Proof. We have

$$L\{D_1X(c,t)\}(z) = \lim_{b\to c} L\{(X(b,t)-X(c,t))/(b-c)\}(z)$$
$$= \lim_{b\to c}\big(L\{X(b,t)\}(z) - L\{X(c,t)\}(z)\big)/(b-c)$$
$$= \frac{\partial}{\partial b} L\{X(c,t)\}(z);$$

hence the result.

These limit interchanges have been used formally to solve partial differential equations by Laplace transform methods.

Finally, we have the following

THEOREM 10.3. *If $D_1X(c,t)$ exists, then for each $Y(t)$ in $G$*

$$D_1\big(X(c,t)*Y(t)\big) = Y(t)*D_1(c,t).$$

Proof. Since

$$\lim_{b\to c} L\{(X(b,t)*Y(t)-X(c,t)*Y(t))\}/(b-c)$$
$$= L\{Y(t)\}\cdot L\{D_1X(c,t)\} = L\{Y(t)*D_1X(c,t)\},$$

$D_1\big(X(c,t)*Y(t)\big)$ exists and is equal to $Y(t)*D_1X(c,t)$.


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# A kernel associated with certain multiply connected domains and its applications to factorization theorems

by

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**§ 1. Introduction.** In this paper we introduce a natural extension of the kernel

$$\mathscr{P}(z,\zeta) = \frac{1}{2\pi}\frac{\zeta+z}{\zeta-z}, \qquad |z|<1,\ |\zeta|=1,$$

associated with the unit disc. It is well known that many properties of analytic functions on this domain can be derived by making use of $\mathscr{P}(z,\zeta)$, whose real part is the Poisson kernel and whose imaginary part is the conjugate Poisson kernel. The theory of classical $H^p$ spaces, for example, can be easily developed by making use of its basic properties (see [11]). We construct similar kernels associated with multiply connected domains of connectivity $n$. We shall show how they can be used in order to obtain some basic conformal mappings of such domains onto canonical "slit" domains. Our main result is a generalization of the classical factorization theorem for functions in the Nevanlinna class of the unit disc.

Let $\mathscr{D}$ be a bounded domain whose boundary $\Gamma$ consists of $n$ disjoint simple closed analytic curves $\gamma_1,\dots,\gamma_n$ (we shall indicate later in § 9 how to include more general domains in these considerations). We shall always let $\gamma_n$ denote the curve whose interior contains $\mathscr{D}$ (thus $\gamma_1,\dots,\gamma_{n-1}$ are contained in the interior of $\gamma_n$). The existence and basic properties of the kernel associated with such domains will be the subject of § 3 where we shall prove the following theorem:

THEOREM I. *Let $z_0\in\mathscr{D}$ be fixed, then there exists a unique jointly continuous function, $\mathscr{P}$, defined on $\mathscr{D}\times\Gamma$ satisfying*

(1) *for $\zeta\in\Gamma$ fixed $\mathscr{P}(z,\zeta)$ defines an analytic function of $z\in\mathscr{D}$;*

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(2) $\int_{\gamma_k} \mathscr{P}(z, \zeta)\, ds(\zeta)$ *is identically zero for* $k = 1, 2, \ldots, n-1$ *and is* 1 *for* $k = n$, *where* $ds(\zeta)$ *denotes the element of arc length on* $\Gamma$;

(3) *if* $F = u+iv$ *is a continuous function on* $\overline{\mathscr{D}}$ *that is analytic on* $\mathscr{D}$, *then*

$$F(z) = \int_{\Gamma} \mathscr{P}(z, \zeta)\, u(\zeta)\, ds(\zeta) + iv(z_0).$$

The original aspects of this paper consist mainly in the uses of this kernel. Though we have not encountered it in the literature, its existence, even if not previously discovered, is not surprising. As we shall see in § 4, $\mathscr{P}(z, \zeta)$ is easily expressed in terms of the conformal mappings $B(z, a)$ of $\mathscr{D}$ onto the interior of the unit circle with circular slits (centered at 0) removed and such that $B(a, a) = 0$. Conversely, we also have the following formula:

$$B(z, a) = (z-a)\exp\Big\{-\int_{\Gamma} \mathscr{P}(z, \zeta) \log|\zeta - a|\, ds(\zeta)\Big\}. \tag{1.1}$$

