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Derivatives of Fourier series and integrals

by

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I. Notation and definitions. Throughout this paper we shall be dealing with n-dimensional Euclidean space E^n, $n \geqslant 2$. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ denote points of E^n, we use the standard notation $x+y = (x_1+y_1, \ldots, x_n+y_n)$, $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$ for λ real, $(x\cdot y) = x_1 y_1 + \ldots + x_n y_n$, $|x| = (x\cdot x)^{1/2}$, $a = (a_1, \ldots, a_n)$ where the a_j are non-negative integers, $a! = a_1! \ldots a_n!$, $|a| = a_1 + \ldots + a_n$, $x^a = x_1^{a_1} \ldots x_n^{a_n}$, $D^a = (\partial/\partial x)^a = (\partial/\partial x_1)^{a_1} \ldots (\partial/\partial x_n)^{a_n}$, $\mu = (\mu_1, \ldots, \mu_n)$ where the μ_j are positive or negative integers, $Q = \{x \mid -\pi < x_j \leqslant \pi, j = 1, \ldots, n\}$, $Q_\mu = Q$ translated by $2\pi\mu$. A real-valued function f on E^n will be called *periodic* if it is periodic 2π in each variable. If $f \in L(Q)$,

$$S[f] = \sum c_\mu e^{i(\mu\cdot x)} \quad \text{where} \quad c_\mu = \frac{1}{(2\pi)^n} \int_Q f(x)\, e^{-i(\mu\cdot x)}\, dx.$$

If $f \in L(E^n)$ is any integrable function,

$$\hat{f}(x) = \frac{1}{(2\pi)^n} \int_{E^n} f(y)\, e^{-i(x\cdot y)}\, dy.$$

We say $S[f]$ is *Bochner-Riesz a-summable of order γ at x to sum s* if

$$\sigma_a^{(\gamma)}(x, \varepsilon) = \left(\frac{\partial}{\partial x}\right)^a \sum_{\varepsilon|\mu| \leqslant 1} c_\mu e^{i(\mu\cdot x)} (1-\varepsilon^2 |\mu|^2)^\gamma$$

tends to s as $\varepsilon \to 0$. We say $S[f]$ is *Abel a-summable at x to sum s* if

$$f_a(x, \varepsilon) = \left(\frac{\partial}{\partial x}\right)^a \sum c_\mu e^{i(\mu\cdot x)} e^{-\varepsilon|\mu|}$$

tends to s as $\varepsilon \to 0$. Alternately, we say

$$\int_{E^n} \hat{f}(y)\, e^{i(x\cdot y)}\, dy$$

is *Bochner-Riesz α-summable of order γ at x to sum s* if

$$\sigma_\alpha^{(\gamma)}(x, \varepsilon) = \left(\frac{\partial}{\partial x}\right)^\alpha \int\limits_{\varepsilon|y|\leqslant 1} \hat{f}(y) e^{i(x\cdot y)} (1-\varepsilon^2|y|^2)^\gamma dy$$

tends to s as $\varepsilon \to 0$. We say it is *Abel α-summable at x to s* if

$$f_\alpha(x, \varepsilon) = \left(\frac{\partial}{\partial x}\right)^\alpha \int\limits_{E^n} \hat{f}(y) e^{i(x\cdot y)} e^{-\varepsilon|y|} dy$$

tends to s as $\varepsilon \to 0$. We use the notations σ_α and f_α for both series and integrals since no confusion will arise.

We shall also need a notion of differential for functions defined and integrable in the neighborhood of a point x. If f is such a function and k is a non-negative integer, we say f has a k^{th} *differential in L at x* if

$$(1) \qquad \varepsilon^{-n} \int\limits_{|y|\leqslant\varepsilon} \Big| f(x+y) - \sum_{|\alpha|\leqslant k} \frac{a_\alpha(x)}{\alpha!} y^\alpha \Big| dy = o(\varepsilon^k)$$

as $\varepsilon \to 0$ for some $a_\alpha(x)$, $a_0(x) = f(x)$. The a_α are uniquely determined by (1). More generally, we say f has a k^{th} *symmetric differential in L at x* if

