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On operational calculus of germs and the Laplace transform of germs

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1. J. Mikusiński introduced, in the paper [2] and in the book [3], the notion of an operator. As the starting point for his operator theory, the author considers the ring C of continuous functions defined in the interval $[0, \infty)$ under usual addition and convolution as multiplication. From Titchmarsh's theorem on convolution it follows that C is an integral domain and may therefore be embedded in a field M_∞. This embedding is analogous to the embedding of the domain of integers in the field of rational numbers. The field M_∞ will be called the *field of Mikusiński's operators*.

For the theory of differential equations, it is useful to have the notion of an operator in a finite interval $[0, T]$ (1). We observe that the analogous ring C_T has divisors of zero. In view of Titchmarsh's theorem on convolution, we can characterize the set Z of divisors of zero in the ring C_T [4], and we can embed this ring in the ring M_T of fractions p/q such that q is not a divisor of zero. This ring will be called the *ring of Mikusiński's operators* in a finite interval.

In the paper [5] J. Mikusiński introduced another ring, which will be denoted by C_0. This ring is an integral domain. The extension of the integral domain C_0 to a fraction field will be denoted by M_0 and the elements of this field will be called *operator germs*.

In this note we shall show some connections between the field M_∞ and the field M_0 and the ring M_T. In the last section there will be given a characterization of the elements of the field M_0 by means of the Laplace transform.

2. Definitions of rings C, C_T and C_0. Let C be a ring of continuous functions defined in the interval $[0, \infty)$ under usual addition and con-

(1) In the paper [4], C_T is a ring of continuous functions in an interval $[0, T)$. In this note C_T denotes a ring of continuous functions in an interval $[0, T]$. Antosik [1] showed that the ring of fractions formed by rings of continuous functions in the closed interval $[0, T]$ and the open interval $[0, T)$ are isomorphic.

volution regarded as multiplication. Each of the following sets: I_∞ — the set consisting of the single function $f(t) \equiv 0$, I_T — the set of functions vanishing in a given common interval $[0, T]$, I_0 — the set of functions such that each of them is equal to zero in some interval $[0, a]$ (depending on the function) is an ideal in the ring C.

From Titchmarsh's theorem it follows that I_∞ and I_0 are prime ideals. The quotient ring C/I_T is isomorphic to the ring C_T. The quotient ring C/I_0 will be called the ring of *function germs* and denoted by C_0 (see [5]). The quotient ring C/I_∞ is identical with the ring C. The rings C and C_0 can be extended to fields M_∞ and M_0.

3. The set of operators M_∞^+. For every function $f \in C$ we can find a number a which is the endpoint of the longest interval in which the function f is equal to zero. Let the numbers α, β, α_1 and β_1 correspond in this sense to the functions a, b, a_1 and b_1. If two pairs a/b and a_1/b_1 determine the same operator $p \in M_\infty$, then it follows from Titchmarsh's theorem that $\alpha-\beta = \alpha_1-\beta_1 = \gamma$. The number γ will be called the *characteristic number* of the operator p. From well-known properties of convolution it follows that

(i) $$\gamma_{pq} = \gamma_p + \gamma_q,$$

where γ_{pq} is the characteristic number of the operator pq, γ_p is the characteristic number of the operator p and γ_q is the characteristic number of the operator q. In addition, from another property of convolution it follows that

(ii) $$\gamma_{p\pm q} \geqslant \min(\gamma_p, \gamma_q),$$

where $\gamma_{p\pm q}$ denotes the characteristic number of the operator $p\pm q$.

Let M_∞^+ denote the set of operators $p \in M_\infty$ whose characteristic numbers are non-negative. From (i) and (ii) it follows that M_∞^+ is a ring if we adopt the same operations as in M_∞. Let J_T denote the set of operators $p \in M_\infty^+$ such that $\gamma_p \geqslant T$. From (i) and (ii) we may conclude that J_T is an ideal in the ring M_∞^+. We shall prove the following

PROPOSITION 1. *The quotient ring M_∞^+/J_T of elements $\overline{p}$ is isomorphic to the ring M_T of operators in the interval* $[0, T]$.

Proof. Let p and q belong to a class $\overline{p}$ of equivalent elements of the ring M_∞^+/J_T. Furthermore, let γ_p and γ_q denote the characteristic numbers of operators p and q. We take representations $p = a/b$ and $q = c/d$ such that a is equal to zero in the interval $[0, \gamma_p]$ and c is equal to zero in the interval $[0, \gamma_q]$. Here b and d are not equal to zero in any right-hand neighbourhood of the origin. Since the operators p and q belong to the class $\overline{p}$ of the quotient ring M_∞^+/J_T, we have $(ad-bc)/bd \in J_T$. The function bd is not equal to zero in any right-hand neighbourhood of the origin.

Hence, $(ad-bc) \in I_T$. This implies that a/b and c/d determine the same element $p \in M_T$. Similarly, we can show that if the pairs a/b and c/d determine the same element p of the ring M_T, then the operators a/b and c/d belong to the same class of the quotient ring M_∞^+/J_T. Hence the theorem.

Each element $\overline{p}$ of the ring M_∞^+/J_T is a class of equivalent elements $p \in M_\infty^+$. If the operators p and q are equivalent, $\gamma_p < T$ and $\gamma_q < T$, then $\gamma_p = \gamma_q$. Conversely, if the characteristic numbers γ_p and γ_q of the operators p and q are not equal, $\gamma_p < T$ and $\gamma_q < T$, then p and q are not equivalent. Hence, the elements p and q determining $\overline{p} \neq J_T$ have the same characteristic numbers. The number $\gamma_{\overline{p}}$, which is the same for all the operators determining the element $\overline{p}$, will be called the *characteristic number of the element* $\overline{p}$. In the case of $\overline{p} = J_T$, we shall take $\gamma_{J_T} = T$ as the characteristic number.

