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Singular integrals and partial differential equations of parabolic type

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Introduction

Important in the study of partial differential equations of parabolic type are classes of singular integrals of the form

$$\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K(x, t; x-y, t-s)f(y, s)\,dy\,ds \tag{1}$$

and

$$\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K(y, s; x-y, t-s)f(y, s)\,dy\,ds. \tag{2}$$

Here x and y denote points in E^n and t and s belong to $(0, \infty)$. The basic assumptions on $K(x, t; y, s)$ are

(i) $K(x, t; y, s) = 0$ for $s < 0$,

(ii) there is an $\alpha \geqslant 2$ such that for every $\lambda > 0$, $K(x, t; \lambda y, \lambda^\alpha s) = \lambda^{-n-\alpha} K(x, t; y, s)$,

(iii) $\int_{E^n} K(x, t; y, 1)\,dy = 0$.

Because of the "homogeneity" of $K(x, t; y, s)$, it is easy to see that $K(x, t; y, s) = K(x, t; y/s^\beta, 1)s^{-n\beta-1}$, where $\beta = 1/\alpha$. The L^p-convergence of (1) as $\varepsilon \to 0$ when $K(x, t; y, s) = K(y, s)$ is independent of (x, t) was considered by B. F. Jones, Jr. in [5]. Also in [5] Jones pointed out the analogy of his class of kernels with those of Zygmund and Calderón in [1]. Likewise here the kernels, $K(x, t; y, s)$, are analogous to the "variable" kernels discussed by Zygmund and Calderón in [2].

Chapter I of this work examines again the question of the convergence of (1) and (2) as $\varepsilon \to 0$ in the L^p-sense over $E^n \times (0, \infty)$ for kernels, $K(x, t; y, s) = K(y, s)$. Here $1 < p < \infty$. The conditions for L^p-convergence are different than in [5]. In particular, no smoothness condition on $K(y, 1)$ will be needed for the case $p = 2$.

In chapter II we return to the kernels $K(x, t; y, s)$ and give sufficient conditions for the L^p convergence of (1) and (2) as $\varepsilon \to 0$ for $1 < p < \infty$. This result is applied in chapter III to obtain the existence and uniqueness of generalized solutions, $u(x, t)$, of parabolic partial differential equations, satisfying, in some sense, $u(x, 0) = 0$.

Most of the notation which will be used will be defined during the course of this work. We will, therefore, only list here a few basic ones:

$$x = (x_1, \ldots, x_n), \qquad y = (y_1, \ldots, y_n),$$

$$x \circ y = \sum_{i=1}^{n} x_i y_i, \qquad \sum = \{x \in E^n : |x| = 1\},$$

$$x' = \frac{x}{|x|} \quad (x \neq 0), \qquad \hat{f}(x) = \frac{1}{(2\pi)^n} \int_{E^n} f(y) e^{ix\circ y}\,dy.$$

We will let C stand for a positive constant, not necessarily the same at each occurrence, depending only on the dimension n of E^n and p.

Unless otherwise stated, the functions $f(x, t)$, which we will consider will belong to $L^p(E^n \times (0, \infty))$, $1 < p < \infty$, and we will consider them extended to all of E^{n+1} by setting $f(x, t) = 0$ for $t < 0$. The terms "L^p-norm" and "L^p-limit" will generally refer to the L^p-norm and L^p-limit over $E^n \times (0, \infty)$.

I. L^p-CONVERGENCE

1.1. L^2-theory. We begin by studying the case when $f \in L^2(E^n \times (0, \infty))$. Extend $f(y, s)$ to all of E^{n+1} by setting $f(y, s) = 0$ for $s < 0$. Set

$$K_{\varepsilon,R}(x, t) = \begin{cases} K(x, t) & \text{for } \varepsilon < t < R, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tilde{f}_{\varepsilon,R}(x, t) = \int_0^\infty \int_{E^n} K_{\varepsilon,R}(x-y, t-s) f(y, s)\,dy\,ds = f * K_{\varepsilon,R}(x, t)$$

and, finally, set $\Omega(x) = K(x, 1)$. We will always assume conditions (i), (ii), and (iii) on $K(x, t)$, given in the introduction.

LEMMA 1. *We have*

(i) $|\hat{K}_{\varepsilon,R}(x, t)| \leqslant C \int_{E^n} |\Omega(y)| \left\{1 + |y| + \left|\log \frac{1}{|x' \circ y|}\right|\right\} dy$, $x' = \frac{x}{|x|}$.

(ii) *If* $\int_{E^n} (1 + |y|)|\Omega(y)|\,dy < \infty$, *then for* $t \neq 0$,

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \hat{K}_{\varepsilon,R}(x, t), \qquad \lim_{\varepsilon\to 0} \lim_{R\to\infty} \hat{K}_{\varepsilon,R}(x, t), \qquad \text{and} \qquad \lim_{\substack{\varepsilon\to\infty \\ R\to\infty}} \hat{K}_{\varepsilon,R}(x, t)$$

all exist and are the same.

Proof. Assume $t \geqslant 0$. We have

$$\hat{K}_{\varepsilon,R}(x, t) = \int_\varepsilon^R \int_{E^n} \Omega(y/s^\beta) \frac{e^{i(x\circ y)} e^{its}}{s^{n\beta+1}}\,dy\,ds = \int_{E^n} \Omega(y) \int_\varepsilon^R \frac{e^{i(x\circ y)s^\beta} e^{its}}{s}\,ds\,dy.$$

Hence

$$\hat{K}_{\varepsilon,R}(x, t) = \alpha \int_{E^n} \Omega(y) \int_{\varepsilon^\beta|x|}^{R^\beta|x|} \frac{e^{i(x'\circ y)s} e^{i(t/|x|^\alpha)s^\alpha}}{s}\,ds\,dy, \qquad x' = \frac{x}{|x|}.$$

Before proceeding to look at the last expression, we will state a lemma which will be useful and whose proof is given in the appendix of this work.

LEMMA 2. *For* $\alpha \geqslant 2$, *the integrals*

$$\int_1^R \frac{e^{\pm ivs} e^{is^\alpha}}{s}\,ds \qquad \text{and} \qquad \int_1^R \frac{e^{\pm is} e^{i(vs)^\alpha}}{s}\,ds$$

are uniformly bounded in $v > 0$, $R \geqslant 1$.

Recall that

$$\hat{K}_{\varepsilon,R}(x, t) = \alpha \int_{E^n} \Omega(y) \int_{\varepsilon^\beta|x|}^{R^\beta|x|} \frac{e^{i(x'\circ y)s} e^{i(t/|x|^\alpha)s^\alpha}}{s}\,ds\,dy.$$

If $R^\beta|x| \leqslant 1$, then, since $\int_{E^n} \Omega(x)\,dx = 0$,

$$\hat{K}_{\varepsilon,R}(x,t) = a\int_{E^n}\Omega(y)\int_{\varepsilon^\beta|x|}^{R^\beta|x|} e^{i(t/|x|^\alpha)s^\alpha}\frac{[e^{i(x'\circ y)s}-1]}{s}\,ds\,dy.$$

Hence

$$|\hat{K}_{\varepsilon,R}(x,t)| \leqslant a\int_{E^n}|y|\,|\Omega(y)|\,dy.$$

If $\varepsilon^\beta|x| \geqslant 1$, we can write

$$\text{(3)}\qquad \hat{K}_{\varepsilon,R}(x,t) = a\int_{E^n}\Omega(y)\int_{R_1|x'\circ y|}^{R_2|x'\circ y|}\frac{e^{\pm is}e^{i(vs)^\alpha}}{s}\,ds\,dy,$$

where $R_2 \geqslant R_1 \geqslant 1$ and $v = t^\beta/|x|\,|x'\circ y|$.

If $R_1|x'\circ y| \geqslant 1$, then using the second integral in Lemma 2 we see from (3) that

$$|\hat{K}_{\varepsilon,R}(x,t)| \leqslant C\int_{E^n}|\Omega(y)|\,dy.$$

If $R_2|x'\circ y| \leqslant 1$, again from (3) we have

$$|\hat{K}_{\varepsilon,R}(x,t)| \leqslant \int_{E^n}|\Omega(y)|\int_{R_1|x'\circ y|}^{1}\frac{1}{s}\,ds\,dy \leqslant \int_{E^n}|\Omega(y)|\log\frac{1}{R_1|x'\circ y|}\,dy.$$

Since $R_1 \geqslant 1$,

$$\log\frac{1}{R_1|x'\circ y|} = \log\frac{1}{R_1}+\log\frac{1}{|x'\circ y|} \leqslant \log\frac{1}{|x'\circ y|}.$$

We conclude that

$$|\hat{K}_{\varepsilon,R}(x,t)| \leqslant \int_{E^n}|\Omega(y)|\left|\log\frac{1}{|x'\circ y|}\right|dy.$$

If $R_1|x'\circ y| < 1 < R_2|x'\circ y|$, we write

$$\hat{K}_{\varepsilon,R}(x,t) = \int_{E^n}\Omega(y)\left\{\int_{R_1|x'\circ y|}^{1}\frac{e^{\pm is}e^{i(vs)^\alpha}}{s}\,ds + \int_{1}^{R_2|x'\circ y|}\frac{e^{\pm is}e^{i(vs)^\alpha}}{s}\,ds\right\}dy.$$

As in the above situations, we can write

$$|\hat{K}_{\varepsilon,R}(x,t)| \leqslant C\int_{E^n}|\Omega(y)|\left\{1+\left|\log\frac{1}{|x'\circ y|}\right|\right\}dy.$$

Finally we consider the case when $\varepsilon^\beta|x| < 1 < R^\beta|x|$. But again

$$\hat{K}_{\varepsilon,R}(x,t) = \int_{E^n}\Omega(y)\left\{\int_{\varepsilon^\beta|x|}^{1}+\int_{1}^{R^\beta|x|}\frac{e^{i(x'\circ y)}e^{i(t/|x|^\alpha)s^\alpha}}{s}\,ds\right\}dy.$$

Hence

$$\text{(4)}\qquad |\hat{K}_{\varepsilon,R}(x,t)| \leqslant C\int_{E^n}|\Omega(y)|\left\{1+|y|+\left|\log\frac{1}{|x'\circ y|}\right|\right\}dy.$$

For $t<0$ we first note that $\hat{K}_{\varepsilon,R}(x,t) = \overline{\hat{\bar{K}}_{\varepsilon,R}(-x,-t)}$ and that $\bar{K}(y,s) = \bar{\Omega}(y/s^\beta)/s^{n\beta+1}$, $s>0$. Since

$$\int_{E^n}\bar{\Omega}(y)\,dy = 0$$

we see that equation (4) is also valid in this case.

This completes the first part of Lemma 1. We will now show the pointwise convergence of $\hat{K}_{\varepsilon,R}(x,t)$ for $t\neq 0$ under the assumption that

$$\int_{E^n}(1+|y|)\,|\Omega(y)|\,dy < \infty.$$

We have

$$\hat{K}_{\varepsilon,R}(x,t) = \int_{E^n}\Omega(y)\int_{\varepsilon}^{R}\frac{e^{its}e^{i(x\circ y)s^\beta}}{s}\,ds\,dy$$
$$= \int_{E^n}\Omega(y)\left\{\int_{\varepsilon}^{1}\frac{e^{its}[e^{i(x\circ y)s^\beta}-1]}{s}\,ds + \int_{1}^{R}\frac{e^{its}e^{i(x\circ y)s^\beta}}{s}\,ds\right\}dy.$$

$$\left|\int_{\varepsilon}^{1}\frac{e^{its}[e^{i(x\circ y)s^\beta}-1]}{s}\,ds\right| \leqslant C|x|\,|y|.$$

$$\int_{1}^{R}\frac{e^{its}e^{i(x\circ y)s^\beta}}{s}\,ds = \frac{1}{it}\int_{1}^{R}\left(\frac{d}{ds}e^{its}\right)\frac{e^{i(x\circ y)s^\beta}}{s}\,ds.$$

Integrating this last expression by parts we see that for (x,t) fixed, $t\neq 0$,

$$\left|\int_{1}^{R}\frac{e^{its}e^{i(x\circ y)s^\beta}}{s}\,ds\right| \leqslant C_{x,t}(1+|y|).$$

Since $(1+|y|)\,|\Omega(y)| \in L(E^n)$, Lebesgue's dominated convergence theorem applies and the second part of Lemma 1 follows.

THEOREM 1. *If* $f \in L^2(E^n \times (0, \infty))$ *and if*

$$\int_{E^n} |\Omega(y)| \left\{1 + |y| + \log \frac{1}{|x' \circ y'|} + |\log|y||\right\} dy \leqslant C$$

where C *is independent of* x', *then*

(i) $\|\tilde{f}_{\varepsilon,R}\|_2 \leqslant C\|f\|_2$;

(ii) *there exists* $\tilde{f}_\varepsilon \in L^2$ *such that* $\|\tilde{f}_{\varepsilon,R} - \tilde{f}_\varepsilon\|_2 \to 0$ *as* $R \to \infty$;

(iii) *there exists* $\tilde{f} \in L^2$ *such that* $\|\tilde{f}_\varepsilon - \tilde{f}\|_2 \to 0$ *as* $\varepsilon \to 0$.

Proof. Since $K_{\varepsilon,R} \in L(E^{n+1})$, we have $\hat{\tilde{f}}_{\varepsilon,R} = \hat{f}\hat{K}_{\varepsilon,R}$ and from Lemma 1 it follows that $\|\tilde{f}_{\varepsilon,R}\|_2 = \|\hat{\tilde{f}}_{\varepsilon,R}\|_2 \leqslant C\|f\|_2$.

From Lemma 1 and Lebesgue's dominated convergence theorem we have

$$\|\tilde{f}_{\varepsilon,R} - \tilde{f}_{\varepsilon,R'}\|_2 = \|\hat{\tilde{f}}_{\varepsilon,R} - \hat{\tilde{f}}_{\varepsilon,R'}\|_2 \to 0 \quad \text{as} \quad R, R' \to \infty.$$

Hence (ii) follows. Note that

$$\hat{\tilde{f}}_\varepsilon = \lim_{R\to\infty} \hat{K}_{\varepsilon,R}\hat{f}$$

almost everywhere. Therefore $\|\tilde{f}_\varepsilon - \tilde{f}_\delta\|_2 = \|\hat{\tilde{f}}_\varepsilon - \hat{\tilde{f}}_\delta\|_2 \to 0$, again using Lemma 1. So (iii) follows.

REMARK. *For almost every* $(x, t) \in E^n \times (0, \infty)$,

$$\tilde{f}_\varepsilon(x, t) = \int_0^{t-\varepsilon} \int_{E^n} K(x-y, t-s) f(y, s)\, dy ds.$$

Proof. There exists a sequence $\tilde{f}_{\varepsilon,R_i} \to \tilde{f}_\varepsilon$ pointwise almost everywhere as $R_i \to \infty$. Observe that for t fixed and $R_i > t - \varepsilon$,

$$\int_0^\infty \int_{E^n} K_{\varepsilon,R_i}(x-y, t-s) f(y, s)\, dy ds = \int_0^{t-\varepsilon} \int_{E^n} K(x-y, t-s) f(y, s)\, dy ds.$$

1.2. Weak type (1,1). In this section we will use the notation $|F|$ to denote the Lebesgue measure of the measurable set F.

Definition. Suppose T is a linear operation from $L^1(E^n \times (0, \infty))$ into the set of measurable functions on $E^n \times (0, \infty)$. T is said to be of *weak type* (1,1) if for every number $M > 0$

$$|\{(x, t) \mid |Tf| > M\}| \leqslant A/M \|f\|_1,$$

where A is some positive constant independent of f.

REMARK. *For* $f \in L^1(E^n \times (0, \infty))$, $\tilde{f}_\varepsilon$ *is finite for almost every* (x, t).

Proof. It will be sufficient to show that $\tilde{f}_\varepsilon(x, t)$ is finite for almost every $(x, t) \in E^n \times (0, R)$, $0 < R < \infty$, and we may assume $f \geqslant 0$. We have

$$\int_0^R \int_{E^n} |\tilde{f}_\varepsilon|\, dx dt \leqslant \int_0^\infty \int_{E^n} \left\{\int_0^R \int_{E^n} |K_\varepsilon(x-y, t-s)|\, dx dt\right\} f(y, s)\, dy ds.$$

Since

$$\int_0^R \int_{E^n} |K_\varepsilon(x-y, t-s)|\, dx dt = \int_0^R \int_{E^n} |K_\varepsilon(x, t-s)|\, dx dt$$

is 0 for $s > R$, we have

$$\int_0^R \int_{E^n} |\tilde{f}_\varepsilon(x, t)|\, dx dt \leqslant C_{\varepsilon,R}\|f\|_1.$$

For the next theorem we introduce the following set: $\gamma > 0$, $\alpha = 1/\beta$,

$$W_\gamma(y, s) = \{(x, t) \mid t > 2|s|, t > |s| + \gamma|y|^\alpha\}.$$

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *assume that*

$$\int_{W_\gamma(y,s)} |K(x-y, t-s) - K(x, t)|\, dx dt \leqslant C_\gamma,$$

where C_γ *is a constant depending only on* γ *and* K *and not on the point* (y, s). *Then the operation* $f \to \tilde{f}_\varepsilon$ *is of weak type* (1,1). *More specifically, for any* $M > 0$,

$$|\{(x, t) \mid |\tilde{f}_\varepsilon(x, t)| > M\}| \leqslant A/M \|f\|_1$$

with A *independent of* ε.

Proof. We may assume $f \geqslant 0$. Given any $M > 0$, there is a sequence of non-overlapping rectangles, $I_k \subset E^n \times (0, \infty)$, of the form $I_k^{(n)} \times I_k^{(1)}$ where $I_k^{(n)}$ is an n-dimensional cube, and $I_k^{(1)}$ is a 1-dimensional interval, satisfying

(a) $|I_k^{(n)}|^{1/n}/|I_k^{(1)}|^\beta$ is contained between two positive absolute constants independent of k and M;

(b) $M \leqslant \dfrac{1}{|I_k|}\displaystyle\int_{I_k} f \leqslant CM$, C absolute constant;

(c) $f \leqslant M$ almost everywhere in the complement of $D_M = \bigcup_{k=1}^\infty I_k$.

(See [4], p. 224, and [5].)