$$(2) \qquad \begin{aligned} &k \text{ even:} \quad \varepsilon^{-n} \int\limits_{|y|\leqslant\varepsilon} \Big| \frac{f(x+y)+f(x-y)}{2} - \sum_{\substack{|\alpha|\leqslant k \\ |\alpha| \text{ even}}} \frac{a_\alpha(x)}{\alpha!} y^\alpha \Big| dy = o(\varepsilon^k), \\ &k \text{ odd:} \quad \varepsilon^{-n} \int\limits_{|y|\leqslant\varepsilon} \Big| \frac{f(x+y)-f(x-y)}{2} - \sum_{\substack{|\alpha|\leqslant k \\ |\alpha| \text{ odd}}} \frac{a_\alpha(x)}{\alpha!} y^\alpha \Big| dy = o(\varepsilon^k) \end{aligned}$$

as $\varepsilon \to 0$. When $k = 0$, we assume $a_0(x) = f(x)$. If f has an ordinary k^{th} differential it also has one in L. If f has a k^{th} differential in L, it has a symmetric k^{th} differential in L.

II. Results. We will prove three kinds of theorems. Here we give their versions for periodic functions. In the body of the paper, however, we state and prove analogues for non-periodic functions. The first kind of result is given in the following two theorems.

THEOREM A. *Let $f(x)$ be periodic, $f \in L(Q)$, and let f satisfy* (2) *at x for some integer $k \geqslant 0$. Then for any α, $|\alpha| = k$, $S[f]$ is Abel α-summable at x to sum $a_\alpha(x)$.*

THEOREM B. *Under the same hypotheses as in theorem* A, *$S[f]$ is Bochner-Riesz α-summable of order γ at x to sum $a_\alpha(x)$ for any $|\alpha| = k$, provided $\gamma > k+(n-1)/2$.*

For $k = 0$, theorem B is Bochner's classical theorem ([1], p. 189) while both theorems A and B are proved in [5], p. 50-57. The proofs there are under the hypothesis (1), i.e., x is a Lebesgue point. However, all that is really needed is the symmetric condition (2) for $k = 0$. For the one-dimensional analogues of A and B, see [7], vol. II, p. 60.

The second type of result is the following theorem:

THEOREM C. *Let $f(x)$ be periodic, $f \in L(Q)$, and let f satisfy* (1) *for some $k \geqslant 1$ at each point of a subset E of E^n. Then for almost every $x \in E$ and any $|\alpha| = k$, $S[f]$ is Bochner-Riesz α-summable of order $\gamma_0 = k+(n-1)/2$ to sum $a_\alpha(x)$.*

For $k = 0$, theorem C is, of course, false. For the one-dimensional analogue of C, see [7], vol. II, p. 81.

The final result is a localization theorem used in the proof of theorem C.

THEOREM D. *If $f \in L(Q)$ is periodic 2π and vanishes in the neighborhood of x, then $S[f]$ is Bochner-Riesz α-summable of order $\gamma_0 = k+(n-1)/2$ at x to zero, $|\alpha| = k \geqslant 1$.*

For $k = 0$, theorem D is false. From theorem B, it is clear that D remains true if the order of summability is increased to $\gamma > \gamma_0$.

In section III, we prove theorems A and B and their non-periodic analogues. In section IV, we prove theorems C and D and their non-periodic analogues.

III. Proofs of theorems A and B. In the main, we restrict ourselves to proving theorem B, the proof for theorem A being only technically different.

1) We begin with the non-periodic case.

THEOREM 1. *Let $f(x) \in L(E^n)$ satisfy* (2) *at some x for an integer $k \geqslant 0$. Then for any α, $|\alpha| = k$,*

$$\int\limits_{E^n} \hat{f}(y) e^{i(x\cdot y)} dy$$

is

a) *Abel α-summable at x to $a_\alpha(x)$,*

b) *Bochner-Riesz α-summable of order $\gamma > k+(n-1)/2$ at x to $a_\alpha(x)$.*

We will prove b) for $n > 2$, the case $n = 2$ being somewhat less involved. We begin with two technical lemmas.

LEMMA 1. *If $Y_m(\xi)$, $\xi \in E^n$ $(n > 2)$, is any spherical harmonic of order m and $J_\nu(s)$, $-\infty < s < \infty$, is the Bessel function of order ν then given any unit vector η,*

$$\int\limits_{|\xi|=1} Y_m(\xi) e^{-is(\xi\cdot\eta)} d\xi = i^m (2\pi)^{\beta+1} s^{-\beta} J_{m+\beta}(s) Y_m(-\eta), \qquad \beta = (n-2)/2.$$

This formula follows from the Funk-Hecke theorem ([4], p. 247) and [4], p. 175, plus the Legendre duplication formula ([3], p. 5).

