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The converse of Wiener-Levy-Marcinkiewicz Theorem

by

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Introduction. Let $F(x)$ be a function of a single real variable x, defined in the open set I. We say $F\in G_s$, $0<s\leqslant 1$, if and only if, for every compact set $\bar{I}'$ contained in I,

$$|F^{(n)}(x)|\leqslant B^n n^{n/s},\qquad x\in\bar{I}', \tag{1.1}$$

where $F^{(n)}$ denotes the n-th derivative of $F(x)$, and B is a constant depending only on $\bar{I}'$.

For $0<p<\infty$, we call

$$A_p=\Big\{f, f(x)=\sum_{-\infty}^{\infty}a_n e^{inx},\ \text{such that}\ \Big(\sum_{-\infty}^{\infty}|a_n|^p\Big)^{1/p}=A_p[f]<\infty\Big\}.$$

Marcinkiewicz [1] (p. 588-594) proved that *if*

(i) *the domain of* $F(x)$ *contains the range of* $f(x)$: $(D(F)\supset R(f))$, *and*

(ii) $f(x)\in A_s$, $F(x)\in G_s$,

then $F(f(x))\in A_1$.

A. Zygmund has pointed out that the proof of Marcinkiewicz can readily be extended to show that actually $F(f(x))\in A_s$.

In this paper we will prove the converse of Marcinkiewicz's theorem, in a stronger form; more precisely:

THEOREM. *Let* $F(x)$ *be defined in an open set* I *and let* $0<s\leqslant 1$. *Suppose that if* $f\in A_s$ *with* $D(F)\supset R(f)$, *then* $F(f(x))\in A_p$, $p<2$ *(*p *depending on* f*). Then* $F\in G_s$.

This result has been proved when $s=1$ by Helson, Kahane, Katznelson, and Rudin in [2], [3], and [4]. The result is also true on any infinite compact abelian group, although our proof will be restricted to the unit circle.

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The above statement will be proved in section 1. In section 2, we discuss the stability of A_s under composition of its elements with a function $F(x)$.

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1. We will divide the proof of the theorem into several lemmas.

LEMMA 1. *Let*

$$F(x) = \sum_{-\infty}^{\infty} a_n e^{inx} \quad \textit{and} \quad |a_n| \leqslant Ce^{-c|n|^s}, \quad C > 0, \quad c > 0$$

(c independent of n). Then $F \in G_s$.

Proof. We have

$$F^{(k)}(x) = \sum_{-\infty}^{\infty} (in)^k a_n e^{inx},$$

$$|F^{(k)}(x)| \leqslant \sum_{-\infty}^{\infty} |n|^k |a_n| \leqslant 2C \sum_{1}^{\infty} n^k e^{-cn^s}.$$

To estimate this series set $f(x) = x^k e^{-cx^s}$, and observe that $f(x)$ increases in $(0, k^{1/s}/(cs)^{1/s})$ and decreases in $(k^{1/s}/(cs)^{1/s}, \infty)$ and that $f(k^{1/s}/(cs)^{1/s}) = k^{k/s} e^{-k/s}$.

Thus

$$\sum_{1}^{\infty} n^k e^{-cn^s} \leqslant \int_0 x^k e^{-cx^s} dx + (e^{-1/s})^k k^{k/s} \leqslant B^k k^{k/s},$$

and the lemma follows.

The converse of this lemma has been proved by Marcinkiewicz [1].

Remark 1. We claim $A_p(e^{i\cos x}) = K > 1$ for $p < 2$. For

$$A_2[e^{i\cos x}] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{i\cos x}|^2 dx = 1$$

and if

$$e^{i\cos x} = \sum_{-\infty}^{\infty} a_n e^{inx},$$

then $|a_n| < 1$ and $|a_n|^2 < |a_n|^p$. The claim now follows.