Set

$$h = \begin{cases} f(x) & \text{in } D_M^{*\prime} = \text{complement of } D_M, \\ \dfrac{1}{|I_k|}\displaystyle\int_{I_k} f & \text{in each } I_k. \end{cases}$$

Claim. $h \in L^1 \cap L^2$ and $\|h\|_2 \leqslant C\sqrt{M\|f\|_1}$. It is clear that $h \in L^1(E^n \times \times (0, \infty))$.

$$\int_0^\infty \int_{E^n} |h(x,t)|^2 dxdt = \int_{D_{M'}} |h|^2 dxdt + \int_{D_M} |h|^2 dxdt.$$

Using (b), (c), and definition of h, we have

$$\int_0^\infty \int_{E^n} |h|^2 dxdt \leqslant M \int_{D_{M'}} |f|\, dxdt + \sum_{k=1}^\infty \int_{I_k} \left\{\frac{1}{|I_k|}\int_{I_k} f\right\}^2 dxdt$$

$$\leqslant M\|f\|_1 + CM \sum_{k=1}^\infty \int_{I_k} \left\{\frac{1}{|I_k|}\int_{I_k} f\right\} dxdt$$

$$\leqslant CM\|f\|_1.$$

Now set $g = f - h$. g satisfies the following:
1) $g = 0$ in $D_{M'}$, 2) $\int_{I_k} g\, dxdt = 0$, 3) $\|g\|_1 \leqslant 2\|f\|_1$.
1) and 2) are immediate; 3) follows since

$$\|g\|_1 \leqslant \int_{D_M} f + \int_{D_M} h \leqslant \|f\|_1 + \sum_k \int_{I_k} \left\{\frac{1}{|I_k|}\int_{I_k} f\right\} dxdt.$$

Hence $\|g\|_1 \leqslant 2\|g\|_1$ and

$$|\{(x,t) |\, |\tilde{f}_\varepsilon| > M\}| \leqslant \left|\left\{(x,t) \,|\, |\tilde{g}_\varepsilon| > \frac{M}{2}\right\}\right| + \left|\left\{(x,t) \,|\, |\tilde{h}_\varepsilon| > \frac{M}{2}\right\}\right|.$$

Set $U = \{(x,t) |\, |\tilde{g}_\varepsilon| > M/2\}$, $V = \{(x,t) |\, |\tilde{h}_\varepsilon| > M/2\}$.
Using Theorem 1, we see that

$$|V| \frac{M^2}{4} \leqslant \|\tilde{h}_\varepsilon\|_2^2 \leqslant C\|h\|_2^2 \leqslant CM\|f\|_1.$$

Hence $|V| \leqslant A/M\|f\|_1$. We are left to show that $|U| \leqslant A/M\|f\|_1$. For this we first expand, concentrically, each side of I_k to a length of five times the original length. We denote the resulting interval by I_k^*.
Set

$$D_M^* = \bigcup_{k=1}^\infty I_k^*.$$

Clearly $D_M \subset D_M^*$ and

$$|D_M^*| \leqslant C|D_M| \leqslant C\sum_k \frac{1}{M}\int_{I_k} f \leqslant C/M\|f\|_1,$$

$$U = (U \cap D_M^*) \cup (U \cap D_M^{*\prime})$$

and

$$|U \cap D_M^{*\prime}| \leqslant 2/M \int_{D_M^{*\prime}} |\tilde{g}_\varepsilon|\, dxdt.$$

Therefore, we will be through once we show that

$$\int_{D_M^{*\prime}} |\tilde{g}_\varepsilon| \leqslant C\|f\|_1.$$

The remaining proof of Theorem 2 is devoted to this fact. We have

$$\tilde{g}_\varepsilon(x,t) = \sum_k \int_{I_k^{(1)}} \int_{I_k^{(n)}} K_\varepsilon(x-y, t-s) f(y,s)\, dyds,$$

$$\tilde{g}_\varepsilon(x,t) = \sum_k \int_{I_k^{(1)} \cap (0,t-\varepsilon)} \int_{I_k^{(n)}} K(x-y, t-s) g(y,s)\, dyds.$$

Set $I_k^{(1)} = (a_k, b_k)$.

Case 1. $(a_k, b_k) \cap (0, t-\varepsilon) = (a_k, b_k)$ and $t > b_k + (b_k - a_k)$.

Let (y_k, s_k) be the symmetric center of I_k. From property (b) of g, we have

$$\tilde{g}_\varepsilon(x,t) = \sum_k \int_{I_k^{(1)}} \int_{I_k^{(n)}} [K(x-y, t-s) - K(x-y_k, t-s_k)] g(y,s)\, dyds.$$

In this case we have $t - s_k > 2|s - s_k|$ and since $t > b_k + (b_k - a_k)$, $t - s_k > |s - s_k| + \gamma|y - y_k|^\alpha$, γ a positive constant. (Recall that $|I_k^{(1)}| \geqslant C|I_k^{(n)}|^{\alpha/n}$.)

Case 2. $(0, t-\varepsilon) \cap (a_k, b_k) \subsetneqq (a_k, b_k)$ and $t > b_k + (b_k - a_k)$.

In this case, I_k is entirely contained in the half-space,

$$\{(y,s) \mid s > t - 3\varepsilon\}.$$

Let $\delta(t)$ denote the characteristic function of the interval $(0, 3)$. Therefore

$$|K_\varepsilon(x-y, t-s)| \leqslant \frac{|\Omega(x-y/(t-s)^\beta)|}{(t-s)_\varepsilon^{n\beta+1}}\, \delta\left(\frac{t-s}{\varepsilon}\right)$$

where $1/r_\varepsilon$ is the function equal to $1/r$ for $r > \varepsilon$ and zero for $r \leqslant \varepsilon$.

Case 3. $a_k \leqslant t \leqslant b_k + (b_k - a_k)$.

For all s, $|t-s| \leqslant 2|I_k^{(1)}|$; but since (x,t) lies outside I_k^*, it is clear that for all $y \in I_k$, $|x - y| > C|I_k^{(n)}|^{1/n}$, C an absolute constant.

From cases 1, 2, 3 we see that

$$\int\limits_{D_M^{*'}} |\tilde{g}_\varepsilon|\,dxdt \leqslant \sum_k \int\limits_{I_k} |g(y,s)|\,dyds \times$$

$$\times \Big[\int\limits_{\substack{t-s_k > 2|s-s_k| \\ t-s_k > |s-s_k| + \gamma|y-y_k|^\alpha}} |K(x-y, t-s) - K(x-y_k, t-s_k)\,dxdt \Big] +$$

$$+ \sum_k \int\limits_{I_k} |g(y,s)|\,dyds \Big[\int\limits_{t-s \geqslant \varepsilon} \delta\Big(\frac{t-s}{\varepsilon}\Big) \int\limits_{E^n} \frac{|\Omega(x-y/(t-s)^\beta)|}{(t-s)^{n\beta+1}}\,dxdt \Big] +$$

$$+ \sum_k \int\limits_{I_k} |g(y,s)|\,dyds \Big[\int\limits_0^{2|I_k^{(1)}|} \int\limits_{|x-y| > C|I_k^{(n)}|^{1/n}} \frac{|\Omega(x-y/(t-s)^\beta)|}{(t-s)^{n\beta+1}}\,dxdt \Big].$$

Using the hypothesis of the theorem with y replaced by $y-y_k$ and s replaced by $s-s_k$, it follows that the first sum is majorized by a constant times

$$\sum_k \int\limits_{I_k} |g|\,dyds \leqslant 2\|f\|_1.$$

The same conclusion holds for the second sum once we note that

$$\int\limits_{t-s \geqslant \varepsilon} \delta(t-s/\varepsilon) \int\limits_{E^n} \frac{|\Omega(x-y/(t-s)^\beta)|}{(t-s)^{n\beta+1}}\,dxdt = \int\limits_\varepsilon^{3\varepsilon} \frac{1}{t} \int\limits_{E^n} |\Omega(x)|\,dxdt$$

$$= C \int\limits_{E^n} |\Omega(x)|\,dx.$$

For the third sum we have

$$\int\limits_0^{2|I_k^{(1)}|} \int\limits_{|x-y| > C|I_k^{(n)}|^{1/n}} \frac{|\Omega(x-y/(t-s)^\beta)|}{(t-s)^{n\beta+1}}\,dxdt = \int\limits_0^{2|I_k^{(1)}|} \frac{1}{t} \int\limits_{|x| > \frac{C|I_k^{(n)}|^{1/n}}{t^\beta}} |\Omega(x)|\,dxdt.$$

Since

$$\int\limits_{E^n} |x|\,|\Omega(x)|\,dx < \infty,$$

the last integral is majorized by a constant times

$$\frac{1}{C|I_k^{(n)}|^{1/n}} \int\limits_0^{2|I_k^{(1)}|} t^{\beta-1}\,dt = C\frac{|I_k^{(1)}|^\beta}{|I_k^{(n)}|^{1/n}} \leqslant C.$$

We conclude that

$$\int\limits_{D_{M'}^*} |g| \leqslant C\|f\|_1.$$

1.3. L^p-convergence. $1 < p < \infty$. Before proceeding to the L^p-convergence of $\tilde{f}_\varepsilon$, we list the conditions on $\Omega(x) = K(x, 1)$ to be assumed in this section.

(a) $\int\limits_{E^n} \Omega(x)\,dx = 0$.

(b) $\int\limits_{E^n} \Big(1 + |y| + \Big|\log\frac{1}{|x' \circ y|}\Big|\Big)|\Omega(y)|\,dy < C$, independent of $x' = x/|x|$.

(c) $\int\limits_{W_\gamma(y,s)} |K(x-y, t-s) - K(x,t)|\,dxdt \leqslant C_\gamma$, depending only on γ, not on (y,s). (Recall $W_\gamma(y,s) = \{(x,t) \mid t > 2|s|,\ t > |s| + \gamma|y|^\alpha\}$.)

(d) $|\Omega(x)| \leqslant C(1+|x|)^{-n-\delta}$, $\delta > 1$.

THEOREM 3. *Under the above conditions*

1) $\|\tilde{f}_\varepsilon\|_p \leqslant A\|f\|_p$, $1 < p \leqslant 2$ *with A independent of ε and f.*

2) *There is a function, $\tilde{f}$, belonging to $L^p(E^n \times (0,\infty))$ such that*

$$\|\tilde{f}_\varepsilon - \tilde{f}\|_p \to 0, \quad 1 < p \leqslant 2.$$

Proof. (1) follows immediately from theorems 1 and 2 and the interpolation theorem of Marcinkiewicz [8]. The proof of the second part will be accomplished in a series of four easily proved remarks.

$C_0^\infty(E^n \times (0,\infty))$ will denote the class of function $f(x,t) \in C^\infty(E^n \times (0,\infty))$ and with compact support contained in $E^n \times (0,\infty)$.

REMARK 1. *If $f \in C_0^\infty(E^n \times (0,\infty))$, there is a number R, depending only on f, such that $\tilde{f}_\varepsilon(x,t) = \tilde{f}_\delta(x,t)$ for $t > R$ and $\varepsilon, \delta < 1$.*

Proof. We need only observe that if $\{(x,t) : t > R-1\}$ is contained in the complement of the support of f, then

$$\tilde{f}_\varepsilon(x,t) = \int\limits_\varepsilon^{R-1} \int\limits_{E^n} K(x-y, t-s) f(y,s)\,dyds.$$

REMARK 2. *If $f \in C_0^\infty(E^n \times (0,\infty))$, then*

(i) *$\tilde{f}_\varepsilon(x,t)$ converges pointwise as $\varepsilon \to 0$.*

(ii) *$|\tilde{f}_\varepsilon(x,t)| \leqslant M$, depending only on f.*

(iii) *$|\tilde{f}_\varepsilon(x,t) - \tilde{f}_\delta(x,t)| \leqslant C|x|^{-n}$ for $|x|$ sufficiently large, depending only on f, and for $0 < \varepsilon$, $\delta < 1$.*

Proof. Using condition (a) on Ω,

$$\tilde{f}_\varepsilon(x,t) = \int_\varepsilon^t \int_{E^n} K(y,s)[f(x-y,t-s)-f(x,t)]\,dyds.$$

Since $|f(x-y,t-s)-f(x,t)| \leqslant C(|y|+|s|)$, we see that

$$|\tilde{f}_\varepsilon(x,t)| \leqslant C\int_0^R \int_{E^n} \frac{|\Omega(y/s^\beta)|}{s^{n\beta+1}}(|y|+|s|)\,dyds = M < \infty$$

(R as in first remark).

Assume that R above is so chosen that $f(y,s)=0$ if $s > R-1$ or if $|y| > R-1$. Now take x such that $|x|>2R$. Since $\tilde{f}_\varepsilon(x,t)-\tilde{f}_\delta(x,t)=0$ for $t>R$, $0<\varepsilon$, $\delta<1$, to show (iii) we may assume $t \leqslant R$. Hence

$$|\tilde{f}_\varepsilon(x,t)-\tilde{f}_\delta(x,t)| \leqslant C\int_0^R \frac{1}{s^{n\beta+1}} \int_{|y|\leqslant R} \left|\Omega\left(\frac{x-y}{s^\beta}\right)\right| dyds.$$

Using condition (d), and noting that $|y|/s^\beta \leqslant \frac{1}{2}|x|/s^\beta$, we have

$$|\tilde{f}_\varepsilon(x,t)-\tilde{f}_\delta(x,t)| \leqslant C_R\int_0^R (1+|x|/2s^\beta)^{-n-\delta}s^{-n\beta-1}\,ds$$

$$\leqslant C_R|x|^{-n} \quad \text{for} \quad |x|>2R.$$

Remark 3. *If* $f\in C_0^\infty(E^n\times(0,\infty))$, *then* $\|\tilde{f}_\varepsilon-\tilde{f}_\delta\|_p\to 0$ *as* $\varepsilon,\delta\to 0$, $1<p<\infty$.

Proof. From (1) and (2) it follows that

$$\int_0^\infty \int_{E^n} |\tilde{f}_\varepsilon-\tilde{f}_\delta|^p\,dxdt = \int_0^R \int_{E^n} |\tilde{f}_\varepsilon-\tilde{f}_\delta|^p\,dxdt, \quad 0<\varepsilon,\ \delta<1,$$

and the last expression tends to 0 as $\varepsilon,\delta\to 0$ since

$$|\tilde{f}_\varepsilon(x,t)-\tilde{f}_\delta(x,t)|^p \leqslant C(1+|x|)^{-np}.$$

Remark 4. *For* $f\in L^p(E^n\times(0,\infty))$, $1<p\leqslant 2$, $\|\tilde{f}_\varepsilon-\tilde{f}_\delta\|_p\to 0$, *as* $\varepsilon,\delta\to 0$.

Proof. Let $\{f_n\}$ be a sequence of functions converging to f in L^p and such that $f_n\in C_0^\infty(E^n\times(0,\infty))$. We have

$$\|\tilde{f}_\varepsilon-\tilde{f}_\delta\|_p \leqslant \|\tilde{f}_\varepsilon-\tilde{f}_{n,\varepsilon}\|_p+\|\tilde{f}_\delta-\tilde{f}_{n,\delta}\|_p+\|\tilde{f}_{n,\varepsilon}-\tilde{f}_{n,\delta}\|_p.$$

By the first part of Theorem 3, $\|\tilde{f}_\varepsilon-\tilde{f}_{n,\varepsilon}\|_p \leqslant A\|f-f_n\|_p$ and $\|\tilde{f}_\delta-\tilde{f}_{n,\delta}\|_p \leqslant A\|f-f_n\|_p$. Hence the first two terms tend to zero as $n\to\infty$, uniformly in ε,δ. By Remark 3, for n fixed, $\|\tilde{f}_{n,\varepsilon}-\tilde{f}_{n,\delta}\|_p\to 0$ as $\varepsilon,\delta\to 0$. This completes Theorem 3.

We now want to consider the case $p>2$. The usual proof (see [1]) for this case uses the "inverse of Hölder's inequality" and the fact that the conjugate operation is of the same form as the original operation. Here the situation is different, but only slightly. For let $f\in L^p(E^n\times(0,\infty))$, $p>2$, and let q be the conjugate of p, that is $1/p+1/q=1$. Then the "inverse of Hölder's inequality" says that

$$\|\tilde{f}_\varepsilon\|_p = \sup_{\substack{g\in C_0^\infty(E^n\times(0,\infty))\\ \|g\|_q\leqslant 1}} \left|\int_0^\infty \int_{E^n} \tilde{f}_\varepsilon(x,t)g(x,t)\,dxdt\right|.$$

Now

$$\int_{E^{n+1}} \tilde{f}_\varepsilon g\,dxdt = \int_{E^{n+1}} (K_\varepsilon * f)g\,dxdt$$

$$= \int_{E^{n+1}} f_\varepsilon(y,s)\Big[\int_{E^{n+1}} K_\varepsilon(x-y,t-s)g(x,t)\,dxdt\Big]dyds.$$

Looking close at the inner integral we see that we want to consider now kernels $K^*(x,t)$ satisfying:

(i) $K^*(x,t)=0$ for $t>0$;

(ii) $K^*(\gamma x,\gamma^a t)=\gamma^{-n-a}K(x,t)$, $\gamma>0$;

(iii) $\int_{E^n} K^*(x,-1)\,dx=0$.

If we set

$$K_\varepsilon^*(x,t) = \begin{cases} K^*(x,t) & \text{for} \quad t<-\varepsilon<0, \\ 0 & \text{for} \quad t\geqslant -\varepsilon, \end{cases}$$

then

$$\int_{E^{n+1}} K_\varepsilon(x-y,t-s)g(x,t)\,dxdt = (g*K_\varepsilon^*)(y,s),$$

where $K^*(x,t)=K(-x,-t)$.

Set

$$g_\varepsilon^*(x,t) = (K_\varepsilon^* * g)(x,t) = \int_{s+\varepsilon}^\infty \int_{E^n} K^*(x-y,t-s)g(y,s)\,dyds$$

and let $\Omega^*(x)=K(x,-1)$. We will consider the mapping $g\to\tilde{g}_\varepsilon$ as an operation from $L^p(E^n\times(0,\infty))\to L^p(E^{n+1})$.

Proposition 1. *If* $g\in L^2(E^n\times(0,\infty))$ *and if*

$$\int_{E^n} |\Omega^*(y)|\left(1+|y|+\left|\log\frac{1}{|x'\circ y|}\right|\right)dy < C,$$

independent of $x'=x/|x|$, *then* $\|g_\varepsilon^*\|_2 \leqslant AC\|g\|_2$, A *an absolute constant independent of* ε, g, *and* K^*.

Proof. Set

$$K^*_{\varepsilon,R}(x,t) = \begin{cases} K^*(x,t) & \text{if } \quad -R \leqslant t \leqslant -\varepsilon, \\ 0 & \text{otherwise}. \end{cases}$$

If we also let $K_{\varepsilon,R}(x,t) = K^*_{\varepsilon,R}(-x,-t)$, then Proposition 1 easily follows from Lemma 1 in Section (1.1) noting that $\hat{K}^*_{\varepsilon,R}(x,t) = \hat{K}_{\varepsilon,R}(-x,-t)$.

