k-1)/2$) at almost every point x in each of the sets A_n $(n = 1, 2, \ldots)$, it is also summable at almost every point x in the union $\bigcup_{n=1}^{\infty} A_n$. From the definition of the sets A_n it is clear that

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid x\in E_k,\ 0\leqslant f(x) < \infty\}.$$

Since $f(x)$ is periodic and integrable over Q_k, $f(x)$ is finite-valued almost everywhere, i.e. the set of all points in E_k where $f(x) = \infty$ has measure zero. This means that the union $\bigcup_{n=1}^{\infty} A_n$ differs from the whole space E_k by a set of measure zero. Therefore $f(x)$, being summable (B-R, $(k-1)/2$) at almost every point x in the union $\bigcup_{n=1}^{\infty} A_n$, is also summable at almost every point x in the whole space E_k.

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Некоторые слабые методы суммирования

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Известно, что для любой неограниченной последовательности $\{S_n\}$ и любого числа a можно построить регулярный матричный метод суммирования, суммирующий последовательность $\{S_n\}$ к числу a и не суммирующий ни одной последовательности, отличной от вида $\{AS_n+\alpha_n\}$, где $A = \text{const}$, а $\{\alpha_n\}$ — сходящаяся последовательность ([2], теорема 2). Однако, если $\{S_n\}$ — ограниченная последовательность, то построение подобного матричного метода оказывается невозможным. Более того, всякий регулярный матричный метод, суммирующий одну расходящуюся ограниченную последовательность, суммирует континуальное множество ограниченных последовательностей, расходящихся одновременно с любой их нетривиальной конечной линейной комбинацией ([3], теорема 1).

В настоящей работе показано, что для полунепрерывных методов суммирования (обобщенный предел последовательности $\{S_n\}$ определяется как $\lim\limits_{x\to\infty}\sum_{n=1}^{\infty} C_n(x)S_n$) оказывается возможным построение регулярных методов, суммирующих к заданному числу „только одну" расходящуюся последовательность, независимо от того, ограничена она или нет.

Доказательству основной теоремы — теоремы 1 — предпошлем следующую лемму:

Лемма. *Пусть $\{S_n\}$ — последовательность, принимающая два значения:*

$$S_n = \begin{cases} b & \text{при}\quad n = n_k, \\ b' & \text{при}\quad n = n'_k\ (b'\neq b \ \text{и}\ \{n_k\}\cup\{n'_k\} = \{n\}) \end{cases}$$

и a — произвольное число, отличное от b и b'. Для чисел вида $x = k+(2^m-1)/2^m$ $(k, m = 1, 2, \ldots)$ определим функции

$$(1)\qquad C_n(x) = \begin{cases} \dfrac{b'-a}{b'-b} & \text{при}\quad n = n_k,\ k, m = 1, 2, \ldots, \\[2mm] \dfrac{a-b}{m(b'-b)} & \text{при}\quad n = n'_k, n'_{k+1}, \ldots, n'_{k+m-1},\ k, m = 1, 2, \ldots, \\[2mm] 0 & \text{в остальных случаях.} \end{cases}$$