Let $\Phi_\gamma(t) = (1-t^2)^\gamma$ for $0 \leqslant t < 1$, $\Phi_\gamma(t) = 0$ for $t > 1$, and if m is a non-negative integer, let

$$\mu_m^{(\gamma)}(r) = \int_0^\infty s^{k+\beta+1}\Phi_\gamma(rs)J_{m+\beta}(s)\,ds.$$

For definiteness, consider the case when k is an even integer and $m = k-2l$, $0 \leqslant 2l \leqslant k$, l an integer.

LEMMA 2. *For k even, $0 \leqslant 2l \leqslant k$,*

$$|\mu_{k-2l}^{(\gamma)}(r)| \leqslant \begin{cases} Ar^{-n-k}, & 0 < r < \infty, \\ A_\gamma r^{\gamma-\gamma_0}, & 0 < r \leqslant 1, \end{cases} \qquad \gamma_0 = k+\frac{n-1}{2}.$$

Proof. Since $|J_\nu(s)| \leqslant 1$, $|J_\nu(s)| \leqslant s^\nu$ $(s > 0)$, it follows $|J_{m+\beta}(s)| \leqslant s^{\beta}$ $(s > 0)$ and

$$|\mu_m^{(\gamma)}(r)| \leqslant \int_0^{r^{-1}} s^{k+2\beta+1}\,ds = Ar^{-n-k}.$$

To prove the second estimate, write $s^2 = r^{-2}-r^{-2}(1-r^2s^2)$. Then

$$\mu_{k-2l}^{(\gamma)}(r) = r^{-2}\int_0^\infty s^{k+\beta-1}\Phi_\gamma(rs)J_{k-2l+\beta}(s)\,ds - r^{-2}\int_0^\infty s^{k+\beta-1}\Phi_{\gamma+1}(rs)J_{k-2l+\beta}(s)\,ds.$$

Applying the same argument to each integral on the right and continuing the process, we obtain at the l^{th} stage a sum (with coefficients ± 1) of integrals

$$r^{-2l}\int_0^\infty s^{k-2l+\beta+1}\Phi_{\gamma+j}(rs)J_{k-2l+\beta}(s)\,ds \qquad (j = 0, 1, \ldots, l).$$

By [6], p. 373, the expression above is

$$2^{\gamma+j}\Gamma(\gamma+j+1)r^{\gamma-(k+n/2)+j}J_{k-2l+\beta+j+1}(r^{-1}).$$

Since $|J_\nu(s)| \leqslant s^{-1/2}$ $(s > 0)$, this is bounded by a constant (depending on γ) times $r^{\gamma-\gamma_0+j} \leqslant r^{\gamma-\gamma_0}$ for $0 < r \leqslant 1$.

We now pass to the proof of theorem 1. Let k be even and $n > 2$. Subtracting from f a function g with k continuous derivatives and compact support, $D^\alpha g(x) = a_\alpha(x)$ for $|\alpha| \leqslant k$, $|\alpha|$ even, it is enough to prove the theorem for both g and $f-g$—that is, we may consider separately the two cases (i) $f \in C^k$ with compact support and (ii) $a_\alpha(x) = 0$ for all α. In case (i), if $|\alpha| = k$,

$$\left(\frac{\partial}{\partial x}\right)^\alpha \int_{E^n} \hat{f}(y)e^{i(x\cdot y)}\Phi_\gamma(\varepsilon|y|)\,dy = \int_{E^n} [D^\alpha f]^\wedge(y)e^{i(x\cdot y)}\Phi_\gamma(\varepsilon|y|)\,dy.$$

Since $D^\alpha f$ is continuous, this tends to $D^\alpha f(x) = a_\alpha(x)$ as $\varepsilon \to 0$ by Bochner's result, for $\gamma > (n-1)/2$. In case (ii), condition (2) becomes

$$\int_{|y|\leqslant\varepsilon} |f(x+y)+f(x-y)|\,dy = o(\varepsilon^{n+k}) \tag{3}$$

and we must show that

$$\lim_{\varepsilon\to 0}\sigma_\alpha^{(\gamma)}(x, \varepsilon) = \lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial x}\right)^\alpha \int_{E^n} \hat{f}(y)e^{i(x\cdot y)}\Phi_\gamma(\varepsilon|y|)\,dy = 0$$

for $\gamma > k+(n-1)/2$, $|\alpha| = k$. Fix such α and γ for the remainder of the proof and write $\Phi_\gamma = \Phi$, etc.