Now assume $\Omega^*(x)$ satisfies conditions (a), (b), (d). We also assume (c') $\int\limits_{W^*_\gamma(y,s)} |K^*(x-y,t-s)-K^*(x,t)|\,dxdt < C_\gamma$, depending on γ and K^*, but not on (y,s), where $W^*_\gamma(y,s) = \{(x,t): t < -2|s|, t < -(|s|+\gamma|y|^a)\}$.

PROPOSITION 2. *If $g \in L^1(E^n \times (0,\infty))$, then the operation $g \to g^*_\varepsilon$ is of weak type* (1,1) *with constant independent of ε and g.*

Proof. We again decompose $E^n \times (0,\infty)$ into a sequence of non-overlapping rectangles, $I_k = I_k^{(n)} \times I_k^{(1)}$, satisfying the conditions stated in Theorem 2. The proof now follows exactly as before with cases 1, 2, 3 respectively replaced by

1) $t+\varepsilon \leqslant a_k$, $t < a_k-(b_k-a_k)$,
2) $t+\varepsilon > a_k$ and $t < a_k-(b_k-a_k)$,
3) $a_k-(b_k-a_k) \leqslant t \leqslant b_k$.

PROPOSITION 3. *If $\Omega^*(x)$ satisfies conditions* (a), (b), (c'), (d), *then for $1 < p \leqslant 2$, $\|g^*_\varepsilon\|_p \leqslant A\|g\|_p$, A independent of ε, g.*

Proof. This follows from propositions 1 and 2 and the Marcinkiewicz interpolation theorem.

THEOREM 3'. *If $\Omega(x)$ satisfies conditions* (a)-(d), *then Theorem* 3 *holds for $p > 2$.*

Proof. We have

$$\|\tilde{f}_\varepsilon\|_p = \sup_{\substack{g \in C_0^\infty(E^n \times (0,\infty)) \\ \|g\|_q \leqslant 1}} \Big| \int\limits_{E^{n+1}} \tilde{f}_\varepsilon(x,t) g(x,t)\,dxdt \Big|,$$

$$\int\limits_{E^{n+1}} \tilde{f}_\varepsilon(x,t) g(x,t)\,dxdt = \int\limits_{E^{n+1}} f(x,t) g^*_\varepsilon(x,t)\,dxdt,$$

where

$$g^*_\varepsilon(x,t) = \int\limits_{E^{n+1}} K^*_\varepsilon(x-y,t-s) g(y,s)\,dyds, \quad \text{with} \quad K^*(x,t) = K(-x,-t).$$

Since

$$\int\limits_{W^*_\gamma(y,s)} |K^*(x-y,t-s)-K^*(x,t)|\,dxdt = \int\limits_{W_\gamma(-y,-s)} |K(x+y,t+s)-K(x,t)|\,dxdt \leqslant C_\gamma$$

by (c), condition (c') holds and hence

$$\Big| \int\limits_{E^{n+1}} \tilde{f}_\varepsilon(x,t) g(x,t)\,dxdt \Big| \leqslant \|f\|_p \|g^*_\varepsilon(x,t)\|_q \leqslant A\|f\|_p.$$

Since remarks 1-3 did not depend on the fact that $1 < p \leqslant 2$, we see that for $p > 2$, $\|\tilde{f}_\varepsilon - \tilde{f}_\delta\|_p \to 0$ as $\varepsilon, \delta \to 0$.

Observation. Theorems 2, 3, and 3', and the interpolation theorem of Marcinkiewicz show that for all p, $1 < p < \infty$,

$$\|\tilde{f}_\varepsilon\|_p \leqslant C(A+B_\gamma)\|f\|_p,$$

where C and γ are absolute constants depending only on p and n, and A and B_γ satisfy the conditions

$$\int\limits_{E^n} |\Omega(y)| \Big(1+|y|+\Big|\log\frac{1}{|x' \circ y|}\Big|\Big)\,dy \leqslant A,$$

$$\int\limits_{w_\gamma(y,s)} |K(x-y,t-s)-K(x,t)|\,dxdt \leqslant B_\gamma.$$

1.4. A special case involving Hermite functions. In this section we shall derive bounds for the norms of a special sequence of operators, which will be useful in Chapter II.

Assume $r \in (-\infty,\infty)$ and denote by $H_j(r)$ $(j = 0,1,2,\ldots)$ the Hermite polynomial of degree j, that is, the polynomials, $H_j(r)$, are defined so that

$$\int\limits_{-\infty}^{\infty} H_j(r) H_m(r) e^{-r^2}\,dr = \sqrt{\pi}\, 2^j j!\, \delta_{m,j},$$

where $\delta_{m,j}$ denotes the Kronecker delta. We state here, without proof, properties of $H_j(r)$ which will be used in this section and in Chapter II (see [7]).

1) $e^{-r^2} H_j(r) = (-1)^j (d/dx)^j e^{-r^2}$.
2) $H_{2j}(0) = (-1)^j (2j)!/j!$.
3) $|H_j(r) e^{-r^2/2}| \leqslant C 2^{j/2} (j!)^{1/2}$.
4) $(d/dr) H_{j+1}(r) = 2(j+1) H_j(r)$.

The function $H_j(r) e^{-r^2/2}$ is called an *Hermite function*, and it is well known that the sequence of Hermite functions form a complete, orthogonal system over $(-\infty,\infty)$ [7]. So the sequence of functions

$$H_{k_1}(x_1) H_{k_2}(x_2) \ldots H_{k_n}(x_n) e^{-|x|^2/2}$$

(k_i a non-negative integer) form a complete orthogonal system over E^n.

We will now adopt the following notation: $k = (k_1, \ldots, k_n)$, k_i non-negative integer; $k! = k_1!\, k_2! \ldots k_n!$; $|k| = \sum k_i$; $x^k = x_1^{k_1} \ldots x_n^{k_n}$.

Set

$$H_k(x)e^{-|x|^2/2} = \prod_{i=1}^{n} H_{k_i}(x_i)e^{-x_i^2/2}, \quad \text{and} \quad \Omega_k(x) = \widehat{[H_k(y)-H_k(0)]e^{-|y|^2}}(x).$$

Note that

$$\int_{E^n} \Omega_k(x)\,dx = 0.$$

Finally, set $T_k(x,t) = \Omega_k(x/t^\beta)/t^{n\beta+1}$.

PROPOSITION 1. *We have*

$$\int_{E^n} |\Omega_k(y)|\left(1+|y|+\left|\log\frac{1}{|x'\circ y|}\right|\right)dy \leqslant C2^{|k|/2}(k!)^{1/2}\prod_{k_i>0}k_i^{(n+1)/2}.$$

Proof. Using formula 1 for Hermite polynomials we see that

$$\Omega_k(y) = C(i)^{|k|}y^k e^{-|y|^2/4} - H_k(0)e^{-|y|^2/4}.$$

From formula 2,

$$|H_k(0)| \leqslant C2^{|k|}\prod_{k_i>0}k_i\Gamma(k_i/2).$$

Hence

$$\int_{E^n}|\Omega_k(y)|\,dy \leqslant C\prod_{k_i>0}\int_{-\infty}^{\infty}|y_i|^{k_i}e^{-y_i^2/4}\,dy + C2^{|k|}\prod_{k_i>0}k_i\Gamma(k_i/2)$$
$$\leqslant C2^{|k|}\Big(\prod_{k_i>0}\Gamma(k_i/2) + \prod_{k_i>0}k_i\Gamma(k_i/2)\Big).$$

Now

$$\int_{E^n}|y^k|e^{-|y|^2/4}|y|\,dy \leqslant \sum_{j=1}^{n}\int_{\{y:\,|y_j|\geqslant|y_m|,\,1\leqslant m\leqslant n\}}|y^k|e^{-|y|^2/4}|y|\,dy$$
$$\leqslant C\sum_{j=1}^{n}\int_{E^n}|y^k|e^{-|y|^2/4}|y_j|\,dy \leqslant C2^{|k|}\prod_{k_i>0}k_i\Gamma(k_i/2).$$

Therefore,

$$\int_{E^n}|\Omega_k(y)|\,|y|\,dy \leqslant C2^{|k|/2}(k!)^{1/2}\prod_{k_i>0}k_i,$$

$$\int_{E^n}|\Omega_k(y)|\,\big|\log|y|\big|\,dy \leqslant C\int_{|y|<1}\big|\log|y|\big|\,dy\,\big(1+|H_k(0)|\big) + C\int_{|y|>1}|y|\,|\Omega_k(y)|\,dy$$

and again,

$$\int_{E^n}|\Omega_k(y)|\,\big|\log|y|\big|\,dy \leqslant C2^{|k|}\prod_{k_i>0}k_i\Gamma(k_i/2) \leqslant C2^{|k|/2}(k!)^{1/2}\prod_{k_i>0}k_i.$$

Finally we want to consider

$$\int_{E^n}|\Omega_k(y)|\log\frac{1}{|x'\circ y'|}\,dy,$$

$$\int_{E^n}|\Omega_k(y)|\log\frac{1}{|x'\circ y'|}\,dy \leqslant C\int_{E^n}|y^k|e^{-|y|^2/4}\log\frac{1}{|x'\circ y'|}\,dy + C|H_k(0)|.$$

In the last integral, multiply and divide the integrand by $(1+|y|^{(n+1)/2})$ and apply Schwartz's inequality to the functions

$$|y^k|e^{-|y|^2/4}(1+|y|^{(n+1)/2}) \quad \text{and} \quad \left(\log\frac{1}{|x'\circ y'|}\right)(1+|y|^{(n+1)/2})^{-1}.$$

Then this integral becomes majorized by

$$C\Big(\int_{E^n}|y^{2k}|e^{-|y|^2/2}(1+|y|^{n+1})^{1/2},$$

C independent of x'. We have

$$\int_{E^n}|y^{2k}|e^{-|y|^2/2}\,dy \leqslant C\prod_{k_i>0}\int_{-\infty}^{\infty}|y_i|^{2k_i}e^{-y_i^2/2}\,dy_i \leqslant C2^{|k|}\prod_{k_i>0}\Gamma(k_i)k_i,$$

$$\int_{E^n}|y^{2k}|e^{-|y|^2/2}|y|^{n+1}\,dy \leqslant C\sum_{j=1}^{n}\int_{E^n}|y^{2k}|e^{-|y|^2/2}|y_j|^{n+1}\,dy.$$

Hence

$$\int_{E^n}|y^{2k}|e^{-|y|^2/2}|y|^{n+1}\,dy \leqslant C2^{|k|}\prod_{k_i>0}\Gamma(k_i+n)(k_i+n)$$
$$\leqslant C2^{|k|}\prod_{k_i>0}k_i^{n+1}\Gamma(k_i).$$

We conclude that

$$\int_{E^n}|\Omega_k(y)|\log\frac{1}{|x'\circ y'|}\,dy \leqslant C2^{|k|/2}\prod_{k_i>0}k_i^{(n+1)/2}\Gamma(k_i)^{1/2}.$$

Collecting the above results, we see that Proposition 1 follows.

PROPOSITION 2.

$$\int_{W_\gamma(y,s)}|T_k(x-y,t-s)-T_k(x,t)|\,dxdt \leqslant C2^{|k|/2}(k!)^{1/2}\prod_{k_i>0}k_i^2.$$

Proof. Set

$$P = \int_{W_\gamma(y,s)} |T_k(x-y,t-s)-T_k(x,t)|\,dx\,dt.$$

We get

$$P \leqslant \int_{t>|s|+\gamma|y|^\alpha} |T_k(x-y,t-s)-T_k(x,t-s)|\,dx\,dt+$$

$$+\int_{t>2|s|} |T_k(x,t-s)-T_k(x,t)|\,dx\,dt,$$

$$P \leqslant \mathrm{I}+\mathrm{II}.$$

Case I.

(1) $T_k(x-y,t-s)-T_k(x,t-s)$

$$=\left[\prod_{i=1}^{n} C_i\left(\frac{x_i-y_i}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{|x_i-y_i|^2}{(t-s)^{2\beta}}\right\}\right](t-s)^{-n\beta-1}-$$

$$-\left[\prod_{i=1}^{n} C_i\left(\frac{x_i}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\right\}\right](t-s)^{-n\beta-1}+$$

$$+(t-s)^{-n\beta-1}H_k(0)\left[\exp\left\{-\frac{1}{4}\frac{|x-y|^2}{(t-s)^{2\beta}}\right\}-\exp\left\{-\frac{1}{4}\frac{|x|^2}{(t-s)^{2\beta}}\right\}\right].$$

$$|H_k(0)|\int_{W_\gamma(y,s)}(t-s)^{-n\beta-1}\left|\exp\left\{-\frac{1}{4}\frac{|x-y|^2}{(t-s)^{2\beta}}\right\}-\exp\left\{-\frac{1}{4}\frac{|x|^2}{(t-s)^{2\beta}}\right\}\right|\,dx\,dt$$

$$\leqslant |H_k(0)|\int_{t>|s|+\gamma|y|^\alpha}(t-s)^{-1}\int_{E^n}\left|\exp\left\{-\frac{1}{4}\left|x-\frac{y}{(t-s)^\beta}\right|^2\right\}-\exp\left\{-\frac{1}{4}|x|^2\right\}\right|\,dx\,dt$$

$$\leqslant |H_k(0)|\int_{t>\gamma|y|^\alpha}t^{-1}\int_{E^n}\left|\exp\left\{-\frac{1}{4}\left|x-\frac{y}{t^\beta}\right|^2\right\}-\exp\left\{-\frac{1}{4}|x|^2\right\}\right|\,dx\,dt.$$

Applying the mean-value theorem to the function $e^{-|x|^2/4}$ and noting that $|y|/t^\beta \leqslant C_\gamma$, we see that the last integral is bounded by

$$C|H_k(0)|\,|y|\int_{t>\gamma|y|^\alpha}t^{-1-\beta}\int_{E^n}\exp\left\{-\frac{1}{4}(|x|-C_\gamma)^2\right\}dx\,dt \leqslant C_\gamma|H_k(0)|.$$

Now

$$(2)\quad \int_{t>|s|+\gamma|y|^\alpha}\int_{E^n}(t-s)^{-n\beta-1}\left|\prod_{i=1}^{n}C_i\left(\frac{(x_i-y_i)}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{|x_i-y_i|^2}{(t-s)^{2\beta}}\right\}-\right.$$

$$\left.-\prod_{i=1}^{n}C_i\left(\frac{x_i}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\right\}\right|\,dx\,dt$$

$$=\int_{t>|s|+\gamma|y|^\alpha}(t-s)^{-1}\int_{E^n}\left|\prod_{i=1}^{n}C_i\left(x_i-\frac{y_i}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\left|x_i-\frac{y_i}{(t-s)^\beta}\right|^2\right\}-\right.$$

$$\left.-\prod_{i=1}^{n}C_i x_i^{k_i}e^{-x_i^2/4}\right|\,dx\,dt.$$

Claim. *(2) is majorized by* $2^{|k|/2}(k!)^{1/2}\prod_{k_i>0}k_i^2$.

Proof. We will use induction on the dimension, n.

Case $n=1$. We want to consider

$$\int_{t>|s|+\gamma|y|^\alpha}(t-s)^{-1}\int_{-\infty}^{+\infty}\left|\left(x-\frac{y}{(t-s)^\beta}\right)^k\exp\left\{-\frac{1}{4}\left|x-\frac{y}{(t-s)^\beta}\right|^2\right\}-\right.$$

$$\left.-x^k\exp\left\{-\frac{1}{4}x^2\right\}\right|\,dx,$$

where $x, y\in(-\infty,+\infty)$ and k is a non-negative integer. Applying the mean-value theorem to this integral, we obtain the following majorization:

$$C|y|\int_{t>\gamma|y|^\alpha}t^{-1-\beta}\int_{-\infty}^{+\infty}\left|x-\frac{\theta y}{t^\beta}\right|^{k+1}\exp\left\{-\frac{1}{4}\left|x-\frac{\theta y}{t^\beta}\right|^2\right\}dx\,dt+$$

$$+Ck|y|\int_{t>\gamma|y|^\alpha}t^{-1-\beta}\int_{-\infty}^{+\infty}\left|x-\frac{\theta y}{t^\beta}\right|^{k-1}\exp\left\{-\frac{1}{4}\left|x-\frac{\theta y}{t^\beta}\right|^2\right\}dx\,dt.$$

Here $\theta=\theta(x,y,t)$ satisfies $|\theta|\leqslant 1$. Now set $A=|y|/t^\beta\leqslant C_\gamma$. We will now show that for any non-negative integer m

$$\int_{-\infty}^{+\infty}\left|x-\frac{\theta y}{t^\beta}\right|^m\exp\left\{-\frac{1}{4}\left|x-\frac{\theta y}{t^\beta}\right|^2\right\}dx\leqslant Cm2^m\Gamma\left(\frac{m}{2}\right).$$

For $r\in(0,\infty)$ the function $r^m e^{-r^2/4}$ is increasing for $r<\sqrt{2m}$ and decreasing for $r>\sqrt{2m}$. Thus

$$\int_{-\infty}^{\infty}\left|x-\frac{\theta y}{t^{\beta}}\right|^{m}\exp\left\{-\frac{1}{4}\left|x-\frac{\theta y}{t^{\beta}}\right|^{2}\right\}dx=\int\limits_{|x-\theta y/t^{\beta}|\leqslant\sqrt{2m}}+\int\limits_{\sqrt{2m}<|x-\theta y/t^{\beta}|\leqslant\sqrt{2m}+2A}+$$

$$+\int\limits_{|x-\theta y/t^{\beta}|>\sqrt{2m}+2A}\left|x-\frac{\theta y}{t^{\beta}}\right|^{m}\exp\left\{-\frac{1}{4}\left|x-\frac{\theta y}{t^{\beta}}\right|^{2}\right\}dx=B_1+B_2+B_3.$$

$B_1\leqslant C2^{m/2}m^{m/2}e^{-m/2}m^{1/2}$. The same inequality holds for B_2. Since $r^m e^{-r^2/4}$ is decreasing for $r>\sqrt{2m}$ and since $|x|-A>\sqrt{2m}$ in B_3, we have

$$B_3\leqslant\int\limits_{|x|>\sqrt{2m}+A}(|x|-A)^m\exp\left\{-\frac{1}{4}(|x|-A)^2\right\}$$

$$\leqslant 2\int_{\sqrt{2m}+A}^{\infty}(s-A)^m\exp\left\{-\frac{1}{4}(s-A)^2\right\}ds$$

$$\leqslant 2\int_{0}^{\infty}s^m\exp\left\{-\frac{1}{4}s^2\right\}ds.$$

Therefore $B_3\leqslant Cm2^m\Gamma(m/2)$.