Observe that if $f \in A_p$ $(1 < p < \infty)$ and $g \in A_1$, then

(1.2) $$A_p[f \cdot g] \leqslant A_p[f] A_1[g] \quad \text{(Young's Inequality)}$$

KAHANE'S LEMMA. *If* $f_1, \ldots f_n \in A_1$, *then given* $\varepsilon > 0$ *there exist* $\lambda_1, \ldots, \lambda_n$ *such that*

$$A_p\Big[\prod_{j=1}^{n} f_j(\lambda_j x)\Big] \geqslant (1-\varepsilon) \prod_{j=1}^{n} A_p[f_j], \quad 1 \leqslant p \leqslant \infty.$$

Actually, the result also holds for $0 < p < 1$ although we shall not prove it. For the case $p = 1$, see [2].

Proof. In the case of two functions f_1 and f_2 where f_1 is a trigonometric polynomial of degree N,

$$A_p[f_1(x) f_2(2Nx)] = A_p[f_1] A_p[f_2].$$

Now let $f_1 \in A_1$ be arbitrary. Given $\varepsilon > 0$, choose a trigonometric polynomial $P(x)$ such that

$$A_p[f_1 - P] \leqslant A_1[f_1 - P] < \delta, \quad \delta = \frac{\varepsilon}{2} A_p[f_1].$$

If the degree of $P(x)$ is N, then

$$A_p[f_1(x) f_2(2Nx)] \geqslant A_p[P(x) f_2(2Nx)] - A_p[(f_1 - P) f_2]$$
$$\geqslant A_p[P] A_p[f_2] - \delta A_p[f_2] \geqslant (1-\varepsilon) A_p[f_1] A_p[f_2].$$

Using induction and the same technique the result follows. Observe that the λ_j's may be chosen to be all different.

COROLLARY. *Let R be an integer, $0 < s \leqslant 1$, $p \geqslant 1$, then* $\sup A_p[e^{if}] \geqslant K^{r^s/2}$, *where* $r = 2^{1/s-1} R^{1/s}$, *the supremum being taken over all f with* $A_s[f] \leqslant r$.

K as defined in remark 1.

Proof. Using Kahane's lemma with all λ_j's different

$$A_p\Big[\exp\Big\{i \sum_{j=1}^{R} \cos \lambda_j x\Big\}\Big] \geqslant (1-\varepsilon) K^R \quad \text{where} \quad A_s\Big[\sum_{j=1}^{R} \cos \lambda_j x\Big] = 2^{1/s-1} R^{1/s}$$

and the corollary follows.

We are now in position to prove the theorem announced in the introduction. The proof follows the line of that in [4].

Proof of the theorem. First of all we observe that it is enough to prove (1.1) in a neighborhood of every point $x \in I$ and then use the fact that $\bar{I}'$ is compact. We may also assume that $F(0) = 0$, and that $I = [-1, 1]$.

The proof will be divided into four steps:

(1) There exists an interval $(-\beta, \beta)$ and numbers $p < 2$, $\delta > 0$, and $M < \infty$ such that $A_p[F(f(x))] \leqslant M$ for all $f \in A_s$ which vanish outside $(-\beta, \beta)$ and satisfy $A_s[f] < \delta$.

(2) With p as above there exist numbers $\eta > 0$, and $B < \infty$ such that $A_p[F(f)] \leqslant B$ for all f with $A_s[f] \leqslant \eta$.

(3) $F(x)$ is continuous.

(4) $F \in G_s$ in a neighborhood of the origin.