$$\sigma_\alpha^{(\gamma)}(x, \varepsilon) = i^k \int_{E^n} f(x+y)[y^\alpha\Phi(\varepsilon|y|)]^\wedge\,dy, \tag{4}$$

$$[y^\alpha\Phi(\varepsilon|y|)]^\wedge = (2\pi)^{-n}\int_{E^n} x^\alpha\Phi(\varepsilon|x|)e^{-i(x\cdot y)}\,dx$$

$$= (2\pi)^{-n}\int_0^\infty t^{n+k-1}\Phi(\varepsilon t)\Big[\int_{|\xi|=1}\xi^\alpha e^{-it|y|(\xi\cdot\eta)}\,d\xi\Big]dt,$$

$x = t\xi$, $y = |y|\eta$ in polar coordinates. We may write the homogeneous even polynomial $\xi^\alpha = \sum Y_{k-2l}(\xi)$ where $Y_m(\xi)$ is a spherical harmonic of degree m. By lemma 1, the inner integral above is

$$\sum \int_{|\xi|=1} Y_{k-2l}(\xi)e^{it|y|(\xi\cdot\eta)}\,d\xi = i^k(2\pi)^{\beta+1}\sum(-1)^l Y_{k-2l}(\eta)(t|y|)^{-\beta}J_{k-2l+\beta}(t|y|).$$

After a change of variable we obtain

$$i^k[y^\alpha\Phi(\varepsilon|y|)]^\wedge = (2\pi)^{-n/2}|y|^{-n-k}\sum_{0\leqslant 2l\leqslant k}(-1)^l Y_{k-2l}(\xi)\,\mu_{k-2l}^{(\gamma)}\left(\frac{\varepsilon}{|y|}\right), \tag{5}$$

$y = |y|\xi$ in polar coordinates. Since k is even, it follows that $[y^\alpha\Phi(\varepsilon|y|)]^\wedge$ is even and (4) may be written

$$\sigma_\alpha^{(\gamma)}(x, \varepsilon) = i^k\int_{E^n}\frac{f(x+y)+f(x-y)}{2}[y^\alpha\Phi(\varepsilon|y|)]^\wedge\,dy.$$

Each $Y_{k-2l}(\xi)$ is bounded on $|\xi| = 1$ and, by lemma 2, $\sigma_\alpha^{(\gamma)}(x, \varepsilon)$ is bounded in absolute value by a constant multiple of

$$\int_{|y|\leqslant\varepsilon}|f(x+y)+f(x-y)|\left(\frac{\varepsilon}{|y|}\right)^{-n-k}|y|^{-n-k}\,dy +$$

$$+\int_{|y|>\varepsilon}|f(x+y)+f(x-y)|\left(\frac{\varepsilon}{|y|}\right)^{\gamma-\gamma_0}|y|^{-n-k}\,dy,$$

$\gamma_0 = k+(n-1)/2$. By (3), the first of these integrals tends to zero with ε. To show that the second tends to zero, let $s = \gamma-\gamma_0 > 0$ and observe

$$\varepsilon^s \int_{|y|>\delta} |f(x+y)+f(x-y)|\,|y|^{-n-k-s}dy \leqslant \frac{2\varepsilon^s}{\delta^{n+k-s}} \int_{E^n} |f(y)|\,dy$$

tends to zero with ε for any fixed $\delta > 0$. Let

$$G(t) = \int_{|y|\leqslant t} |f(x+y)+f(x-y)|\,dy, \qquad t > 0.$$

By hypothesis, $G(t) = o(t^{n+k})$ as $t \to 0$. But

$$\varepsilon^s \int_{\varepsilon\leqslant|y|\leqslant\delta} |f(x+y)+f(x-y)|\,|y|^{-n-k-s}dy = \varepsilon^s \int_\varepsilon^\delta t^{-n-k-s}dG(t)$$

$$= \varepsilon^s \Big[t^{-n-k-s}G(t)|_\varepsilon^\delta + (n+k+s)\int_\varepsilon^\delta t^{-n-k-s-1}G(t)\,dt\Big]$$

$$= o(1)+O(\varepsilon^s)\int_\varepsilon^\delta t^{-n-k-s-1}o(t^{n+k})\,dt = o(1).$$

This completes the proof of theorem 1 (b).