This concludes the proof for $n=1$. Now assume $n>1$. We have

$$\int\limits_{t>|s|+\gamma|y|^{\alpha}}(t-s)^{-1}\int\limits_{E^n}\left|\prod_{i=1}^{n}\left(x_i-\frac{y_i}{(t-s)^{\beta}}\right)^{k_i}\exp\left\{-\frac{1}{4}\left|x_i-\frac{y_i}{(t-s)^{\beta}}\right|^2\right\}-\right.$$

$$\left.-\prod_{i=1}^{n}C_i x_i^{k_i}\exp\left\{-\frac{1}{4}x_i^2\right\}\right|dxdt$$

$$\leqslant C_n\int\limits_{t>|s|+\gamma|y|^{\alpha}}(t-s)^{-1}\int\limits_{E^n}\left|\left[\prod_{i=1}^{n-1}C_i\left(x_i-\frac{y_i}{(t-s)^{\beta}}\right)^{k_i}\exp\left\{-\frac{1}{4}\left|x_i-\frac{y_i}{(t-s)^{\beta}}\right|^2\right\}-\right.\right.$$

$$\left.\left.-\prod_{i=1}^{n-1}C_i x_i^{k_i}\exp\left\{-\frac{1}{4}x_i^2\right\}\right]\right|\left|x_n^{k_n}\exp\left\{-\frac{1}{4}x_n^2\right\}\right|+$$

$$+C_n\int\limits_{t>|s|+\gamma|y|^{\alpha}}(t-s)^{-1}\int\limits_{E^n}\left|\prod_{i=1}^{n}\left(x_i-\frac{y_i}{(t-s)^{\beta}}\right)^{k_i}\exp\left\{-\frac{1}{4}\left|x_i-\frac{y_i}{(t-s)^{\beta}}\right|^2\right\}\right|\times$$

$$\times\left|\left(x_n-\frac{y_n}{(t-s)^{\beta}}\right)^{k_n}\exp\left\{-\frac{1}{4}\left|x_n-\frac{y_n}{(t-s)^{\beta}}\right|^2\right\}-x_n^{k_n}\exp\left\{-\frac{1}{4}x_n^2\right\}\right|.$$

Applying the inductive assumption for the first integral and the case $n=1$ for the second integral, we see that our claim is proved.

Therefore we have proved that

$$\int\limits_{W_\gamma(y,s)}|T_k(x-y,t-s)-T_k(x,t-s)|\,dxdt\leqslant C2^{|k|/2}k!\prod_{k_i>0}k_i^2.$$

Finally we want to consider

$$\mathrm{II}=\int\limits_{W_\gamma(y,s)}|T_k(x,t-s)-T_k(x,t)|\,dxdt.$$

Assume first that $s>0$. Thus

$$T_k(x,t-s)-T_k(x,t)=(t-s)^{-n\beta-1}\prod_{i=1}^{n}C_i\left(\frac{x_i}{(t-s)^{\beta}}\right)^{k_i}\times\tag{3}$$

$$\times\exp\left\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\right\}-t^{-n\beta-1}\prod_{i=1}^{n}C_i\left(\frac{x_i}{t^{\beta}}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{t^{2\beta}}\right\}+$$

$$+H_k(0)\left[(t-s)^{-n\beta-1}\exp\left\{-\frac{1}{4}\frac{|x|^2}{(t-s)^{2\beta}}\right\}-t^{-n\beta-1}\exp\left\{-\frac{1}{4}\frac{|x|^2}{t^{2\beta}}\right\}\right].$$

$$|H_k(0)|\int\limits_{W_\gamma(y,s)}\left|(t-s)^{-n\beta-1}\exp\left\{-\frac{1}{4}\frac{|x|^2}{(t-s)^{2\beta}}\right\}-t^{-n\beta-1}\exp\left\{-\frac{1}{4}\frac{|x|^2}{t^{2\beta}}\right\}\right|dxdt$$

$$\leqslant C|H_k(0)|\,|s|\int\limits_{t>2|s|}\int\limits_{E^n}\left[\frac{1}{(t-\theta s)^{n\beta+2}}\exp\left\{-\frac{1}{4}\frac{|x|^2}{(t-\theta s)^{2\beta}}\right\}+\right.$$

$$\left.+\frac{|x|^2}{(t-\theta s)^{(n+2)\beta+2}}\exp\left\{-\frac{1}{4}\frac{|x|^2}{(t-\theta s)^{2\beta}}\right\}\right]dxdt,\qquad 0<\theta<1.$$

This last expression is majorized by

$$C|H_k(0)|\,|s|\left[\int\limits_{t>2|s|}\frac{1}{(t-s)^{n\beta+2}}\int\limits_{E^n}\exp\left\{-\frac{1}{4}\frac{|x|^2}{t^{2\beta}}\right\}+\right.$$

$$\left.+\int\limits_{t>2|s|}\frac{1}{(t-s)^{(n+2)\beta+2}}\int\limits_{E^n}|x|^2\exp\left\{-\frac{1}{4}\frac{|x|^2}{t^{2\beta}}\right\}\right].$$

Since $t>2|s|$, $t\leqslant 2(t-s)$, and therefore,

$$|H_k(0)|\int\limits_{W_\gamma(y,s)}\left|(t-s)^{-n\beta-1}\exp\left\{-\frac{1}{4}\frac{|x|^2}{(t-s)^{2\beta}}\right\}-\right.$$

$$\left.-t^{-n\beta-1}\exp\left\{-\frac{1}{4}\frac{|x|^2}{t^{2\beta}}\right\}\right|dxdt\leqslant C|H_k(0)|\,|s|\int\limits_{t>|s|}\frac{1}{t^2}ds\leqslant C|H_k(0)|.$$

Now the first two terms in (3) equals

$$(4)\qquad \prod_{i=1}^{n} C_i(t-s)^{-\beta-1/n}\left(\frac{x_i}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\right\}-$$

$$-\prod_{i=1}^{n} C_i t^{-\beta-1/n}\left(\frac{x_i}{t^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{t^{2\beta}}\right\}.$$

Claim. *For* $t>2|s|$,

$$\int_{E^m}\left|\prod_{i=1}^{m} C_i(t-s)^{-\beta-1/n}\left(\frac{x_i}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\right\}-\right.$$

$$\left.-\prod_{i=1}^{m} C_i t^{-\beta-1/n}\left(\frac{x_i}{t^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{t^{2\beta}}\right\}\right|dx\leqslant C2^{|k|/2}(k!)^{1/2}\Big(\prod_{k_i>0}k_i^2\Big)|s|\,(t-s)^{-m/n-1}.$$

Proof. We will use the induction on the dimension, m.

Case $m=1$ (k now denotes a non-negative integer). We have

$$\int_{-\infty}^{+\infty}\left|(t-s)^{-\beta-1/n}\left(\frac{x}{(t-s)^\beta}\right)^k\exp\left\{-\frac{1}{4}\frac{x^2}{(t-s)^{2\beta}}\right\}-t^{-\beta-1/n}\left(\frac{x}{t^\beta}\right)^k\times\right.$$

$$\left.\times\exp\left\{-\frac{1}{4}\frac{x^2}{t^{2\beta}}\right\}\right|dx\leqslant C|s|\int_{-\infty}^{+\infty}k(t-\theta s)^{-\beta-1/n-1}\left(\frac{|x|}{(t-\theta s)^\beta}\right)^k\times$$

$$\times\exp\left\{-\frac{1}{4}\frac{x^2}{(t-\theta s)^{2\beta}}\right\}(t-\theta s)^{-\beta-1/n-1}\left(\frac{|x|}{(t-\theta s)^\beta}\right)^{k+2}\exp\left\{-\frac{1}{4}\frac{x^2}{(t-\theta s)^{2\beta}}\right\}dx.$$

In general let us consider the integral

$$(5)\qquad \int_{-\infty}^{+\infty}(t-\theta s)^{-\beta-1/n-1}\left(\frac{|x|}{(t-\theta s)^\beta}\right)^j\exp\left\{-\frac{1}{4}\frac{x^2}{(t-\theta s)^{2\beta}}\right\}dx,\qquad t>2|s|.$$

Here j is a non-negative integer.

Let us recall that the function $r^j\exp\{-\frac{1}{4}r^2\}$ is increasing for $r\in(0,\sqrt{2j})$ and decreasing for $r>\sqrt{2j}$. Now the integral (5) is majorized by

$$(t-s)^{-\beta-1/n-1}\int_{-\infty}^{+\infty}\left|\frac{x}{(t-\theta s)^\beta}\right|^j\exp\left\{-\frac{1}{4}\left|\frac{x}{(t-\theta s)^\beta}\right|^2\right\}dx.$$

Thus

$$\int_{-\infty}^{+\infty}\left|\frac{x}{(t-\theta s)^\beta}\right|^j\exp\left\{-\frac{1}{4}\left|\frac{x}{(t-\theta s)^\beta}\right|^j\right\}dx$$

$$\leqslant\int\limits_{\{x:|x/(t-\theta s)^\beta|\leqslant\sqrt{2j}\}}+\int\limits_{\substack{\{x:|x/(t-\theta s)^\beta|>\sqrt{2j}\\ \text{and }|x|/t^\beta\leqslant\sqrt{2j}\}}}+\int\limits_{\{x:|x|/t^\beta>\sqrt{2j}\}}\left|\frac{x}{(t-\theta s)^\beta}\right|^j\exp\left\{-\frac{1}{4}\left|\frac{x}{(t-\theta s)^\beta}\right|^2\right\}dx.$$

The first integral on the right side of the above inequality is majorized by $C2^{j/2}j^{j/2}e^{-j/2}j^{1/2}t^\beta\leqslant C2^{j/2}j\Gamma(j)^{1/2}t^\beta$. This bound holds for the second integral once we note that

$$\left|\{x\colon |x|/t^\beta\leqslant\sqrt{2j}\}\right|\leqslant Cj^{1/2}t^\beta.$$

For the third integral we note that

$$\left|\frac{x}{(t-\theta s)^\beta}\right|\geqslant\frac{|x|}{t^\beta}>\sqrt{2j}$$

and therefore this integral is bounded by

$$\int_{-\infty}^{+\infty}\left(\frac{|x|}{t^\beta}\right)^j\exp\left\{-\frac{1}{4}\frac{|x|^2}{t^{2\beta}}\right\}dx.$$

Hence

$$\int_{-\infty}^{+\infty}(t-\theta s)^{-\beta-1/n-1}\left(\frac{|x|}{(t-\theta s)^\beta}\right)^j\exp\left\{-\frac{1}{4}\frac{x^2}{(t-\theta s)^{2\beta}}\right\}dx$$

$$\leqslant C2^{j/2}(j!)^{1/2}j\,(t-s)^{-1-1/n}.$$

Case $m>1$. We have

$$\prod_{i=1}^{m}C_i(t-s)^{-\beta-1/n}\left(\frac{x_i}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\right\}-\prod_{i=1}^{m}C_it^{-\beta-1/n}\left(\frac{x_i}{t^\beta}\right)^{k_i}\times$$

$$\times\exp\left\{-\frac{1}{4}\frac{x_i}{t^\beta}\right\}=\left[C_m(t-s)^{-\beta-1/n}\left(\frac{x_m}{(t-s)^\beta}\right)^{k_m}\exp\left\{-\frac{1}{4}\frac{x_m^2}{(t-s)^{2\beta}}\right\}-\right.$$

$$\left.-t^{-\beta-1/n}\left(\frac{x_m}{t^\beta}\right)^{k_m}\exp\left\{-\frac{1}{4}\frac{x_m^2}{t^{2\beta}}\right\}\right]\prod_{i=1}^{m-1}C_it^{-\beta-1/n}\exp\left\{-\frac{1}{4}\frac{x_i^2}{t^{2\beta}}\right\}+$$

$$+C_m(t-s)^{-\beta-1/n}\left(\frac{x_m}{(t-s)^\beta}\right)^{k_m}\exp\left\{-\frac{1}{4}\frac{x_m^2}{(t-s)^{2\beta}}\right\}\left[\prod_{i=1}^{m-1}C_i(t-s)^{-\beta-1/n}\times\right.$$

$$\left.\times\left(\frac{x_i}{(t-s)^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\right\}-\prod_{i=1}^{m-1}C_it^{-\beta-1/n}\left(\frac{x_i}{t^\beta}\right)^{k_i}\exp\left\{-\frac{1}{4}\frac{x_i^2}{t^{2\beta}}\right\}\right]$$

$$=A(x,t,s)+B(x,t,s).$$

Using the case $m=1$, we see that

$$\int_{E^m} |A(x,t,s)|\,dx \leqslant C2^{|k|/2}(k!)^{1/2}\Big(\prod_{k_i>0} k_i^2\Big)\, t^{-(m-1)/n}(t-s)^{-1-1/n}|s|.$$

Since $t>2|s|$,

$$\int_{E^m} |A(x,t,s)|\,dx \leqslant C2^{|k|/2}(k!)^{1/2}\Big(\prod_{k_i>0} k_i^2\Big)|s|\,(t-s)^{-m/n-1}.$$

Applying the inductive assumption in $B(x,t,s)$ we have for $t>2|s|$,

$$\int_{E^m} |B(x,t,s)|\,dx \leqslant C2^{|k|/2}(k!)^{1/2}\Big(\prod_{k_i>0} k_i^2\Big)|s|\,(t-s)^{-m/n-1}.$$

Our claim is now established.

Going back to equation (4) and applying the above for $m=n$, we have

$$\int_{W_\gamma(y,s)} \Big|\prod_{i=1}^n C_i(t-s)^{-\beta-1/n}\Big(\frac{x_i}{(t-s)^\beta}\Big)^{k_i}\exp\Big\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\Big\} - \prod_{i=1}^n C_i t^{-\beta-1/n}\Big(\frac{x_i}{t^\beta}\Big)^{k_i}\exp\Big\{-\frac{1}{4}\frac{x_i^2}{t^{2\beta}}\Big\}\Big|\,dxdt$$

$$\leqslant \int_{t>2|s|}\int_{E^n} \Big|\prod_{i=1}^n C_i(t-s)^{-\beta-1/n}\Big(\frac{x_i}{(t-s)^\beta}\Big)^{k_i}\exp\Big\{-\frac{1}{4}\frac{x_i^2}{(t-s)^{2\beta}}\Big\} - \prod_{i=1}^n C_i t^{-\beta-1/n}\Big(\frac{x_i}{t^\beta}\Big)^{k_i}\exp\Big\{-\frac{1}{4}\frac{x_i^2}{t^{2\beta}}\Big\}\Big|\,dxdt$$

$$\leqslant C2^{|k|/2}(k!)^{1/2}|s|\int_{t>2|s|}\frac{1}{(t-s)^2}\,dt \leqslant C2^{|k|/2}(k!)^{1/2}\Big(\prod_{k_i>0} k_i^2\Big).$$

For $s<0$ we observe that

$$\int_{W_\gamma(y,s)} |T_k(x,t-s)-T_k(x,t)|\,dxdt = \int_{\substack{t>-2s\\ t>-s+\gamma|y|^\alpha}}\int_{E^n} |T_k(x,t-s)-T_k(x,t)|\,dxdt.$$

Setting $r=t-s$, the last integral becomes

$$\int_{\substack{r>-3s\\ r>-2s+\gamma|y|^\alpha}}\int_{E^n} |T_k(x,r)-T_k(x,r+s)|\,dxdr \leqslant \int_{W_\gamma(y,s)} |T_k(x,r)-T_k\big(x,r-(-s)\big)|\,dxdr.$$

Hence the case $s<0$ is reduced to the case $s>0$. Proposition 2 is now complete.

PROPOSITION 3. *Set*

$$T_{k,\varepsilon}(f)(x,t) = \int_0^{t-\varepsilon}\int_{E^n} T_k(x-y,t-s)f(y,s)\,dyds$$

and let

$$T_k(f) = \lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}} T_{k,\varepsilon}(f).$$

$$\|T_{k,\varepsilon}f\|_p \leqslant C_{p,n}2^{|k|/2}(k!)^{1/2}\prod_{k_i>0} k_i^{(n+4)/2}\|f\|_p.$$

Proof. This follows immediately from propositions 1 and 2 and the final observation in Section 1.3.

II. OPERATORS WITH VARIABLE KERNELS

We are now in a position to study the L^p-norms and the limit in L^p as $\varepsilon\to 0$ of functions defined by

$$\int_0^{t-\varepsilon}\int_{E^n} K(x,t;x-y,t-s)f(y,s)\,dyds$$

and

$$\int_0^{t-\varepsilon}\int_{E^n} K(y,s;x-y,t-s)f(y,s)\,dyds.$$

2.1. Definition. We will call a function, $K(x,t;y,s)$, a *variable kernel* if it satisfies the following conditions:

(1) There is $\alpha\geqslant 2$ such that if $\lambda>0$, $K(x,t;\lambda y,\lambda^\alpha s) = \lambda^{-n-\alpha}K(x,t;y,s)$. Hence $K(x,t;y,s) = K(x,t;y/s^\beta,1)s^{-n\beta-1}$, $\beta=1/\alpha$.

(2) If $\Omega(x,t;y) = K(x,t;y,1)$, then $\int_{E^n}\Omega(x,t;y)\,dy = 0$.

(3) Set

$$\hat{\Omega}(x,t;y) = \frac{1}{(2\pi)^{n/2}}\int_{E^n}\Omega(x,t;z)e^{iy\cdot z}\,dz.$$

Then for every a, $|a|\leqslant mn$, we assume $|(\partial/\partial y)^a\hat{\Omega}(x,t;y)|\leqslant A_m e^{-|y|^2}$.

Set

$$K_\varepsilon f(x,t) = \int_0^{t-\varepsilon}\int_{E^n} K(x,t;x-y,t-s)f(y,s)\,dyds$$

and

$$\overline{K}_\varepsilon f(x,t) = \int_0^{t-\varepsilon}\int_{E^n} K(y,s;x-y,t-s)f(y,s)\,dyds.$$

2.2. L^p-convergence. Before we begin to study the continuity of the above operators, we need to consider a few properties of the Fourier development of $\hat{\Omega}(x,t;y)e^{|y|^2/2}$ with respect to the Hermite functions over E^n. We get

$$\hat{\Omega}(x,t;y)e^{|y|^2/2}\sim\sum_k C_k(x,t)H_k(y)e^{-|y|^2/2}.$$

$$C_k(x,t)=C\frac{1}{2^{|k|}k!}\int\limits_{E^n}\hat{\Omega}(x,t;y)H_k(y)\,dy.$$

LEMMA 1. $\sup\limits_{(x,t)}|C_k(x,t)|\leqslant C_mA_m(2^{|k|}k!)^{-1/2}\big(\prod\limits_{k_i>0}k_i^m\big)^{-1/2}$. C_m *depends only on* m *and* n.