Proof of (1). If (1) is false, there exists a sequence of disjoint intervals $[a_j, b_j] \subset [-\pi, \pi]$ (obtained by translating the origin), a sequence of functions f_j with support in (a_j, b_j) and $A_s^s[f_j] \leqslant 1/2^j$, and a sequence of numbers $p_j \to 2$ with the following property. If $\Phi_j \in A_1$ is equal to 1 in (a_j, b_j) and zero in (a_k, b_k) for $k \neq j$, then $A_{p_j}[F(f_j)] > jA_1[\Phi_j]$. Let

$$f = \sum_{j=1}^{\infty} f_j.$$

Then $f \in A_s$ and $F(f) \in A_p$ for some $p < 2$ since $\max|f(x)| \leqslant A_s[f] \leqslant 1$. Observe that $F(f_j(x)) = \Phi_j F(f(x))$ and therefore using (1.2)

$$jA_1[\Phi] \leqslant A_{p_j}[F(f_j)] = A_{p_j}[\Phi_j F(f)] \leqslant A_{p_j}[F(f)]A_1[\Phi_j].$$

Hence $A_{p_j}[F(f)] \geqslant j$, which is impossible.

For the proofs of (2) and (3) we refer the reader to [4], theorem 5.

Proof of (4). With p, η, and B as before, observe that if $f, g \in A_s$, then $f \cdot g \in A_s$ and

$$A_s[f\cdot g] \leqslant A_s[f]A_s[g], \quad 0 < s \leqslant 1.$$

Let

$$\Phi(t) = F\left(\frac{\eta}{d^{1/s}}(\sin t)\right), \quad d = \sum \left(\frac{1}{n!}\right)^s.$$

If $A_s[f] \leqslant 1$ and a is a real number, then

$$A_s^s[\sin(f+a)] \leqslant |\sin a|^s A_s^s[\cos f] + |\cos a|^s A_s^s[\sin f] \leqslant \sum_{n=0}^{\infty} \frac{(A_s^s[f])^n}{(n!)^s} \leqslant d.$$

That is to say, if $A_s[f] \leqslant 1$, then $A_s[\sin(f+a)] \leqslant d^{1/s}$. Hence if $A_s[f] \leqslant 1$, then $A_p[\Phi(f+a)] \leqslant B$. Using (3), $\Phi(t)$ is a continuous periodic function. If

$$\Phi(x) \approx \sum_{-\infty}^{\infty} a_n e^{inx},$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(f(x)+a)\,e^{ina}\,da = a_n e^{inf(x)},$$

and therefore $A_p[a_n e^{inf}] \leqslant B$ for all f with $A_s[f] \leqslant \eta$. Then

$$|a_n| \leqslant B\left(\begin{matrix}\sup A_p[e^{inf}] \\ A_s[nf] \leqslant n\end{matrix}\right)^{-1} \leqslant Bq^{-n^s}, \quad q > 1.$$

The continuity of Φ and lemma 1 now imply that

$$\Phi(x) = \sum_{-\infty}^{\infty} a_n e^{inx} \quad \text{and} \quad \Phi \in G_s.$$

Finally, $F(x) = \Phi(\text{arc.sin.}(d^{1/s}x/\eta))$ and since $\text{arc.sin.}(x) \in G_1$ in a neighborhood of the origin, it follows that $F \in G_s$ in a neighborhood of the origin. This completes the proof of the theorem.

2. The theorem proved in section 1 shows that if

$$F(f(x)) \in \bigcup_{s \leqslant p < 2} A_p$$

for all $f \in A_s$ with $D(F) \supset R(f)$, then we actually have $F(f(x)) \in A_s$ for all these f. Thus, we cannot "lift" the algebra A_s by composing its elements with a function $F(x)$.

In this section we show that the only functions $F(x)$ that "lower" the algebra A_s, are the constant functions. This will follow rather easily from the following

Lemma. *Let $F(x) \in G_s$, $0 < s \leqslant 1$. Let U be an interval where $F'(x) > 0$. Let W be an open set contained in the image of U under F. Then $F^{-1}(y)$, the composition inverse of the restriction of F to U, is in G_s in W.*