If we restrict ourselves to the representation of operators $p \in M_\infty^+$ in the form $p = a/b$, where b is not equal to zero in any right-hand neighbourhood of point zero, then the ring M_∞^+/J_T is identical to the ring M_T. Throughout our discussion we shall take only such representations.

4. The quotient ring M_T/J_0. From (i) and (ii) it follows that the set J_0 of elements $\overline{p} \in M_T$ such that $\gamma_{\overline{p}} > 0$ is an ideal in M_T. Hence M_T/J_0 is a field.

PROPOSITION 2. *The field M_T/J_0 is isomorphic to the field M_0.*

Proof. See [5].

The above discussion can be recapitulated as follows:

The ring M_∞^+ is homomorphic to the ring M_T with the kernel J_T.

The ring M_T is homomorphic to the field M_0 with the kernel J_0.

5. Laplace transform of operator germs. Some functions $f \in C$ have their Laplace transform but not all. Nevertheless, we shall show that, in some way, it is possible to define the Laplace transform for every element of the field M_0. In order to show this, we shall define a ring A and an ideal I_A in A. We shall prove that the quotient ring A/I_A is an integral domain and that we can extend it to a fraction field M^*. The fraction field M^* s isomorphi c to the field M_0.

Definition of the ring A. Let A be a set of functions defined in the interval $[0, \infty)$, locally square integrable and such that $\int\limits_0^\infty |f(t)|^2 e^{-2\gamma t} < \infty$ for some γ.

We shall show that A is a *ring* under usual addition and convolution regarded as multiplication. It is easy to show that the sum f_1+f_2 of functions $f_1 \in A$ and $f_2 \in A$ belongs to A. Moreover, we shall show that the convolution $f_1 * f_2$ of functions f_1 and f_2, belonging to A, is an element of A.

The functions f_1 and f_2 are locally integrable; therefore their convolution $f_1 * f_2$ is also a locally integrable function (see [6]). In order to prove that $f_1 * f_2$ belongs to A it suffices to show that there exists a number γ such that the integral $\int_0^\infty \left|\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right|^2 e^{-2\gamma t} dt$ is finite.

Let the functions f_1 and f_2 belong to A. Then there exist numbers γ_1 and γ_2 such that

$$\int_0^\infty |f_1(t)|^2 e^{-2\gamma_1 t} dt = K_1 \quad \text{and} \quad \int_0^\infty |f_2(t)|^2 e^{-2\gamma_2 t} dt = K_2.$$

Now we shall show that, for arbitrary $\varepsilon > 0$, the integral

$$\int_0^\infty \left|\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right|^2 e^{-2\gamma t} dt$$

where $\gamma = \gamma_0 + \varepsilon$ and $\gamma_0 = \max(\gamma_1, \gamma_2)$, is finite.

Indeed,

$$\int_0^\infty \left|\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right|^2 e^{-2(\gamma_0+\varepsilon)t} dt$$

$$\leqslant \int_0^\infty e^{-2\varepsilon t} \left|\int_0^t f_1(t-\tau) e^{-\gamma_0(t-\tau)} f_2(\tau) e^{-\gamma_0 \tau} d\tau\right|^2 dt$$

$$\leqslant \int_0^\infty e^{-2\varepsilon t} \left(\int_0^t |f_1(t-\tau) e^{-\gamma_0(t-\tau)}|^2 d\tau \cdot \int_0^t |f_2(\tau) e^{-\gamma_0 \tau}|^2 d\tau\right) dt$$

$$\leqslant \int_0^\infty K_1 K_2 e^{-2\varepsilon t} dt.$$

Each function f of the set A has its Laplace transform $F(s) = \int_0^\infty e^{-st} f(t)\, dt$, which is an analytic function in the half-plane $\operatorname{re} s > \gamma$ and is such that

$$\text{(iii)} \qquad \int_{-\infty}^{\infty} |F(x+iy)|^2 dy < \infty \quad \text{for} \quad x > \gamma.$$

Conversely, every analytic function in a certain half-plane $\operatorname{re} s > \gamma$, satisfying condition (iii) is the Laplace transform of a function $f \epsilon A$ (see [7]). It is possible to use the following symbolic notation: $\mathscr{L}(A) = \tilde{A}$. The Laplace transform maps the ring A onto the isomorphic ring $\tilde{A}$ with usual addition and multiplication. The subset I_A of functions $f \epsilon A$ such that each of them vanishes almost everywhere in some right-hand neighbourhood of the origin is an ideal in the ring A. In view of Titchmarsh's theorem mentioned above I_A is a prime ideal in the ring A. Hence, the quotient ring A/I_A is an integral domain.

Let $\tilde{I}_A$ denote the set of Laplace transforms of functions of the ideal I_A. From a well-known property of the Laplace transform it follows that the function $F \epsilon \tilde{A}$ belongs to $\tilde{I}_A$ if and only if $\lim_{x\to\infty} e^{\alpha x} F(x+iy) = 0$ for some $\alpha > 0$. Hence it follows that $\tilde{I}_A$ is an ideal in the ring A. We observe that the quotient ring A/I_A is isomorphic to the quotient ring $\tilde{A}/\tilde{I}_A$. The ring A/I_A generates the field M^*. Hence it follows that the quotient ring $\tilde{A}/\tilde{I}_A$ generates the field $\tilde{M}^*$, which is isomorphic to the field M_0. Elements of the field $\tilde{M}^*$ can be taken as Laplace transforms of elements of the field M_0.

References

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