To prove (a) we proceed similarly, arriving at (4) and (5) with $\sigma_\alpha^{(\gamma)}$ replaced by f_α, $\Phi(\varepsilon|y|)$ replaced by $e^{-\varepsilon|y|}$ and $\mu_m^{(\gamma)}$ replaced by

$$\nu_m(r) = \int_0^\infty e^{-rs}s^{k+\beta+1}J_{m+\beta}(s)\,ds.$$

Using repeated integration by parts $(d[s^\nu J_\nu(s)]/ds = s^\nu J_{\nu-1}(s))$ and [6], p. 386, it can be shown that $|\nu_m(r)| \leqslant Ar$ for $0 < r \leqslant 1$. Also $|\nu_m(r)| \leqslant Ar^{-n-k}$, and the rest of the proof is the same.

2) We now prove theorem B. The method is very similar to that of theorem 1 (b).

LEMMA 3. *Let $f(x)$, $x \in E^n$, be periodic, $f \in L(Q)$. There is a sequence $S^m(x)$ of trigonometric polynomials converging in $L(Q)$ to $f(x)$, i.e.,*

$$\lim_{m\to\infty} \int_Q |f(x)-S^m(x)|\,dx = 0.$$

For a proof, see [7], vol. II, p. 304.

To prove theorem B, assume k is even and $k \geqslant 2$, the case $k = 0$ being Bochner's theorem. Subtracting from f a periodic function g with k bounded continuous derivatives in E^n and $D^\alpha g(x) = a_\alpha(x)$ for $|\alpha|$ even, we may consider the two cases: (i) f has k bounded continuous derivatives in E^n, $D^\alpha f(x) = a_\alpha(x)$, and (ii) $a_\alpha(x) = 0$ for all α. For case (i) we have

$$S[D^\alpha f] = \Big(\frac{\partial}{\partial x}\Big)^\alpha \sum c_\mu e^{i(\mu\cdot x)}$$

and since $D^\alpha f$ is continuous,

$$\sigma_\alpha^{(\gamma)}(x,\varepsilon) = \Big(\frac{\partial}{\partial x}\Big)^\alpha \sum c_\mu e^{i(\mu\cdot x)}\Phi_\gamma(\varepsilon|\mu|)$$

tends to $a_\alpha(x)$ as $\varepsilon \to 0$, $\gamma > (n-1)/2$. In case (ii), condition (2) becomes (3) and we must show that

$$\sigma_\alpha^{(\gamma)}(x,\varepsilon) = \Big(\frac{\partial}{\partial x}\Big)^\alpha \sum c_\mu e^{i(\mu\cdot x)}\Phi_\gamma(\varepsilon|\mu|)$$

tends to zero. Fix α, $|\alpha| = k$, and $\gamma > k+(n-1)/2$ for the rest of the proof and write $\Phi = \Phi_\gamma$, etc. We claim that (4) remains true in the periodic case. To show this, let $H_\varepsilon(y) = i^k[y^\alpha\Phi(\varepsilon|y|)]^\wedge$, H_ε depending on α and γ. By lemma 2 and formula (5) for H_ε, $|H_\varepsilon(y)| \leqslant A\varepsilon^{-n-k}$ and, for $|y| \geqslant \varepsilon$, $|H_\varepsilon(y)| \leqslant A\varepsilon^s|y|^{-n-k-s}$, $s = \gamma-\big(k+(n-1)/2\big) > 0$. From the second of these inequalities, it is easy to see that the integral

$$\int_{E^n} f(x+y)H_\varepsilon(y)\,dy$$

converges absolutely for periodic $f \in L(Q)$. Moreover, $H_\varepsilon \in L(E^n)$ for each $\varepsilon > 0$ and from the continuity of $y^\alpha\Phi(\varepsilon|y|)$,

$$i^k x^\alpha \Phi(\varepsilon|x|) = \int_{E^n} H_\varepsilon(y)e^{i(x\cdot y)}dy.$$