Proof. Suppose $k_i>0$. We have

$$\int\limits_{E^n}\hat{\Omega}(x,t;y)H_k(y)\,dy$$
$$=\int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty}\prod_{i=2}^{n}H_{k_i}(y_i)\,dy_2\ldots dy_n\Big\{\int\limits_{-\infty}^{\infty}\hat{\Omega}(x,t;y_1,\ldots,y_n)H_{k_1}(y_1)\,dy_1\Big\},$$

$$\int\limits_{\infty}^{\infty}\Omega(x,t;y_1,\ldots,y_n)H_{k_1}(y_1)\,dy_1$$
$$=\frac{1}{2^m(k_1+1)\ldots(k_1+m)}\int\limits_{-\infty}^{\infty}\hat{\Omega}(x,t;y_1,\ldots,y_n)\frac{d^m}{dy_1^m}H_{k_1+m}(y_1)$$
$$=\frac{(-1)^mk_1!}{2^m(k_1+m)!}\int\limits_{-\infty}^{\infty}(\partial^m/\partial y_1^m)\,\hat{\Omega}(x,t;y_1,\ldots,y_n)H_{k_1+m}(y_1).$$

Setting $M=(m,\ldots,m)$ we see that

$$\int\limits_{E^n}\hat{\Omega}(x,t;y)H_k(y)\,dy=\frac{(-1)^{nm}k!}{2^{nm}(k+M)!}\int\limits_{E^n}(\partial/\partial y)^M\hat{\Omega}(x,t;y)H_{k+M}(y)\,dy.$$

Hence

$$|C_k(x,t)|\leqslant\frac{C_m}{2^{|k|/2}(k+M)!^{1/2}}\Big[\int\limits_{E^n}|(\partial/\partial y)^M\hat{\Omega}(x,t;y)|^2e^{|y|^2}\,dy\Big]^{1/2}$$
$$\leqslant C_mA_m\Big[\frac{k!}{(k+M)!}\Big]^{1/2}(2^{|k|}k!)^{-1/2}.$$
$$|C_k(x,t)|\leqslant C_mA_m(2^{|k|}k!)^{-1/2}\Big(\prod_{k_i>0}k_i^m\Big)^{-1/2}.$$

Recall that $\Omega_k(y)=\widehat{[H_k(z)-H_k(0)]e^{-|z|^2}}(y)$.

LEMMA 2.

$$\|\Omega_k(y)\|_p\leqslant C\Big(\prod_{k_i>0}k_i\Big)(2^{|k|}k!)^{1/2},\quad 1\leqslant p\leqslant\infty,$$

and

$$\sum_k|C_k(x,t)|\,|H_k(y)e^{-|y|^2/2}|<C,$$

C *independent of* (x,t), *provided* $m>2$.

Proof. We have

$$\|\Omega_k(y)\|_p\leqslant\Big\|\prod_{i=1}^{n}|y_i|^{k_i}e^{-y_i^2/4}\Big\|_p+|H_k(0)|\,\|e^{-|y|^2/4}\|_p.$$

The first term is majorized by a constant times $\big(\prod\limits_{k_i>0}k_i\big)(2^{|k|}k!)^{1/2}$. The same inequality holds for the last term.

The second part of Lemma 2 follows from Lemma 1 once we note that

$$\operatorname*{Max}_x|H_k(x)|e^{-|x|^2/2}\leqslant C(2^{|k|}k!)^{1/2}.$$

THEOREM 4. *If* $f\in L^p\big(E^n\times(0,\infty)\big)$, $1<p<\infty$, *and if* $m>(n+6)$, *then*

1) $\|K_\varepsilon f\|_p\leqslant C\|f\|_p$; $\|\bar{K}_\varepsilon f\|_p\leqslant C\|f\|_p$;

2) $K_\varepsilon f$ *and* $\bar{K}_\varepsilon f$ *converge in* $L^p\big(E^n\times(0,\infty)\big)$ *as* $\varepsilon\to0$.

Proof.

$$K_\varepsilon f(x,t)=\int\limits_{\varepsilon}^{t}\int\limits_{E^n}\frac{\Omega(x,t;y/s^\beta)}{s^{n\beta+1}}f(x-y,t-s)\,dy\,ds.$$

$$\hat{\Omega}(x,t;y)e^{|y|^2/2}=\sum_kC_k(x,t)H_k(y)e^{-|y|^2/2}.$$

Since $\hat{\Omega}(x,t;0)=0$,

$$\sum_kC_k(x,t)[H_k(y)-H_k(0)]e^{-|y|^2}=\hat{\Omega}(x,t;y).$$

By Lemma 2, this last series converges in every $L^q(E^n)$, $1\leqslant q<\infty$, to $\hat{\Omega}(x,t;y)$. Also from Lemma 2, the series

$$\sum_kC_k(x,t)\widehat{[H_k(z)-H_k(0)]e^{-|z|^2}}(y)=\sum_kC_k(x,t)\Omega_k(y)$$

converges in every $L^q(E^n)$, $1\leqslant q<\infty$, to $\Omega(x,t;y)$. Hence

$$\sum_kC_k(x,t)\frac{\Omega_k(y/s^\beta)}{s^{n\beta+1}}$$

converges in every $L^q\big(E^n\times(\varepsilon,t)\big)$, $\varepsilon>0$, to $\Omega(x,t;y/s^\beta)/s^{n\beta+1}$.

Therefore

$$K_\varepsilon f(x,t) = \sum_k C_k(x,t) T_{k,\varepsilon} f(x,t)$$

where

$$T_{k,\varepsilon} f(x,t) = \int_\varepsilon^t \int_{E^n} \frac{\Omega_k(y/s^\beta)}{s^{n\beta+1}} f(x-y, t-s)\,dy\,ds.$$

(See Chapter I, Section 1.4.)

$$\|K_\varepsilon f\|_p \leqslant \sum_k \operatorname{Sup}_{(x,t)} |C_k(x,t)|\, \|T_{k,\varepsilon} f\|_p.$$

Using Proposition (1) of Chapter I, Section 1.4, and Lemma 1 of this section, we have

$$\|K_\varepsilon f\|_p \leqslant C \sum_k \Big(\prod_{k_i>0} k_i^m\Big)^{-1/2} \Big(\prod_{k_i>0} k_i\Big)^{(n+4)/2}.$$

Hence if $m > n+6$, we have $\|K_\varepsilon f\|_p \leqslant C\|f\|_p$, C independent of ε and t.

Now

$$\bar{K}_\varepsilon f = \sum_k T_{k,\varepsilon}(C_k f)(x,t)$$

and therefore

$$\|\bar{K}_\varepsilon f\|_p \leqslant \sum_k \|T_{k,\varepsilon}\|\, \|C_n f\|_p \leqslant \sum_k \operatorname{Sup}_{x,t} |C_k(x,t)|\, \|T_{k,\varepsilon}\|\, \|f\|_p.$$

$$\|\bar{K}_\varepsilon f\|_p \leqslant C\|f\|_p, \quad C \text{ independent of } \varepsilon \text{ and } f.$$

Concerning the L^p-convergence of $K_\varepsilon f$, we first note that $T_{k,\varepsilon} f$ converges in L^p as $\varepsilon \to 0$ to $T_k f$:

$$\sum_k C_k(x,t)[T_{k,\varepsilon} f - T_k f] = \sum_{|k|<M} + \sum_{|k|>M} C_k(x,t)[T_{k,\varepsilon} f - T_k f].$$

Clearly

$$\Big\| \sum_{|k|<M} C_k(x,t)(T_{k,\varepsilon} f - T_k f)(x,t) \Big\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

But

$$\Big\| \sum_{|k|>M} C_k(x,t)(T_{k,\varepsilon} f - T_k f)(x,t) \Big\|_p \leqslant C \sum_{|k|>M} \prod_{k_i>0} k_i^{(n-m)/2}$$

is small for M large. Hence we have shown that $K_\varepsilon f$ converges in L^p to $\sum_k C_k(x,t) T_k f(x,t)$. Similarly it can be shown that $\bar{K}_\varepsilon f$ converges in L^p to $\sum_k T_k(C_k f)(x,t)$.

Remark. *If there exists a fixed $A > 0$ such that*

$$|(\partial/\partial y)^\alpha \hat{\Omega}(x,t;y)| \leqslant C_\alpha e^{-A|y|^2}, \qquad |\alpha| \leqslant nm,$$

then Theorem 7 holds for

$$K_\varepsilon f(x,t) = \int_0^{t-\varepsilon} \int_{E^n} K(x,t;x-y,t-s) f(y,s)\,dy\,ds$$

and

$$\bar{K}_\varepsilon f(x,t) = \int_0^{t-\varepsilon} \int_{E^n} K(y,s;x-y,t-s) f(y,s)\,dy\,ds.$$

Proof. Set $\Omega_1(x,t;y) = \Omega(Ax,t;Ay)$ and

$$K_{1,\varepsilon} f(x,t) = \int_0^{t-\varepsilon} \int_{E^n} K_1(x,t;x-y,t-s) f(y,s)\,dy\,ds.$$

$$K_\varepsilon f(x,t) = \int_0^{t-\varepsilon} \int_{E^n} K(x,t;x-y,t-s) f(y,s)\,dy\,ds$$

$$= A^n \int_0^{t-\varepsilon} \int_{E^n} K\big(A(x/A), t; A((x/A)-y), t-s\big) f(Ay,s)\,dy\,ds.$$

Set $(T_A f)(x,t) = f(Ax,t)$. $K_\varepsilon f = A^n T_{1/A} K_{1,\varepsilon} T_A f$. Since $\hat{\Omega}_1(x,t;y) = A^{-n}\hat{\Omega}(Ax,t;y/A)$, Ω_1 satisfies conditions of Theorem 7.

Similarly by setting $\Omega_1(x,t;y) = \Omega(x,t;Ay)$, we have

$$\bar{K}_\varepsilon f = A^n T_{1/A} \bar{K}_{1/\varepsilon} T_A f.$$

2.3 Hölder continuity. We prove

Lemma. *Suppose $f(x,t) \in L^p(E^n \times (0,\infty))$, $1 < p < \infty$, is continuous in (x,t), and satisfies $|f(x,t) - f(y,t)| \leqslant C|x-y|^\gamma$, $0 < \gamma < 1$, C independent of t, then $(T_k f)(x,t)$ is Hölder continuous in both variables (x,t), of order $\beta\gamma$.*

Proof. It is clear that $\lim_{\varepsilon\to 0} T_{k,\varepsilon} f(x,t) = T_k f(x,t)$ exists pointwise for every (x,t):

$$T_k f(x,t) - T_k f(y,s) = [T_k f(x,t) - T_k f(y,t)] + [T_k f(y,t) - T_k f(y,s)].$$

Consider first $T_k f(x, t) - T_k f(y, t)$. Set $\varrho = |x-y| + |t-s|$. If $t \leqslant \varrho^a$, then

$$T_k f(x,t) - T_k f(y,t)$$
$$= \int_0^t \int_{E^n} T_k(z,r)[f(x-z, t-r) - f(x, t-r) + f(y, t-r) - f(y-z, t-r)]\,dz\,dr.$$

Hence

$$|T_k f(x,t) - T_k f(y,t)| \leqslant C \int_0^{\varrho^a} \int_{E^n} |T_k(z,r)|\,|z|^\gamma\,dz\,dr \leqslant C\varrho^\gamma \|\Omega_k\|_1.$$

Assume now that $t > \varrho^a$:

$$T_k f(x,t) - T_k f(y,t) = \int_0^{\varrho^a} \int_{E^n} T_k(z,r)[f(x-z, t-r) - f(y-z, t-r)]\,dz\,dr +$$
$$+ \int_{\varrho^a}^t \int_{E^n} T_k(z,r)[f(x-z, t-r) - f(y-z, t-r)]\,dz\,dr.$$

The first integral is, of course, to be understood as $\lim_{\varepsilon \to 0} \int_\varepsilon^{\varrho^a} \int_{E^n}$. As in the case $t \leqslant \varrho^a$, the first integral is majorized by $C\varrho^\gamma \|\Omega_k\|_1$. We rewrite the second integral as

$$I = \int_{\varrho^a}^\infty \int_{E^n} [T_k(x-z, r) - T_k(y-z, r)]\{f(z, t-r) - f(x, t-r)\}\,dz\,dr,$$

$$|I| \leqslant C \int_{\varrho^a}^\infty \frac{1}{r^{1-\gamma\beta}} \int_{E^n} \left|\Omega_k\left(\frac{x}{r^\beta} - z\right) - \Omega_k\left(\frac{y}{r^\beta} - z\right)\right| \left|\frac{x}{r^\beta} - z\right|^\gamma dz\,dr$$
$$\leqslant C \sum_{i=1}^n \int_{\varrho^a}^\infty \frac{1}{r^{1-\gamma\beta}} \int_{E^n} \left|\Omega_k\left(\frac{x}{r^\beta} - z\right) - \Omega_k\left(\frac{y}{r^\beta} - z\right)\right| \left|\frac{x_i}{r^\beta} - z_i\right|^\gamma dz\,dr.$$

Claim. *If* $0 \leqslant \gamma < 1$ *and* $|x-y|/r^\beta \leqslant 1$, *then*

$$(+) \qquad \int_{E^n} \left| \prod_j c_j \left(\frac{x_j}{r^\beta} - z_j\right)^{k_j} \right| \exp\left\{-\frac{1}{4}\left|\frac{x_j}{r^\beta} - z_j\right|^2\right\} -$$
$$- \prod_j c_j \left(\frac{y_j}{r^\beta} - z_j\right)^{k_j} \exp\left\{-\frac{1}{4}\left|\frac{y_j}{r^\beta} - z_j\right|^2\right\} \left|\frac{x_i}{r^\beta} - z_i\right|^\gamma dz\,dr$$

is $\leqslant C(2^{|k|} k!)^{1/2} \prod_{k_i > 0} k_i^2 \frac{|x-y|}{r^\beta}$.

Proof. The case $n = 1$ follows the exact same lines as in the proof of a corresponding claim on p. 99. For general n, we observe that (+) is majorized by

$$|C_n| \int_{E^n} \left| \prod_1^{n-1} c_j \left(\frac{x_j}{r^\beta} - z_j\right)^{k_j} \exp\left\{-\frac{1}{4}\left|\frac{x_j}{r^\beta} - z_j\right|^2\right\} - \prod_1^{n-1} c_j \left(\frac{y_j}{r^\beta} - z_j\right)^{k_j} \times\right.$$
$$\left. \times \exp\left\{-\frac{1}{4}\left|\frac{y_j}{r^\beta} - z_j\right|^2\right\}\right| \left|\frac{x_i}{r^\beta} - z_i\right|^\gamma \left|\frac{x_n}{r^\beta} - z_n\right|^{k_n} \exp\left\{-\frac{1}{4}\left|\frac{x_n}{r^\beta} - z_n\right|^2\right\} dz +$$
$$+ |C_n| \int_{E^n} \left| \prod_1^{n-1} c_j \left(\frac{y_j}{r^\beta} - z_j\right)^{k_j} \exp\left\{-\frac{1}{4}\left|\frac{y_j}{r^\beta} - z_j\right|^2\right\}\right| \left| \left(\left(\frac{x_n}{r^\beta} - z_n\right)^{k_n} \times \right.\right.$$
$$\left.\left. \times \exp\left\{-\frac{1}{4}\left|\frac{x_n}{r^\beta} - z_n\right|^2\right\} - \left(\frac{y_n}{r^\beta} - z_n\right)^{k_n} \exp\left\{-\frac{1}{4}\left|\frac{y_n}{r^\beta} - z_n\right|^2\right\}\right)\right| \left|\left(\frac{x_i}{r^\beta} - z_i\right)\right|^\gamma dz.$$

In the first integral use the inductive hypothesis and in the second use the case $n = 1$. The "claim" now follows and from this it is not difficult to see that

$$|I| \leqslant c(2^{|k|} k!)^{1/2} \prod_{k_i > 0} k_i^2 \varrho \int_{\varrho^a}^\infty \frac{dr}{r^{1+\beta-\gamma\beta}} \leqslant c(2^{|k|} k!)^{1/2} \prod_{k_i > 0} k_i^2 \varrho^\gamma.$$

Hence we have shown that

$$\text{(i)} \qquad |T_k f(x,t) - T_k f(y,t)| \leqslant C(2^{|k|} k!)^{1/2} \prod_{k_i > 0} k_i^2 \varrho^\gamma.$$

Consider now $[T_k f(y,t) - T_k f(y,s)]$ and assume, for simplicity, that $t > s$:

$$T_k f(y,t) - T_k f(y,s) = \int_0^s \int_{E^n} [T_k(z, t-r) - T_k(z, s-r)] f(y-z, r)\,dz\,dr +$$
$$+ \int_s^t \int_{E^n} T_k(z, t-r) f(y-z, r)\,dz\,dr = A + B.$$

$$B = \int_0^{t-s} \int_{E^n} T_k(z,r)[f(y-z, t-r) - f(y, t-r)]\,dz\,dr.$$

$$|B| \leqslant C \int_0^\varrho \int_{E^n} |T_k(z,r)|\,|z|^\gamma \leqslant C\|\Omega_k\|_1 \varrho^{\gamma\beta}.$$

$$A = \int_0^s \int_{E^n} [T_k(z, t-r) - T_k(z, s-r)] f(y-z, r)\,dz\,dr$$
$$= \int_0^s \int_{E^n} [T_k(z, r + (t-s)) - T_k(z, r)]\,[f(y-z, s-r) - f(y, s-r)]\,dz\,dr.$$

If $s \leqslant 2\varrho$, we proceed as in B. Assume then that $s > 2\varrho$. Then

$$A = \int_0^{2\varrho}\int_{E^n} + \int_{2\varrho}^{s}\int_{E^n}[T_k(z, r+(t-s)) - T_k(z, r)]\times$$

$$\times[f(y-z, s-r) - f(y, s-r)]\,dz\,dr = A_1 + A_2.$$

A_1 handle as in the case of B.

$$|A_2| \leqslant \int_{2\varrho}^{\infty}\int_{E^n}|T_k(z, r+(t-s)) - T_k(z, r)|\,|z|^\gamma\,dz\,dr.$$

(ii) $\quad T_k(z, r+(t+s)) - T_k(z, r)$

$$= H_k(0)\left[\frac{\exp\{-|z|^2/4(r+(t-s))^{2\beta}\}}{[r+(t-s)]^{n\beta+1}} - \frac{\exp\{-|z|^2/4r^{2\beta}\}}{r^{n\beta+1}}\right] + (r+(t-s))^{-n\beta-1}\times$$

$$\times\prod_{j=1}^{n} c_j\left(\frac{iz_j}{[r+(t-s)]^\beta}\right)^{k_j}\exp\left\{-\frac{|z|^2}{4(r+(t-s))^\beta}\right\} - r^{-n\beta-1}\prod_{j=1}^{n} c_j\left(\frac{iz_j}{r^\beta}\right)^{k_j}\exp\left\{-\frac{|z|^2}{4r^\beta}\right\}.$$

Consider

$$|H_k(0)|\int_{2\varrho}^{\infty}\int_{E^n}\left|\frac{\exp\{-|z|^2/4[r+(t-s)]^{2\beta}\}}{[r+(t-s)]^{n\beta+1}} - \frac{\exp\{-|z|^2/4r^{2\beta}\}}{r^{n\beta+1}}\right|\,|z|^\gamma\,dz\,dr.$$

Since $0 < (t-s) \leqslant \varrho \leqslant \frac{1}{2}r$, this expression is majorized by

$$C|H_k(0)|\int_{2\varrho}^{\infty}\frac{1}{r^{2-\beta\gamma}}\,dr\,(\varrho) \leqslant C(2^{|k|}k!)^{1/2}\varrho^{\beta\gamma}.$$

The first two terms in (ii) equal

$$|z|^\gamma\left(\prod_{j=1}^{n} c_j[r+(t-s)]^{-\beta-1/n}\left(\frac{iz_j}{[r+(t-s)]^\beta}\right)^{k_j}\exp\left\{-\frac{|z|^2}{4[r+(t-s)]^{2\beta}}\right\} - \right.$$

$$\left. - \prod_{j=1}^{n} c_j r^{-\beta-1/n}\left(\frac{iz_j}{r^\beta}\right)^{k_j}\exp\left\{-\frac{|z|^2}{4r^{2\beta}}\right\}\right).$$

Claim. *For* $r > 2(t-s)$, $0 \leqslant \gamma < 1$,

$$\int_{E^m}|z|^\gamma\left|\left(\prod_{j=1}^{m} c_j(r+(t-s))^{-\beta-1/n}\left(\frac{iz_j}{r^\beta}\right)^{k_j}\exp\left\{-\frac{z_j^2}{4(r+(t-s))^{2\beta}}\right\} - \right.\right.$$

$$\left.\left. - \prod_{j=1}^{m} c_j r^{-\beta-1/n}\left(\frac{iz_j}{r^\beta}\right)^{k_j}\exp\left\{-\frac{z_j^2}{4r^{2\beta}}\right\}\right)\right|dz$$

$$\leqslant C(2^{|k|}k!)^{1/2}\left(\prod_{k_i>0} k_i^{2+\gamma/2}\right)(r^{-m/n-1+\beta\gamma})(t-s).$$

Proof. We will again use induction on the dimension m.