Proof. We have for $x \in U$, $F^{-1}(F(x)) = x$. Write

$$G_1(x) = \frac{1}{\dfrac{d}{dx}(F(x))}.$$

Then $G_1(x) \in G_s$ in U, since $G_1(x)$ is a composition of $(d/dx)F(x) \in G_s$ with $1/z$, which is analytic in the range of $(d/dx)F(x)$, for $x \in U$. Write now

$$y = F(x), \quad G_n(x) = G_1(x)G'_{n-1}(x).$$

Then clearly

$$\frac{d^n}{dy^n}(F^{-1}(y)) = G_n(x).$$

Thus, we have to show $|G_n(x)| \leqslant C^n n^{n/s}$. Since $G_1(x) \in G_s$, we have

$$|G_1^{(n)}(x)| \leqslant B^n (n!)^{1/s}.$$

(This is equivalent to the definition of G_s in section 1, and is more suitable for our calculations here.)

$$G_n(x) = G_1(x)G'_{n-1}(x) = G_1(x)[G_1(x)G'_{n-2}(x)]' = \ldots$$

$$= G_1(x)\sum_{k_1=0}^{1}\binom{1}{k_1}G_1^{(1-k_1)}(x)\sum_{k_2=0}^{k_1+1}\binom{k_1+1-k_2}{k_2}G_1^{(k_1+1-k_2)}(x)\sum_{k_3}\ldots$$

$$\ldots\sum_{k_{n-2}=0}^{k_{n-3}+1}\binom{k_{n-3}+1-k_{n-2}}{k_{n-2}}G_1^{(k_{n-3}+1-k_{n-2})}(x)\,G_1^{(k_{n-2}+1)}(x).$$

Theorefore

$$|G_n(x)| \leqslant B^{n-1} \sum_{k_1=0}^{1} \binom{1}{k_1} [(1-k_1)!]^{1/s} \sum_{k_2} \ldots$$
$$\ldots \sum_{k_{n-2}=0}^{k_{n-3}+1} \binom{k_{n-3}+1-k_{n-2}}{k_{n-2}} [(k_{n-3}+1-k_{n-2})!]^{1/s} [(k_{n-2}+1)!]^{1/s}$$
$$\leqslant B^{n-1} \Big[\sum_{k_1=0}^{1} \binom{1}{k_1} (1-k_1)! \sum_{k_2=0}^{k_1+1} \binom{k_1+1-k_2}{k_2} (k_1+1-k_2)! \sum_{k_3} \ldots$$
$$\ldots \sum_{k_{n-2}=0}^{k_{n-3}+1} \binom{k_{n-3}+1-k_{n-2}}{k_{n-2}} (k_{n-3}+1-k_{n-2})!(k_{n-2}+1)! \Big]^{1/s}$$
$$= B^{n-1} \cdot \left[\frac{(2n-3)!}{2^{n-2}(n-2)!}\right]^{1/s} \leqslant C^n \cdot n^{n/s}.$$

Thus, the lemma is proved.

We are now in position to prove

THEOREM. *Let $F(x)$ be defined in an open set I, and let $0 < s \leqslant 1$. Then if $F(f(x)) \in \bigcup_{0<r<s} A_r$, for all $f(x) \in A_s$, with $D(F) \supset R(f)$, then $F(x)$ is a constant.*

Proof. If F satisfies the hypothesis, then since $\bigcup_{0<r<s} A_r \subset A_s$, we have $F \in G_s$. If F is not a constant function, there is an open set $U \subset D(F)$ such that $F'(x) \neq 0$ in U. To fix ideas, we assume $[-1, 1] \subset U$, $F(0) = 0$. Let W be an open neighbourhood of 0 which is contained in the image of $[-1, 1]$ under $F(x)$. Let $F^{-1}(y)$ be the composition inverse of $F(x)$. We have $F^{-1}(y) \in G_s$ in W. Thus, for all $f \in A_s$ with $R(f) \subset W$ we have $F^{-1}(f(x)) \in A_s$, $R(F^{-1}(f(x))) \subset [-1, 1]$. Thus for all these f, $F(F^{-1}(f)) \in \bigcup_{r<s} A_r$. That is, $f \in \bigcup_{r<s} A_r$. This contradicts $\bigcup_{r<s} A_r \neq A_s$, and the theorem is proved.

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