In particular,

$$i^k e^{i(\mu\cdot x)}\mu^\alpha\Phi(\varepsilon|\mu|) = \int_{E^n} H_\varepsilon(y)e^{i(\mu\cdot x+y)}dy$$

and

$$i^k \sum_{|\mu|\leqslant R} c_\mu^m e^{i(\mu\cdot x)}\mu^\alpha\Phi(\varepsilon|\mu|) = \int_{E^n} S^m(x+y)H_\varepsilon(y)\,dy, \tag{6}$$

where

$$S^m(y) = \sum_{|\mu|\leqslant R} c_\mu^m e^{i(\mu\cdot x)}.$$

Fix $\varepsilon > 0$. $H_\varepsilon(y)$ is continuous as the Fourier transform of an integrable function. Since the series $\sum H_\varepsilon(y+2\pi\mu)$ converges absolutely and uniformly over Q, its limit H_ε^* is continuous and bounded on Q. Thus

$$\int_{E^n} S^m(x+y)H_\varepsilon(y)\,dy = \sum \int_{Q_\mu} S^m(x+y)H_\varepsilon(y)\,dy = \int_Q S^m(x+y)H_\varepsilon(y+2\pi\mu)\,dy = \int_Q S^m(x+y)H_\varepsilon^*(y)\,dy.$$

Letting $m\to\infty$ and observing that $c_\mu^m \to c_\mu$ (lemma 3), we obtain from (6),

$$i^k \sum c_\mu e^{i(\mu\cdot x)}\mu^a \Phi(\varepsilon|\mu|) = \int_Q f(x+y)H_\varepsilon^*(y)\,dy = \int_{E^n} f(x+y)H_\varepsilon(y)\,dy.$$

Hence for periodic $f\in L(Q)$,

$$\sigma_a^{(\gamma)}(x,\varepsilon) = i^k \int_{E^n} f(x+y)[y^a\Phi_\gamma(\varepsilon|y|)]^\wedge dy. \tag{7}$$

Let

$$G(t) = \int_{|y|\leqslant t} |f(x+y)+f(x-y)|\,dy, \qquad t>0.$$

By hypothesis, $G(t) = o(t^{n+k})$ as $t\to 0$. Since $f\in L(Q)$, $G(t) = O(t^n)$ as $t\to\infty$. Using (5) and (7), the remainder of the proof follows that of theorem 1 (b).

IV. Proof of theorems C and D.

1) We begin with the non-periodic version of theorem C.

THEOREM 2. *Let $f(x)\in L(E^n)$ and satisfy (1) for $k\geqslant 1$ at each point of a measurable subset $E\subset E^n$. Then for almost every $x\in E$ and any $|\alpha| = k$,*

$$\int_{E^n} \hat f(y)e^{i(x\cdot y)}dy$$

is Bochner-Riesz a-summable of order $\gamma_0 = k+(n-1)/2$ to $a_\alpha(x)$.

To prove this theorem, we need five lemmas.

LEMMA 4. *Given $f\in L(E^n)$ satisfying (1) for some $k\geqslant 1$ and all $x\in E$, E bounded, there is a closed subset $P\subset E$, $|E-P|$ arbitrarily small, and a decomposition $f = g+h$ satisfying*

(i) $g\in C^k$ *with compact support,*

(ii) $f(x) = g(x)$, $a_\alpha(x) = D^\alpha g(x)$ *for* $x\in P$, $|\alpha|\leqslant k$,

(iii) $\varepsilon^{-n}\int_{|y|\leqslant\varepsilon} |h(x+y)|\,dy = o(\varepsilon^k)$ *for* $x\in P$,

(iv) $\int_{E^n} \frac{|h(x+y)|}{|y|^{n+k}}\,dy < \infty$ *for* $x\in P$.

For a proof, see [2], p. 189.

Let $\Phi_\gamma(t)$ be defined as usual. Then

LEMMA 5. *For $\gamma > -1$, $n>2$,*

$$[\Phi_\gamma(\varepsilon|y|)]^\wedge = C_\gamma \varepsilon^{-n}(\varepsilon^{-1}|y|)^{-n/2-\gamma}J_{n/2+\gamma}(\varepsilon^{-1}|y|),$$

with $C_\gamma = 2^\gamma\Gamma(\gamma+1)(2\pi)^{-n/2}$.