Case $m = 1$. Here we proceed exactly as in the case $m = 1$ on p. 102. For the case $m > 1$, we note that $|z|^\gamma \leqslant \sum |z_i|^\gamma$ and that

$$|z_l|^\gamma\left[\prod_{j=1}^{m} c_j(r+(t-s))^{-\beta-1/n}\left(\frac{iz_j}{(r+(t-s))^\beta}\right)^{k_j}\exp\left\{-\frac{z_j^2}{4(r+(t-s))^{2\beta}}\right\} - \right.$$

$$\left. - \prod_{j=1}^{m} c_j r^{-\beta-1/n}\left(\frac{iz_j}{r^\beta}\right)^{k_j}\exp\left\{-\frac{z_j^2}{4r^{2\beta}}\right\}\right]$$

$$= |z_l|^\gamma\left[c_l(r+(t-s))^{-\beta-1/n}\left(\frac{iz_l}{[r+(t-s)]^\beta}\right)^{k_l}\exp\left\{-\frac{z_l^2}{4(r+(t-s))^{2\beta}}\right\} - \right.$$

$$\left. - c_l r^{-\beta-1/n}\left(\frac{iz_l}{r^\beta}\right)^{k_l}\exp\left\{-\frac{z_l^2}{4r^{2\beta}}\right\}\right]\prod_{j\neq l} c_j\left(\frac{iz_j}{r^\beta}\right)^{k_j}\exp\left\{-\frac{z_l^2}{4r^{2\beta}}\right\} +$$

$$+ c_l(r+(t-s))^{-\beta-1/n}\left(\frac{iz_l}{r+(t-s)^\beta}\right)^{k_l}\exp\left\{-\frac{z_l^2}{4(r+(t-s))^{2\beta}}\right\}|z_l|^\gamma$$

$$\left[\prod_{j\neq l} c_j(r+(t-s))^{-\beta-1/n}\left(\frac{iz_j}{[r+(t-s)]^\beta}\right)^{k_j}\exp\left\{-\frac{z_j^2}{4(r+(t-s))^{2\beta}}\right\} - \right.$$

$$\left. - \prod_{\gamma\neq l} c_j r^{-\beta-1/n}\left(\frac{iz_j}{r^\beta}\right)^{k_j}\exp\left\{-\frac{z_j^2}{4r^{2\beta}}\right\}\right].$$

We now proceed using the inductive assumption (for the case $\gamma = 0$) and the result for the case $m = 1$.

Hence

$$|A_2| \leqslant c(2^{|k|}k!)^{1/2}\left(\prod_{k_i>0} k_i^{2+\gamma/2}\right)(t-s)\int_{2\varrho}^{\infty}\frac{1}{r^{-2+\beta\gamma}} \leqslant c(2^{|k|}k!)^{1/2}\left(\prod_{k_i>0} k_i^{2+\gamma/2}\right)\varrho^{\beta\gamma}.$$

This completes the proof for the Hölder continuity of $T_k f$.

Remark. Under the above assumptions on $f(x, t)$, we have

$$|T_k f(x, t)| \leqslant c(2^{|k|}k!)^{1/2}\left(\prod_{k_i>0} k_i^{2+\gamma/2}\right).$$

To see this remark observe that if $g(x, t)\in L^p(E^n\times(0, \infty))$, $|g(x_1, t_1) - g(x_2, t_2)| \leqslant M(|x_1-x_2| + |t_1-t_2|)^\gamma$ and $\|g\|_p \leqslant cM$, then $|g| \leqslant cM$.

We now introduce a new assumption on $\hat{\Omega}(x, t; z)$.

(4) There exists γ, $0 < \gamma < 1$, such that for all $\beta = (\beta_1, \ldots, \beta_n)$, $|\beta| \leqslant mn$.

$$|(\partial/\partial z)^\beta\hat{\Omega}(x, t; z) - (\partial/\partial z)^\beta\hat{\Omega}(y, s; z)| \leqslant C_\beta(|x-y|^\gamma + |t-s|^\gamma)e^{-|y|^2},$$

C_β depends only on β.

THEOREM 5. *If $K(x,t;y,s)$ is a variable kernel satisfying conditions (1)-(4), and if $f(x,t)$ is in $L^p(E^n\times(0,\infty))$, continuous in (x,t), and Hölder continuous in x, uniformly in t, then $Kf(x,t)$ and $\bar{K}f(x,t)$ are Hölder continuous in both the variables x and t.*

Proof. We will consider only Kf, $\bar{K}f$ being completely analogous.

From the above remark and the estimates on $c_k(x,t)$, it is clear that $\sum_k c_k(x,t)T_kf(x,t)$ converges absolutely and uniformly in (x,t) to $Kf(x,t)$. Moreover, if $M=(m,\dots,m)$, then as in the previous section,

$$c_k(x,t)-c_k(y,s)$$
$$=\frac{(-)^{nm}k!}{2^{nm}(k+M)!}\int_{E^n}(\partial/\partial z)^M[\hat{\Omega}(x,t;z)-\hat{\Omega}(y,s;z)]H_{k+M}(z)\,dz.$$

Hence

$$|c_k(x,t)-c_k(y,s)|\leqslant c(2^{|k|}k!)^{-1/2}\Big(\prod_{k_i>0}k_i^m\Big)^{-1/2}(|x-y|+|t-s|)^{\gamma}.$$

Using the previous lemma, we see that there is a $\delta>0$ and a number $l>1$ such that

$$|c_k(x,t)T_kf(x,t)-c_k(y,s)T_kf(y,s)|\leqslant c\Big(\prod_{k_i>0}k_i\Big)^{-l}(|x-y|+|t-s|)^{\delta}.$$

Therefore $|Kf(x,t)-Kf(y,s)|\leqslant C(|x-y|+|t-s|)^{\delta}$.

Remark (i). If $\hat{\Omega}(x,t;y)$ satisfies the condition

$$(4')\qquad |(\partial/\partial z)^{\beta}\hat{\Omega}(x,t;z)-(\partial/\partial z)^{\beta}\hat{\Omega}(y,t;z)|\leqslant c_{\beta}(|x-y|^{\gamma})e^{-|v|^2}$$

for $|\beta|\leqslant mn$ and c_β independent of t, then under the above assumptions on f, $Kf(x,t)$ is Hölder continuous in x, uniformly in t, but $\bar{K}f(x,t)$ is Hölder continuous in both variables. To see this we need only observe that

$$|c_k(x,t)-c_k(y,t)|\leqslant c(2^{|k|}k!)^{-1/2}\Big(\prod_{k_i>0}k_i^m\Big)^{-1/2}|x-y|^{\gamma},$$

c independent of t.

Remark (ii). Theorem 5 and remark (i) also remain valid if conditions (4) and (4′) on $\hat{\Omega}(x,t;z)$ are respectively replaced by

$$(5)\qquad |(\partial/\partial z)^{\beta}\hat{\Omega}(x,t;z)-(\partial/\partial z)^{\beta}\hat{\Omega}(y,s;z)|\leqslant c_{\beta}(|x-y|+|t-s|)^{\gamma}e^{-A_{\beta}|v|^2},$$

c_β, A_β depending only on β, $|\beta|\leqslant mn$.

$$(5')\qquad |(\partial/z)^{\beta}\hat{\Omega}(x,t;z)-(\partial/\partial z)^{\beta}\hat{\Omega}(y,t;z)|\leqslant c_{\beta}(|x-y|)^{\gamma}e^{-A_{\beta}|v|^2},\qquad |\beta|\leqslant mn.$$

Remark (ii) follows immediately from the remark at the conclusion of Theorem 4.

III. APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

3.1. The space $L_0^{p,m,1}(E^n\times(0,R))$. In this section $\alpha=(\alpha_1,\dots,\alpha_n)$ will denote a vector in E^n with each coordinate, α_i, a non-negative integer.

$$(\partial/\partial x)^{\alpha}=\partial^{\alpha_1}/\partial x_1^{\alpha_1}\dots\partial^{\alpha_n}/\partial x_n^{\alpha_n},\qquad x^{\alpha}=x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n},$$
$$|\alpha|=\sum_{i=1}^{n}\alpha_1.$$

Set $S_R=E^n\times(0,R)$.

We let $C_0^{m,1}(S_R)$ stand for the class of functions $u(x,t)$ defined on the strip S_R such that $(\partial/\partial x)^{\alpha}u$, $|\alpha|\leqslant m$, and $(\partial/\partial t)u$ exist in the classical sense for every $(x,t)\in S_R$, are continuous functions in this strip, and $u(x,t)=0$ in $S_\delta=E^n\times(0,\delta)$ for some $\delta>0$.

We will write the L_p-norm of a function, $f(x,t)$, over $E^n\times(0,\infty)$ by $\|f\|_p$ and its p^{th}-norm over S_R by $\|f\|_{p,S_R}$.

For $1<p<\infty$, we set $\tilde{C}_0^{m,1}(S_R)$ equal to the set of functions $u\in C_0^{m,1}(S_R)$ such that

$$\|u\|_{m,1}=\sum_{|\alpha|\leqslant m}\|(\partial/\partial x)^{\alpha}u\|_{p,S_R}+\|(\partial/\partial t)u\|_{p,S_R}<\infty,$$

and finally we define $L_0^{p,m,1}(S_R)$ to be the closure of $\tilde{C}_0^{m,1}(S_R)$ with respect to the norm $\|\cdot\|_{m,1}$.

THEOREM 6. a) *If $|\alpha|\leqslant m$ and $u\in L_0^{p,m,1}(S_R)$, then for every $w(x,t)$, infinitely differentiable in S_R and with compact support in S_R, we have*

$$\int_0^R\int_{E^n}u(\partial/\partial x)^{\alpha}w\,dxdt=(-1)^{|\alpha|}\int_0^R\int_{E^n}(\partial/\partial x)^{\alpha}u\,w\,dxdt$$

and

$$\int_0^R\int_{E^n}u(\partial/\partial t)w\,dxdt=-\int_0^R\int_{E^n}(\partial/\partial t)u\,w\,dxdt.$$

b) *$L_0^{p,m,1}(S_R)$ is a Banach space.*

c) *The set of functions $V(x,t)$, infinitely differentiable and with compact support in $E^n\times(0,\infty)$, are dense in $L_0^{p,m,1}(S_R)$.*

Proof. a) Let $u_k\in\tilde{C}_0^{m,1}(S_R)$ be a sequence of functions such that $\|u-u_k\|_{m,1}\to 0$ as $K\to\infty$. We have

$$\int_0^R\int_{E^n}u(\partial/\partial x)^{\alpha}w\,dxdt=\lim_{k\to\infty}\int_0^R\int_{E^n}u_k(\partial/\partial x)^{\alpha}w\,dxdt.$$

Therefore

$$\int_0^R \int_{E^n} u(\partial/\partial x)^\alpha w\, dxdt = (-1)^{|\alpha|} \lim_k \int_0^R \int_{E^n} (\partial/\partial x)^\alpha u_k w$$

$$= (-1)^{|\alpha|} \int_0^R \int_{E^n} (\partial/\partial x)^\alpha u w.$$

Exactly in the same manner we have

$$\int_0^R \int_{E^n} u(\partial/\partial t) w = (-1) \int_0^R \int_{E^n} (\partial/\partial t) uw.$$

b) From part a we see that $L_0^{p,m,1}(S_R)$ is a linear subspace of the space of functions $u(x,t)$ such that $(\partial/\partial x)^\alpha u$, $|\alpha| \leqslant m$, and $(\partial/\partial t)u$ exist in the sense of distribution and belong to $L^p(S_R)$. This space of functions is known to be a Banach space with respect to the same norm as we have introduced for $L_0^{p,m,1}(S_R)$. Hence part b follows.

c) Suppose $u(x,t) \in L_0^{p,m,1}(S_R)$. For $|\alpha| \leqslant m$, extend $(\partial/\partial x)^\alpha u(x,t)$ and $(\partial/\partial t)u$ to be 0 for $t < 0$. Let $\zeta(x,t)$, $\Phi(x,t)$ be functions infinitely differentiable and with compact support in S_R and such that

$$\int_{E^{n+1}} \zeta(x,t)\, dxdt = 1$$

and $\Phi(x,t) = 1$ in the neighborhood of the origin. Set $\zeta_k(x,t) = k^{n+1}\zeta(kx, kt)$ and $\Phi_k(x,t) = \Phi(x/k, t/k)$. Set

$$u_k(x,t) = u * \zeta_k(x,t) = \int_0^t \int_{E^n} u(y,s)\zeta_k(x-y, t-s)\, dyds.$$

Clearly $u_k(x,t) = 0$ for t near 0 and $u_k \in C^\infty(E^n \times (0,\infty))$. Also $\|u_k - u\|_p \to 0$. By the same method of proof as in part a, it follows that

$$(\partial/\partial x)^\alpha u_k = u * (\partial/\partial x)^\alpha \zeta_k = (\partial/\partial y)^\alpha u * \zeta_k(x,t).$$

Hence for $|\alpha| \leqslant m$, $\|(\partial/\partial x)^\alpha u_k - (\partial/\partial x)^\alpha u\|_p \to 0$, and since

$$(\partial/\partial t) u_k = u * (\partial/\partial t)\zeta_k = \big((\partial/\partial s) u * \zeta_k\big)(x,t)$$

we have $\|(\partial/\partial t)u_k - (\partial/\partial t)u\|_p \to 0$.

Now let $V_k(x,t) = u_k(x,t)\,\Phi_k(x,t)$. Since $\Phi_k(x,t) \to 1$ for each (x,t), and since each derivative of $\Phi_k(x,t)$ converges uniformly to zero, we have that $\|(\partial/\partial x)^\alpha[u_k\Phi_k] - \Phi_k(\partial/\partial x)^\alpha u_k\|_p \to 0$ and $\|\Phi_k(\partial/\partial x)^\alpha u_k - (\partial/\partial x)^\alpha u_k\|_p \to 0$. Hence $\|(\partial/\partial x)^\alpha V_k - (\partial/\partial x)^\alpha u\|_p \to 0$. Similarly $\|(\partial/\partial t)V_k - (\partial/\partial t)u\|_p \to 0$. Therefore $\|u - V_k\|_{m,1} \to 0$.

3.2. The differential operator. The differential operator we wish to consider is:

(1) $$\sum_{|\alpha| \leqslant m} a_\alpha(x,t)(\partial/\partial x)^\alpha u(x,t) - (\partial/\partial t)u(x,t) = Lu.$$

We will always assume that i) if

$$P(x,t;iz) = \sum_{|\alpha|=m} a_\alpha(x,t)(iz)^\alpha,$$

then, for $|z| = 1$, $\mathrm{Re}\big(P(x,t;iz)\big) < -\delta < 0$, $\delta > 0$ independent of (x,t), ii) $a_\alpha(x,t)$ is a bounded function of $(x,t) \in S_R$.

We will think of L as a linear operator from $L_0^{p,m,1}(S_R)$ into $L^p(S_R)$. It is clear from the definition of norms and condition ii) above that L is a continuous operator from $L_0^{p,m,1}(S_R) \to L^p(S_R)$.

We will now discuss some important properties of the differential operator, L, when each of the coefficients satisfy a Hölder condition in (x,t). The properties discussed here have been proved in [3] and [6]. Set

$$W(x,t;y,s) = \frac{1}{(2\pi)^{n/2}} \int_{E^n} e^{P(x,t;iz)s} e^{iy\cdot z}\, dz.$$

$W(x,t;y,s)$ is a fundamental solution of the equation

$$P(x,t;\partial/\partial y)u - (\partial/\partial s)u = 0.$$

It is known that for $|\alpha| \leqslant m$

(2) $$|(\partial/\partial y)^\alpha W(x,t;y,s)| \leqslant C_1 \frac{e^{-C|y/s^{1/m}|^{m/m-1}}}{s^{(n/m)+(|\alpha|/m)}}$$

(3) $$|(\partial/\partial s)W(x,t;y,s)| \leqslant C_1 \frac{e^{-C|y/s^{1/m}|^{m/m-1}}}{s^{(n/m)+1}},$$

C_1, C are absolute constants.

Set now that $K_\alpha(x,t;y,s) = (\partial/\partial y)^\alpha W(x,t;y,s)$, and $K(x,t;y,s) = (\partial/\partial s)W(x,t;y,s)$.

If the coefficients of L are Hölder continuous in (x,t), then for $f(x,t)$ infinitely differentiable and with compact support in S_R, there is a function, $u(x,t)$, with m continuous derivatives in x and one derivative in t satisfying $Lu = -f$ in S_R and the initial condition $u(x,0) = 0$. In fact, $u(x,t)$ can be written as

$$u(x,t) = \int_0^t \int_{E^n} W(y,s;x-y,t-s)[f(y,s) + (Af)(y,s)]\, dyds$$

with

$$(Af)(x,t) = \int_s^t \int_{E^n} \Phi(x,t;y,s) f(y,s)\, dyds.$$

Here $\Phi(x,t;y,s)$ is the solution of the integral equation,

$$\Phi(x,t;y,s) = \sum_{|\alpha|\leqslant m} a_\alpha(x,t)K_\alpha(y,s;x-y,t-s)-K(y,s;x-y,t-s)+$$
$$+\int_s^t\int_{E^n}\Big[\sum_{|\alpha|\leqslant m} a_\alpha(x,t)K_\alpha(M,\theta;x-M,t-\theta)-K(M,\theta;x-M,t-\theta)\Big]\times$$
$$\times\Phi(M,\theta;y,s)\,dM\,d\theta.$$

Since

$$\sum_{|\alpha|=m} a_\alpha(y,s)K_\alpha(y,s;x-y,t-s)-K(y,s;x-y,t-s)=0,$$

we have

(4) $$\Phi(x,t;y,s)=\sum_{|\alpha|=m}\big(a_\alpha(x,t)-a_\alpha(y,s)\big)K_\alpha(y,s;x-y,t-s)+$$
$$+\sum_{|\alpha|=m} a_\alpha(x,t)K_\alpha(y,s;x-y,t-s)+$$
$$+\int_s^t\int_{E^n}\Big[\sum_{|\alpha|<m}\big(a_\alpha(x,t)-a_\alpha(M,\theta)\big)K_\alpha(M,\theta;x-M,t-\theta)+$$
$$+\sum_{|\alpha|<m} a_\alpha(x,t)K_\alpha(M,\theta;x-M,t-\theta)\Big]\Phi(M,\theta;y,s)\,dM\,d\theta.$$

If we set

$$N_0(x,t;y,s)=\sum_{|\alpha|=m}\big(a_\alpha(x,t)-a(y,s)\big)K_\alpha(y,s;x-y,t-s)+$$
$$+\sum_{|\alpha|<m} a_\alpha(x,t)K_\alpha(y,s;x-y,t-s),$$

$$N_1(x,t;y,s)=\int_s^t\int_{E^n}N_0(x,t;M,\theta)N_0(M,\theta;y,s)\,dM\,d\theta,$$

. . . .

$$N_v(x,t;y,s)=\int_s^t\int_{E^n}N_0(x,t;M,\theta)N_{v-1}(M,\theta;y,s)\,dM\,d\theta,$$

then

$$\Phi(x,t;y,s)=\sum_{v=0}^{\infty}N_v(x,t;y,s).$$

In [2] the following estimates are shown:

$$|N_v(x,t;y,s)|\leqslant A_v\frac{e^{-c|(x-y)/(t-s)^{1/m}|^{m/(m-1)}}}{(t-s)^{(n/m)+\gamma}},\qquad 0<\gamma<1,$$

where $\sum A_v<\infty$. The proof in [3] also shows that $\sum A_v$ depends only on R, the bounds of all the coefficients, the constant of parabolicity (i.e. δ), and finally the Hölder exponent and constants involved in the Hölder continuity of $a_\alpha(x,t)$ for $|\alpha|=m$. The $\sum A_v$ does not depend on the Hölder continuity of the lower order coefficients. Hence the operator, A, given by

$$(Af)(x,t)=\int_0^t\int_{E^n}\Phi(x,t;y,s)f(y,s)\,dy\,ds$$

maps $L^p(S_R)$ continuously in $L^p(S_R)$ and $\|A\|\leqslant C(\sum A_v)$, C an absolute constant. Moreover, if the lower order coefficients are Hölder continuous in x, uniformly in t, then there exists numbers μ and λ, $0<\mu\leqslant 1$, $0<\lambda<1$, such that

(5) $$|\Phi(x_1,t;y,s)-\Phi(x_2,t;y,s)|$$
$$\leqslant B_2|x_1-x_2|^\mu\frac{\exp\Big\{-C\Big|\frac{x_1-y}{(t-s)^{1/m}}\Big|^{m/(m-1)}\Big\}+\exp\Big\{-C\Big|\frac{x_2-y}{(t-s)^{1/m}}\Big|^{m/(m-1)}\Big\}}{(t-s)^{n/m+\lambda}},$$

where B_2 and C are absolute constants.

In [3] and in [6] it is also shown that for $|\beta|\leqslant m-1$,

(6) $$(\partial/\partial x)^\beta u(x,t)$$
$$=\int_0^t\int_{E^n}(\partial/\partial x)^\beta W(y,s;x-y,t-s)\,[f(y,s)+(Af)(y,s)]\,dy\,ds,$$

and for $|\alpha|=m$,

(7) $$(\partial/\partial x)^\alpha u(x,t)$$
$$=\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n}(\partial/\partial x)^\alpha W(y,s;s-y,t-s)\,[f(y,s)+(Af)(y,s)]\,dy\,ds,$$

(8) $$(\partial/\partial t)u(x,t)$$
$$=\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n}(\partial/\partial t)W(y,s;x-y,t-s)\,[f(y,s)+$$
$$+(Af)(y,s)]\,dy\,ds-f(x,t)-(Af)(x,t).$$

The above limits are pointwise limits.

3.3. Existence. We are now ready to give our first application of the results of Chapter II to establish an existence theorem for the equation $Lu=f$, $f\in L^p(S_R)$, in the class $L_0^{p,m,1}(S_R)$.

THEOREM 7. *In addition to the boundedness of the coefficients and the parabolicity of L, we assume that the coefficients of highest order only, i.e., $a_\alpha(x,t)$, $|\alpha|=m$, satisfy $|a_\alpha(x,t)-a_\alpha(y,s)|\leqslant C(|x-y|^{\delta_1}+|t-s|^{\delta_2})$ with $0<\delta_1\leqslant 1$, $0<\delta_2\leqslant 1$.*

Then given any $f \in L^p(S_R)$, $1 < p < \infty$, *there is a function* $u(x,t) \in L_0^{p,m,1}(S_R)$ *satisfying* $Lu = f$ *for almost every* $(x,t) \in S_R$.

Proof. Suppose $f(x,t)$ is infinitely differentiable and has compact support in S_R. We first assume $a_\beta(x,t)$, $|\beta| \leqslant m-1$, is Hölder continuous. Define

$$\tilde{L}^{-1} f(x,t) = u(x,t) = -\int_0^t \int_{E^n} W(y,s;x-y,t-s)[f(y,s)+Af(y,s)]\,dy\,ds.$$

From (2) of Section 3.2, it is clear that for $|\alpha| \leqslant m-1$,

$$\|(\partial/\partial x)^\alpha u\|_{p,S_R} \leqslant C(\|f\|_{p,S_R} + \|Af\|_{S,R}) \leqslant C\|f\|_{p,S_R}.$$

Noting now that for $|\alpha| = m$, $K_\alpha(x,t;y,s)$ and $K(x,t;y,s)$ are variable kernels, it follows from Theorem 5, Chapter II, that for $|\alpha| = m$

$$\|(\partial/\partial x)^\alpha u\|_{p,S_R} \leqslant C\|f\|_{p,S_R} \quad \text{and} \quad \|(\partial/\partial t)u(x,t)\|_{p,S_R} \leqslant C\|f\|_{p,S_R}.$$

Since $u(x,t) = 0$ for t near 0, $u(x,t) \in L_0^{p,m,1}(S_R)$ and $\|\tilde{L}^{-1}f\|_{m,1} \leqslant B\|f\|_{p,S_R}$, B independent of the Hölder continuity of the lower order coefficients.

We now extend $\tilde{L}^{-1}$ to a continuous operation from $L^p(S_R)$ into $L_0^{p,m,1}(S_R)$. Since $L(\tilde{L}^{-1}f) = f$ for a dense subset of $L^p(S_R)$, it is clear that L takes $L_0^{p,m,1}(S_R)$ onto $L^p(S_R)$ for the case when the lower order terms satisfy a Hölder continuity in (x,t).

Now assume that $a_\beta(x,t)$, $|\beta| \leqslant m-1$, is merely bounded in S_R. Let $a_\beta^j(x,t)$ denote sequence of functions infinitely differentiable in S_R, such that

$$\sup_{S_R} |a_\beta^j(x,t)| \leqslant \sup_{S_R} |a_\beta(x,t)|$$

and $a_\beta^j \to a_\beta$ pointwise almost everywhere in S_R.

Set

$$N_0^j(x,t;y,s) = \sum_{|\alpha|=m} [a_\alpha(x,t) - a_\alpha(y,s)] K_\alpha(y,s;x-y,t-s) + \sum_{|\alpha|<m} a_\alpha^j(x,t) K_\alpha(y,s;x-y,t-s).$$

Clearly for almost every $(x,t) \in S_R$, $N_0^j(x,t;y,s)$ tends pointwise to the limit

$$N_0(x,t;y,s) = \sum_{|\alpha|=m} [a_\alpha(x,t) - a_\alpha(y,s)] K_\alpha(y,s;x-y,t-s) + \sum_{|\alpha|=m} a_\alpha(x,t) K_\alpha(y,s;x-y,t-s) \quad \text{for every } (y,s),\ s<t.$$

Set

$$N_1^j(x,t;y,s) = \int_s^t \int_{E^n} N_0^j(x,t;M,\theta) N_0^j(M,\theta;y,s)\,dM\,d\theta.$$

Now

$$|N_0^j(x,t;y,s)| \leqslant A_0 \frac{e^{-(|x-y|/(t-s)^{1/m})^{m/m-1}}}{(t-s)^{(n/m)+\gamma}}, \qquad 0<\gamma<1,$$

A_0 independent of j, γ depending only on Hölder exponent for a_α, $|\alpha| = m$. Hence $N_1^j(x,t;y,s)$ tends pointwise for almost every (x,t) to

$$N_1(x,t;y,s) = \int_s^t \int_{E^n} N_0(x,t;M,\theta) N_0(M,\theta;y,s)\,dM\,d\theta \quad \text{for every } (y,s),\ s<t.$$

In general, $N_v^j(x,t;y,s)$ will tend pointwise for almost every (x,t) to

$$N_v(x,t;y,s) = \int_s^t \int_{E^n} N_0(x,t;M,\theta) N_{v-1}(M,\theta;y,s)\,dM\,d\theta \quad \text{for every } (y,s),\ s<t.$$

Since

$$|N_v^j(x,t;y,s)| \leqslant A_v \frac{e^{-C|(x-y)/(t-s)^{1/m}|^{m/m-1}}}{(t-s)^{(n/m)+\gamma}},$$

A_v independent of j, the same inequality holds for $N_v(x,t;y,s)$. Set

$$\Phi^j(x,t;y,s) = \sum_{v=0}^{\infty} N_v^j(x,t;y,s).$$

Clearly $\Phi^j(x,t;y,s)$ tends pointwise for almost every (x,t) to

$$\Phi(x,t;y,s) = \sum_{v=0}^{\infty} N_v(x,t;y,s)$$

and

$$|\Phi(x,t;y,s)| \leqslant \Big(\sum_{v=0}^{\infty} A_v\Big) \frac{e^{-C|(x-y)/(t-s)^{1/m}|^{m/m-1}}}{(t-s)^{(n/m)+\gamma}}.$$

Set

$$A_j(f)(x,t) = \int_s^t \int_{E^n} \Phi_j(x,t;y,s) f(y,s)\,dy\,ds.$$

We have shown that for almost every $(x,t) \in S_R$, $N_v^j(x,t;y,s) \to N_v(x,t;y,s)$ for every (y,s), $s<t$. From this it follows that for almost every $(x,t) \in S_R$, $\Phi_j(x,t;y,s) \to \Phi(x,t;y,s)$ for every (y,s), $s<t$.

Therefore if $f(x,t)$ is infinitely differentiable and has compact support in S_R, $(A_j f)(x,t)$ converges pointwise for almost every (x,t). Since $(A_j f)(x,t)$ is dominated by

$$\Big(\sum A_v\Big) \int_s^t \int_{E^n} e^{-C|(x-y)/(t-s)^{1/m}|^{m/m-1}} (t-s)^{-(n/m)-\gamma} |f(y,s)|\,dy\,ds$$

which is a function in $L^p(S_R)$, A_jf converges in $L^p(S_R)$ as $j\to\infty$ and $\|A_jf\|_{p,S_R}\leqslant C\|f\|_{p,S_R}$, C independent of j. Hence A_jf converges in $L^p(S_R)$ for every $f\in L^p(S_R)$.

Set

$$L_j=\sum_{|\alpha|=m}a_\alpha(x,t)(\partial/\partial x)^\alpha-\partial/\partial t+\sum_{|\alpha|<m}a_\alpha^j(x,t)(\partial/\partial x)^\alpha.$$

Let

$$u_j(x,t)=\tilde{L}_j^{-1}f=-\int_0^t\int_{E^n}W(y,s;x-y,t-s)\big(f+A_j(f)\big)(y,s)\,dyds.$$

For any $f\in L^p(S_R)$, from the first part of the theorem, we know that $u_j(x,t)\in L_0^{p,m,1}(S_R)$ and $L_ju_j=f$ almost everywhere in S_R. More than that, we know for $|\alpha|\leqslant m$

$$(\partial/\partial x)^\alpha u_j=\lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}}\int_0^{t-\varepsilon}\int_{E^n}(\partial/\partial t)W(y,s;x-y,t-s)(f+A_jf)(y,s)\,dyds$$

and that for almost every $(x,t)\in S_R$,

$$(\partial/\partial t)u_j(x,t)=\lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}}\int_0^{t-\varepsilon}\int_{E^n}(\partial/\partial t)W(y,s;x-y,t-s)(f+A_jf)(y,s)\,dyds+\\ +f(x,t)+A_jf(x,t).$$

Since A_jf converges in $L^p(S_R)$ to a function which we denote by Af, it follows that $\|u_j-u_k\|_{m,1}\to 0$. Since $L_0^{p,m,1}(S_R)$ is a Banach space, let $u(x,t)$ denote the limit in this space of the u_j's. Clearly $\|Lu-L_ju_j\|_{p,S_R}\to 0$ as $j\to\infty$. But $L_ju_j=f$ almost everywhere in S_R. Therefore $Lu=f$ almost everywhere in S_R.

3.4. Uniqueness. We first assume that for all α, $a_\alpha(x,t)$ is infinitely differentiable in S_R and every derivative is bounded in S_R.

LEMMA. *If $u(x,t)$ is infinitely differentiable and with compact support in $E^n\times(0,\infty)$, then*

$$u(x,t)=\int_0^t\int_{E^n}W(x,t;x-y,t-s)(Lu)(y,s)\,dyds+$$
$$+\sum_{|\alpha|=m}\int_0^t\int_{E^n}(\partial/\partial y)^\alpha W(x,t;x-y,t-s)\big(a_\alpha(x,t)-a_\alpha(y,s)\big)u(y,s)\,dyds+$$
$$+\sum_{\substack{|\gamma+\beta|\leqslant m\\ |\gamma|\leqslant m-1}}(-1)^{|\gamma+\beta|}\int_0^t\int_{E^n}(\partial/\partial y)^\gamma W(x,t;x-y,t-s)(\partial/\partial y)^\beta\big(a_\alpha(x,t)-a_\alpha(y,s)\big)\times$$
$$\times u(y,s)\,dyds-\int_0^t\int_{E^n}\sum_{|\beta|<m}(-1)^{|\beta|}(\partial/\partial y)^\beta\big(W(x,t;x-y,t-s)\big)u(y,s)\,dyds.$$

Proof.

$$\int_0^t\int_{E^n}W(x,t;x-y,t-s)\Big\{\sum_{|\alpha|\leqslant m}a_\alpha(x,t)(\partial/\partial y)^\alpha u(y,s)-(\partial/\partial s)u(y,s)\Big\}dyds$$
$$=\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n}\Big[\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}(\partial/\partial y)^\alpha\big(W(x,t;x-y,t-s)\big)+$$
$$+\partial/\partial s\big(W(x,t;x-y,t-s)\big)\Big]$$
$$u(y,s)\,dyds-\lim_{\varepsilon\to 0}\int_{E^n}W(x,t;x-y,\varepsilon)u(y,t-\varepsilon)\,dy.$$

Since $\sum_{|\alpha|=m}a_\alpha(x,t)(\partial/\partial y)^\alpha\big(W(x,t;x-y,t-s)\big)+(\partial/\partial s)\big(W(x,t;x-y,t-s)\big)$ $=0$, we have

$$\int_0^t\int_{E^n}W(x,t;x-y,t-s)\Big\{\sum_{|\alpha|\leqslant m}a_\alpha(x,t)(\partial/\partial y)^\alpha u(y,s)-(\partial/\partial s)u(y,s)\Big\}dyds$$
$$=\int_0^t\int_{E^n}\sum_{|\beta|<m}a_\beta(x,t)(-1)^{|\beta|}(\partial/\partial y)^\beta\big(W(x,t;x-y,t-s)\big)u(y,s)\,dyds-$$
$$-\lim_{\varepsilon\to 0}\int_{E^n}W(x,t;x-y,\varepsilon)u(y,t-\varepsilon)\,dy.$$

From the definition of W it is clear that $\int_{E^n}W(x,t;x-y,\varepsilon)\,dy=1$, $\varepsilon>0$. Therefore

$$\int_{E^n}W(x,t;x-y,\varepsilon)u(y,t-\varepsilon)$$
$$=\int_{E^n}W(x,t;x-y,\varepsilon)\big(u(y,t-\varepsilon)-u(x,t)\big)\,dy+u(x,t).$$

Since $|u(x,t)-U(y,t-\varepsilon)|\leqslant C(|x-y|+\varepsilon)$ and since

$$|W(x,t;x-y,\varepsilon)|\leqslant C\frac{e^{-|(x-y)/\varepsilon^{1/m}|^{m/m-1}}}{\varepsilon^{n/m}},$$
$$\lim_{\varepsilon\to 0}\int_{E^n}W(x,t;x-y,\varepsilon)\big(u(y,t-\varepsilon)-u(x,t)\big)\,dy=0.$$

Hence

$$u(x,t)=$$
$$=\int_0^t\int_{E^n}W(x,t;x-y,t-s)\Big\{\sum_{|\alpha|\leqslant m}a_\alpha(x,t)(\partial/\partial y)^\alpha u(y,s)-(\partial/\partial s)u(y,s)\Big\}dyds-$$
$$-\int_0^t\int_{E^n}\Big\{\sum_{|\beta|<m}a_\beta(x,t)(-1)^{|\beta|}(\partial/\partial y)^\beta\big(W(x,t;x-y,t-s)\big)u(y,s)\,dyds.$$

Adding and subtracting

$$(Lu)(y,s)=\sum_{|\alpha|\leqslant m}a_\alpha(y,s)(\partial/\partial y)^\alpha u-(\partial/\partial s)u$$

in the first integral above, we have

$$u(x,t) = \int_0^t \int_{E^n} W(x,t;x-y,t-s)(Lu)(y,s)\,dy\,ds +$$

$$+\int_0^t \int_{E^n} W(x,t;x-y,t-s)\Big\{\sum_{|\alpha|\leqslant m}\big(a_\alpha(x,t)-a_\alpha(y,s)\big)(\partial/\partial y)^\alpha u(y,s)\Big\}dy\,ds -$$

$$-\int_0^t \int_{E^n} \sum_{|\beta|<m} a_\beta(x,t)(-1)^{|\beta|}(\partial/\partial y)^\beta\big(W(x,t;x-y,t-s)\big)u(y,s)\,dy\,ds.$$

Integrating by parts in the second integral, the desired representation of $u(x,t)$ follows.