Proof. We have

$$[\Phi_\gamma(|y|)]^\wedge = (2\pi)^{-n}\int_{|x|\leqslant 1}(1-|x|^2)^\gamma e^{-i(x\cdot y)}dx = (2\pi)^{-n}\int_0^1 t^{n-1}(1-t^2)^\gamma\Big[\int_{|\xi|=1} e^{-it|y|(\xi\cdot\eta)}d\xi\Big]dt,$$

where $x = t\xi$, $y = |y|\eta$ in polar coordinates. By lemma 1, we obtain

$$(2\pi)^{-n/2}\int_0^1 t^{n-1}(1-t^2)^\gamma(t|y|)^{-\beta}J_\beta(t|y|)\,dt, \qquad \beta = \frac{n-2}{2}.$$

By [6], p. 373, this is

$$2^\gamma\Gamma(\gamma+1)(2\pi)^{-n/2}|y|^{-n/2-\gamma}J_{n/2+\gamma}(|y|),$$

and since $[u(\varepsilon y)]^\wedge = \varepsilon^{-n}\hat u(y/\varepsilon)$, the lemma follows.

LEMMA 6. *For $|\alpha| = k\geqslant 0$, $\gamma_0 = k+(n-1)/2$,*

$$|[y^\alpha\Phi_{\gamma_0}(\varepsilon|y|)]^\wedge| \leqslant C\begin{cases}|y|^{-n-k},\\ \varepsilon^{-n-k},\end{cases}$$

with C depending only on n and k.

Proof. By [6], p. 45,

$$\frac{\partial}{\partial y_j}[|y|^{-\nu}J_\nu(|y|)] = -y_j|y|^{-\nu-1}J_{\nu+1}(|y|).$$

Continuing, we see that the derivatives of order k of $|y|^{-\nu}J_\nu(|y|)$ are sums of terms

$$y^\beta|y|^{-\nu-m}J_{\nu+m}(|y|)$$

where β is a multi-index and $0\leqslant|\beta|\leqslant m\leqslant k$. By lemma 5, $D^\alpha[\Phi_{\gamma_0}(\varepsilon|y|)]^\wedge = i^k[y^\alpha\Phi_{\gamma_0}(\varepsilon|y|)]^\wedge$ is a constant times a sum of terms

$$\varepsilon^{-n-k}(\varepsilon^{-1}y)^\beta(\varepsilon^{-1}|y|)^{-\nu-m}J_{\nu+m}(\varepsilon^{-1}|y|) \tag{8}$$

where $\nu = n/2+\gamma_0 = n+k-\frac{1}{2}$. In absolute value, (8) is less than

$$\varepsilon^{-n-k}(\varepsilon^{-1}|y|)^{-\nu-m+|\beta|}|J_{\nu+m}(\varepsilon^{-1}|y|)|.$$

Since $|J_\nu(s)| \leqslant s^\nu$ $(s > 0)$, each such term is bounded by $\varepsilon^{-n-k} \leqslant |y|^{-n-k}$ if $|y| \leqslant \varepsilon$. Since $|J_\nu(s)| \leqslant s^{-1/2}$ $(s > 0)$, each term is majorized in $|y| \geqslant \varepsilon$ by

$$|y|^{-n-k}(\varepsilon|y|^{-1})^{m-|\beta|} \leqslant |y|^{-n-k} \leqslant \varepsilon^{-n-k},$$

since $m-|\beta| \geqslant 0$.

LEMMA 7. *For* $\nu \geqslant -\frac{1}{2}$, $\delta > 0$, $F \in L(\delta, \infty)$,

$$\int\limits_{\delta}^{\infty} F(t)t^{1/2}J_\nu(\varepsilon^{-1}t)\,dt = o(\varepsilon^{1/2})$$

as $\varepsilon \to 0$.

For a proof, see [6], p. 457.

LEMMA 8. *If* $h \in L(E^n)$ *and* $|a| = k \geqslant 0$,

$$\lim_{\varepsilon\to 0} \int\limits_{|y|\geqslant\delta} h(x+y)[y^a\Phi_{\gamma_0}(\varepsilon|y|)]^\wedge dy = 0$$

for any $\delta > 0$.

Proof. From (8), $[y^a\Phi_{\gamma_0}(\varepsilon|y|)]^\wedge$ is a constant times a sum of terms

$$\varepsilon^{-1/2+m-|\beta|}y^\beta|y|^{-\nu-m}J_{\nu+m}(\varepsilon^{-1}|y|),$$

$\nu = n+k-\frac{1}{2}$, $0 \leqslant |\beta| \leqslant m \leqslant k$. Hence it is enough to show each

$$\varepsilon^{-1/2}\int\limits_{|y|\geqslant\delta} h(x+y)y^\beta|y|^{-\nu-m}J_{\nu+m}(\varepsilon^{-1}|y|)\,dy$$

tends to zero. This integral may be written

$$\varepsilon^{-1/2}\int\limits_{\delta}^{\infty} F(t)t^{1/2}J_{\nu+m}(\varepsilon^{-1}t)\,dt$$

with

$$F(t) = t^{-k-1-(m-|\beta|)}\int\limits_{|\xi|=1} h(x+t\xi)\,\xi^\beta\,d\xi.$$

Since $h \in L(E^n)$ and $\delta > 0$, $F \in L(\delta, \infty)$. The lemma follows from lemma 7.