Using the above lemma and the estimates for $(\partial/\partial y)^\alpha\big(W(x,t;x-y,t-s)\big)$ given in Section 3.2, it follows that for u infinitely differentiable and with compact support in $E^n\times(0,\infty)$,

$$\|u\|_{p,S_\varepsilon} \leqslant C\|Lu\|_{p,S_\varepsilon} + C_1(\varepsilon)\|u\|_{p,S_\varepsilon}$$

where $C_1(\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence for all ε sufficiently small $\|u\|_{p,S_\varepsilon} \leqslant C\|Lu\|_{p,S_\varepsilon}$. Since this inequality holds for a dense subset of $L_0^{p,m,1}(S_R)$, ε being independent of u, it is valid for all $u\in L_0^{p,m,1}(S_R)$. Therefore if $Lu=0$ in S_R, then $u=0$ in S_ε. Applying the same argument we see that $u=0$ in $S_{2\varepsilon}$. Hence $u=0$ in S_R.

Now for the general case we assume

$$L = \sum_{|\alpha|\leqslant m} a_\alpha(x,t)(\partial/\partial x)^\alpha - \partial/\partial t$$

where $a_\alpha(x,t)$ is a bounded function in S_R, and for $|\alpha|=m$,

$$|a_\alpha(x,t)-a_\alpha(y,s)| \leqslant M_1(|x-y|^{\gamma_1}+|t-s|^{\gamma_2}),\qquad 0<\gamma_1\leqslant 1,\ 0<\gamma_2\leqslant 1.$$

Suppose $w(x,t)\geqslant 0$ in S_R, infinitely differentiable there, with compact support in S_R and such that $\int w=1$.

Set $a_\alpha^j(x,t) = j^{n+1}\int a_\alpha(x-y,t-s)\,w(jy,js)\,dy\,ds$. Now $a_\alpha^j(x,t)\to a_\alpha(x,t)$ for almost every $(x,t)\in S_R$ and

$$\sup_{(x,t)}|a_\alpha^j(x,t)| \leqslant \sup_{(x,t)}|a_\alpha(x,t)| \leqslant M_2.$$

Also for $|\alpha|=m$ it is clear that $|a_\alpha^j(x,t)-a_\alpha^j(y,s)| \leqslant M_1(|x-y|^{\gamma_1}+|t-s|^{\gamma_2})$. Set

$$L_j = \sum_{|\alpha|\leqslant m} a_\alpha^j(x,t)(\partial/\partial x)^\alpha - \partial/\partial t$$

$$P_j(x,t;iz) = \sum_{|\alpha|=m} a_\alpha^j(x,t)(iz)^\alpha.$$

For $|z|=1$, $\mathrm{Re}\big(P_j(x,t;iz)\big) < -\delta < 0$, δ independent of j.

$$\|u\|_{p,S_R} = \|L_j^{-1}(L_j u)\|_{p,S_R}.$$

From the proof of existence we know that $\|L_j^{-1}\| \leqslant A$, A depending on $M_1, M_2, \delta, \gamma_1, \gamma_2$. Hence A is independent of j. Therefore $\|u\|_{p,S_R} \leqslant A\|L_j u\|_{p,S_R} \leqslant A\|(L_j-L)u\|_{p,S_R} + A\|Lu\|_{p,S_R}$.

Letting $j\to\infty$, we have $\|u\|_{p,S_R} \leqslant A\|Lu\|_{p,S_R}$.

We combine Theorem 7 and Section 3.4 into

THEOREM 8. *Under the hypotheses of Theorem 7, the differential operator L maps $L_0^{p,m,1}(S_R)$ in a continuous, one-to-one manner, onto $L^p(S_R)$.*

We end this chapter with an application of Theorem 5 concerning the Hölder continuity of $(\partial/\partial x)^\alpha u$ and $(\partial/\partial t)u$ where u is a solution of $Lu=f$.

THEOREM 9. *In addition to the hypotheses of Theorem 7, assume that $a_\beta(x,t)$, $|\beta|\leqslant m-1$, is Hölder continuous in x, uniformly in t. Given $f\in L^p(S_R)$, $1<p<\infty$, let $u\in L_0^{p,m,1}(S_R)$ be the solution of $Lu=f$. Then*

(i) *if $f(x,t)$ is Hölder continuous in (x,t), the same holds for $(\partial/\partial x)^\alpha u$, $|\alpha|=m$, and $(\partial/\partial t)u$;*

(ii) *if $f(x,t)$ is bounded, continuous, Hölder continuous in x, uniformly in t, then $(\partial/\partial x)^\alpha u$, $|\alpha|=m$, is Hölder continuous in both variables (x,t) and $(\partial/\partial t)u$ is Hölder continuous in x, uniformly in t.*

Proof. From our previous discussion, it is clear that

$$(\partial/\partial x)^\alpha u(x,t) = -\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K_\alpha(y,s;x-y,t-s)\big(f(y,s)+Af(y,s)\big)dy\,ds$$

and

$$(\partial/\partial t)u(x,t)$$

$$= -\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K(y,s;x-y,t-s)\big(f(y,s)+Af(y,s)\big)dy\,ds + f(x,t)+Af(x,t).$$

The above limits are pointwise limits. In terms of our notation of Chapter II,

$$(\partial/\partial x)^\alpha u = \overline{K}_\alpha(f+Af)\quad\text{ and }\quad(\partial/\partial t)u = \overline{K}(f+Af)+f+Af.$$

It is clear from (5), Section 3.2, that in both (i) and (ii), Af is bounded and Hölder continuous in x, uniformly in t. Using Theorem 5, the conclusions in (i) and (ii) are now immediate.

Appendix

We will now prove Lemma 2 of Chapter I which is stated here in two separate parts.

LEMMA 2′. *For $a\geqslant 2$,*

$$\left|\int_1^R \frac{e^{is^a}e^{\pm ivs}}{s}\,ds\right| \leqslant C,$$

C independent of $R\geqslant 1$, $v>0$.

Proof. If $v \leqslant 1$,

$$\int_1^R \frac{e^{is^\alpha} e^{\pm ivs}}{s}\, ds = \frac{1}{i\alpha} \int_1^R \frac{e^{\pm ivs}}{s^\alpha} \left(\frac{d}{ds}\right) e^{is^\alpha} ds.$$

Integrating the last expression by parts, we have

$$\int_1^R \frac{e^{is^\alpha} e^{\pm ivs}}{s}\, ds = O(1) + O(v) \int_1^R \frac{e^{\pm ivs} e^{is^\alpha}}{s^\alpha}\, ds + O(1) \int_1^R \frac{e^{is^\alpha} e^{\pm ivs}}{s^{\alpha+1}}\, ds.$$

Hence

$$\int_1^R \frac{e^{is^\alpha} e^{\pm ivs}}{s}\, ds = O(1).$$

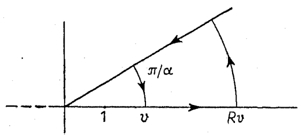
So we may assume $v > 1$. Now

$$\int_1^R \frac{e^{is^\alpha} e^{\pm ivs}}{s}\, ds = \int_v^{Rv} \frac{e^{i(s/v)^\alpha} e^{\pm is}}{s}\, ds.$$

We first consider

$$\int_\Gamma \frac{e^{i(z/v)^\alpha} e^{iz}}{z}\, dz$$

over the contour, Γ, as shown:



(1) $$0 = -\int_v^{Rv} \frac{e^{ise^{i(\pi/\alpha)^\alpha}}}{s} e^{ise^{i(\pi/\alpha)}} ds -$$

(2) $$-i \int_0^{\pi/\alpha} e^{ie^{i\alpha\theta}} e^{ive^{i\theta}} d\theta +$$

(3) $$+i \int_0^{\pi/\alpha} e^{iR^\alpha} e^{i\alpha\theta} e^{iRve^{i\theta}} d\theta +$$

$$+\int_v^{Rv} \frac{e^{i(s/v)^\alpha} e^{is}}{s}\, ds.$$

(1) is majorized by

$$\int_v^{Rv} \frac{e^{-\sin(\pi/\alpha)s}}{s}\, ds \leqslant \int_1^\infty e^{-\sin(\pi/\alpha)s} ds < \infty.$$

(2) is majorized by

$$\int_0^{\pi/\alpha} e^{-\sin(\alpha\theta)} e^{-v\sin(\theta)} d\theta \leqslant \pi/\alpha.$$

(3) is majorized by

$$\int_0^{\pi/\alpha} e^{-R^\alpha \sin(\alpha\theta)} e^{-Rv\sin(\theta)} d\theta \leqslant \pi/\alpha.$$

Therefore

$$\left|\int_v^{Rv} \frac{e^{i(s/v)^\alpha} e^{is}}{s}\, ds\right| \leqslant C$$

independent of R, v.

Now consider

$$\int_v^{Rv} \frac{e^{i(s/v)^\alpha} e^{-is}}{s}\, ds.$$

Let $x = \alpha(\alpha-2)/(\alpha-1)$. We may assume $Rv \leqslant v^{\alpha-x}$ since otherwise

$$\int_v^{Rv} = \int_v^{v^{\alpha-x}} + \int_{v^{\alpha-x}}^{Rv} \qquad \left(\text{Note } \alpha - x = \frac{\alpha}{\alpha-1} > 1\right).$$

$$\int_{v^{\alpha-x}}^{Rv} \frac{e^{i(s/v)^\alpha} e^{-is}}{s}\, ds = \int_{v^{\alpha-x-1}}^{R} e^{is^\alpha} \frac{e^{-ivs}}{s}\, ds = \frac{1}{i\alpha} \int_{v^{\alpha-x-1}}^{R} \frac{e^{-ivs}}{s^\alpha} \frac{d}{ds} e^{is^\alpha} ds$$

$$= O(1) + \frac{v}{\alpha} \int_{v^{\alpha-x-1}}^{R} \frac{e^{-ivs} e^{is^\alpha}}{s^\alpha}\, ds.$$

$$v \int_{v^{\alpha-x-1}}^{\infty} \frac{ds}{s^\alpha} = \frac{v}{\alpha-1} (v^{\alpha-x-1})^{1-\alpha}.$$

Therefore

$$\left|\int_{v^{\alpha-x}}^{Rv} \frac{e^{i(s/v)^\alpha} e^{-is}}{s}\, ds\right| = O(1)$$

if $(\alpha-x-1)(\alpha-1)-1 \geqslant 0$.

$$(a-x-1)(a-1)-1 = 0 \Longleftrightarrow a^2-ax-2a+x = 0 \Longleftrightarrow a(a-2) = x(a-1).$$

Since $x = a(a-2)/a-1$, we have

$$\left|\int_{v^{a-x}}^{Rv} \frac{e^{i(s/v)^a}e^{-is}}{s}\,ds\right| \leqslant C.$$

Since $\int_{v}^{v^{a-x}}$ is of the form $\int_{v}^{Rv}$ with $Rv \leqslant v^{a-x}$, we may begin with this assumption. Hence $R \leqslant v^{a-x-1}$. Set $C = C_a = 2/\pi a \leqslant \sin(\theta)/\sin(a\theta)$, $0 \leqslant \theta \leqslant \pi/a$ $(a \geqslant 2)$. Note $C_a < 1$. If $C_aR \leqslant 1$, then

$$\left|\int_{v}^{Rv} \frac{e^{i(s/v)^a}e^{-is}}{s}\,ds\right| \leqslant \int_{1}^{1/C_a} \frac{1}{s}\,ds.$$

Hence assume $C_aR > 1$. Therefore

$$\int_{v}^{Rv} \frac{e^{i(s/v)^a}e^{-is}}{s}\,ds = \int_{v}^{C_aRv} + \int_{C_aRv}^{Rv} \frac{e^{i(s/v)^a}e^{-is}}{s}\,ds = \int_{1}^{C_aR} \frac{e^{is^a}e^{-ivs}}{s}\,ds + O\left(\log\frac{1}{C_a}\right).$$

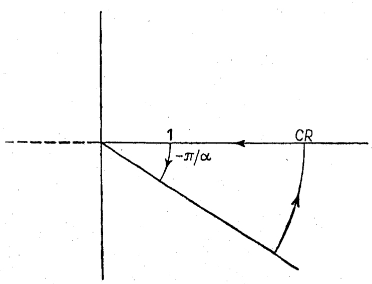
So we want to finally bound the integral

$$\int_{1}^{C_aR} \frac{e^{is^a}e^{-ivs}}{s}\,ds.$$

Consider

$$\int_{1}^{CR} \frac{e^{iz^a}e^{-ivz}}{z}\,dz$$

over the contour given below:



(1) $$0 = \int_{1}^{CR} \frac{e^{is^a}e^{-ia(\pi/a)}e^{-ivse^{-i(\pi/2)}}}{s}\,ds -$$

(2) $$-i\int_{-(\pi/a)}^{0} e^{ie^{ia\theta}}e^{-ive^{i\theta}}\,d\theta +$$

(3) $$+i\int_{-(\pi/a)}^{0} e^{iC^aR^ae^{ia\theta}}e^{-ivCRe^{i\theta}}\,d\theta - \int_{1}^{CR} \frac{e^{is^a}e^{-ivs}}{s}\,ds.$$

(1), in absolute value, is majorized by

$$\int_{1}^{CR} \frac{e^{-vs\sin(\pi/a)}}{s}\,ds = O(1)$$

uniformly in $R > 1$ and $v > 1$.

(2), in absolute value, is majorized by

$$\int_{0}^{\pi/a} e^{\sin(a\theta)}e^{-v\sin(\theta)}\,d\theta = O(1).$$

(3), in absolute value, is majorized by

$$\int_{0}^{\pi/a} e^{[C^aR^a\sin(a\theta)-CRv\sin(\theta)]}\,d\theta \leqslant \int_{0}^{\pi/a} e^{CRv\{(C^{a-1}R^{a-1})/v\sin(a\theta)-\sin(\theta)\}}\,d\theta.$$

Now

$$\frac{C^{a-1}R^{a-1}}{v}\sin(a\theta)-\sin(\theta) \leqslant 0$$

if and only if

$$\frac{C^{a-1}R^{a-1}}{v} \leqslant \frac{\sin(\theta)}{\sin(a\theta)}.$$

But

$$\frac{C^{a-1}R^{a-1}}{v} \leqslant C^{a-1}v^{(a-1)(a-1-x)-1}$$

and $(a-1)(a-1-x)-1 = 0$. Hence

$$\frac{C^{a-1}R^{a-1}}{v} \leqslant C^{a-1} \leqslant C \leqslant \frac{\sin(\theta)}{\sin(a\theta)}.$$

We note that $C^{a-1} \leqslant C$ since $a \geqslant 2$ and $C < 1$. Therefore

$$\left|\int_{1}^{R} \frac{e^{is^a}e^{-ivs}}{s}\,ds\right| \leqslant C.$$

LEMMA 2″.

$$\left|\int_1^R \frac{e^{\pm is} e^{i(vs)^a}}{s} ds\right| \leqslant C,$$

C independent of $R \geqslant 1$, $v > 0$.

Proof.

$$\int_1^R \frac{e^{\pm is} e^{i(s/(1/v))^a}}{s} ds = \int_1^R \frac{e^{\pm is} e^{i(s/u)^a}}{s} ds, \quad u = 1/v.$$

If $u \leqslant 1$, then

$$\int_1^R \frac{e^{\pm is} e^{i(s/u)^a}}{s} ds = \frac{u^a}{ia} \int_1^R \frac{e^{\pm is}}{s^a} \left(\frac{d}{ds} e^{i(s/u)^a}\right) ds.$$

Hence

$$\left|\int_1^R \frac{e^{\pm is} e^{i(s/u)^a}}{s} ds\right| \leqslant O(1)+O(1)\left[\int_1^\infty \frac{1}{s^a}+\frac{1}{s^{a+1}}\right] \leqslant C.$$

So we may assume $u > 1$.

$$\int_1^R \frac{e^{\pm is} e^{i(s/u)^a}}{s} ds = \int_{1/u}^{R/u} \frac{e^{\pm ius} e^{is^a}}{s} ds.$$

Suppose first $R/u > 1$.

$$\int_{1/u}^{R/u} \frac{e^{\pm ius} e^{is^a}}{s} ds = \int_{1/u}^{1} \int_{1}^{R/u} \frac{e^{\pm ius} e^{is^a}}{s} ds = A+B,$$

$|B| \leqslant C$ by Lemma 2′.

$$A = \frac{1}{\pm iu} \int_{1/u}^{1} \left(\frac{d}{ds} e^{\pm ius}\right) \frac{e^{is^a}}{s} ds.$$

Integrating by parts we see that

$$|A| \leqslant O(1)+\frac{1}{u}\int_{1/u}^{1} \frac{ds}{s^2} = O(1).$$

If $R/u \leqslant 1$, then again

$$\int_{1/u}^{R/u} \frac{e^{\pm ius} e^{is^a}}{s} ds = \frac{1}{\pm iu} \int_{1/u}^{R/u} \frac{d}{ds} e^{\pm ius} \frac{e^{is^a}}{s} ds.$$

Integrating by parts once more and using the fact that $R \geqslant 1$, we have

$$\left|\int_{1/u}^{R/u} \frac{e^{\pm ius} e^{is^a}}{s} ds\right| \leqslant C.$$

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