To prove theorem 2, we may assume E is a bounded measurable set and consider (lemma 4) the cases (i) $f = g$, $E = P$, and (ii) $f = h$, $E = P$. By Bochner's theorem,

$$D^a\int\limits_{E^n}\hat{g}(y)e^{i(x\cdot y)}\Phi_{\gamma_0}(\varepsilon|y|)\,dy = \int\limits_{E^n}[D^a g]^\wedge(y)e^{i(x\cdot y)}\Phi_{\gamma_0}(\varepsilon|y|)\,dy$$

tends everywhere to $D^a g(x)$, and so in P to $a_a(x)$, $|a| = k \geqslant 1$.

In case (ii), we will show that for $|a| = k$

$$D^a\int\limits_{E^n}\hat{h}(y)e^{i(x\cdot y)}\Phi_{\gamma_0}(\varepsilon|y|)\,dy$$

tends to zero for $x \in P$. In absolute value, the expression above is

$$\Big|\int\limits_{E^n} h(x+y)[y^a\Phi_{\gamma_0}(\varepsilon|y|)]^\wedge dy\Big| \leqslant \Big|\int\limits_{|y|\leqslant\delta}\Big| + \Big|\int\limits_{|y|>\delta}\Big| = A_\varepsilon + B_\varepsilon.$$

By lemma 6,

$$A_\varepsilon \leqslant C_{n,k}\int\limits_{|y|\leqslant\delta}\frac{|h(x+y)|}{|y|^{n+k}}\,dy.$$

By lemma 4 (iv) we may choose $\delta > 0$ so small that A_ε is small for all ε and a given $x \in P$. With δ fixed, B_ε is small with ε by lemma 8.

Before proving theorem C, we restate lemma 8 in the form of a localization theorem. Since

$$D^a\int\limits_{E^n}\hat{f}(y)e^{i(x\cdot y)}\Phi_{\gamma_0}(\varepsilon|y|)\,dy = i^k\int\limits_{E^n} f(x+y)[y^a\Phi_{\gamma_0}(\varepsilon|y|)]^\wedge dy,$$

we obtain the non-periodic version of theorem D:

THEOREM 3. *If* $f \in L(E^n)$ *vanishes in the neighborhood of* x, *then*

$$\int\limits_{E^n}\hat{f}(y)e^{i(x\cdot y)}dy$$

is Bochner-Riesz a-summable of order $\gamma_0 = k+(n-1)/2$ *at* x *to zero*, $|a| = k \geqslant 0$.

For $k = 0$ this theorem is well-known. It is clear from theorem 1 (b) that the result remains true if we increase the order of summability to $\gamma > \gamma_0$.

2) To prove theorem C, only a few words are necessary. We may assume $E \subset Q$. Since lemma 4 holds for functions defined only on an interval, an application of Bochner's theorem in its periodic form shows we may consider only the case $f = h$, $E = P$. Formula (7) holds for $\gamma = \gamma_0$, which can be seen by replacing lemma 2 by lemma 6 where necessary in proving (7). Moreover, lemma 8 holds for periodic $h \in L(Q)$ with the restriction $k \geqslant 1$. For then

$$\int\limits_{|y|>\delta}\frac{h(x+y)}{|y|^{n+k}}\,dy$$

converges absolutely for all x and $\delta > 0$, and therefore the function F of lemma 8 belongs to $L(\delta, \infty)$. The proof of theorem C is now the same as that of theorem 2. Finally, from (7) with $\gamma = \gamma_0$ and the periodic version of lemma 8 $(k \geqslant 1)$ we obtain theorem D.

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Singular integrals and partial differential equations of parabolic type

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To A. Zygmund

Table of contents

Introduction

Important in the study of partial differential equations of parabolic type are classes of singular integrals of the form

$$\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K(x,t;x-y,t-s)f(y,s)\,dy\,ds \tag{1}$$

and

$$\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K(y,s;x-y,t-s)f(y,s)\,dy\,ds. \tag{2}$$