As $a$ tends to a point, $a'$, on one of the components $\gamma_k$ of $\Gamma$, the expression on the right is well-defined in the limit, and, thus, $B(z, a)$ tends to

$$B(z, a') = B_k(z, a') = (z-a')\exp\Big\{-\int_{\Gamma} \mathscr{P}(z, \zeta) \log|\zeta - a'|\, ds(\zeta)\Big\}\ (^1). \tag{1.2}$$

We shall show that $B_k(z, a')$ is a conformal mapping of $\mathscr{D}$ onto an annular region about 0 whose outer boundary is the unit circle, having $n-2$ concentric circular slits removed. When $k < n$, this mapping transforms $\gamma_n$ onto the outer boundary and $\gamma_k$ onto the inner boundary of this annulus. When $k = n$, $B_n(z, a')$ is constant of modulus 1.

If $z$ is restricted to a compact subset, $C$, of $\mathscr{D}$, then

$$\int_{\Gamma} \mathscr{P}(z, \zeta) \log|\zeta - a|\, ds(\zeta) = \log(z-a)/B(z, a)$$

has a harmonic extension as a function of $a$ to a domain containing $\overline{\mathscr{D}}$ (²). This domain depends on $C$.

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(¹) Despite the singularity at $\zeta = a'$, the function $\log|\zeta - a'|$ is integrable on $\Gamma$.

(²) This extension, however, is *not* given by the integral in (1.1) when $a \notin \overline{\mathscr{D}}$, even though the latter is well defined. For example, when $\mathscr{P}(z, \zeta)$ is the kernel

$$\frac{1}{2\pi}\, \frac{\zeta + z}{\zeta - z}$$

associated with the unit circle we have $B(z, a) = (z-a)/(1-\bar{a}z)$ for $|a| < 1$, while

$$(z-a)\exp\Big\{-\int_{\Gamma} \mathscr{P}(z, \zeta) \log|\zeta - a|\, ds(\zeta)\Big\} = -a/|a|$$

when $|a| > 1$. This also illustrates the above described behavior of $B(z, a)$ as $a$ tends to a point on the outer boundary.



In § 5 we show that these kernels vary continuously with the domains; in § 6 we study more explicitly the case when $\mathscr{D}$ is an annular region.

In these six sections we assume as known the existence of the Green function for these domains. Otherwise the paper is self-contained. Some of the facts proved are well known; however, since we shall need some of the details in their proofs, it is more expedient to give them here than to refer to several articles and books where the notation varies.

An analytic function, $F$, defined on $\mathscr{D}$ is said to belong to the *Nevanlinna class* $\mathfrak{N} = \mathfrak{N}(\mathscr{D})$ if there exists a harmonic function, $h$, such that $\log^+|F(z)| \leqslant h(z)$ for all $z \in \mathscr{D}$ (where $\log^+ x = \max\{\log x, 0\}$ for $x > 0$ and $\log^+ 0 = 0$). In the seventh section we shall establish the following factorization theorem for such functions:

THEOREM II. *Suppose* $F \in \mathfrak{N}(\mathscr{D})$ *and* $F$ *is not identically* 0. *Then* $F = BG$, *where*

(1) $G$ *is an exponential function of the form*

$$G(z) = \exp\Big\{\int_{\Gamma} \mathscr{P}(z, \zeta)\, d\nu(\zeta)\Big\}$$

*where* $\nu$ *is a finite Borel measure on* $\Gamma$ *determined by* $|F|$;

(2) $B$ *is a bounded analytic function having the same zeros as* $F$;

(3) *The limits*

$$B(\zeta) = \lim_{z\to\zeta\in\Gamma} B(z) \quad \text{and} \quad G(\zeta) = \lim_{z\to\zeta\in\Gamma} G(z),$$

*where* $z$ *tends to the boundary point* $\zeta$ *non-tangentially, exist for almost every* $\zeta \in \Gamma$. *Furthermore, there exist constants* $\lambda_1, \ldots, \lambda_{n-1}$, *depending on* $|F|$ *and* $\mathscr{D}$, *such that* $|B(\zeta)| = e^{\lambda_k}$ *for almost every* $\zeta \in \gamma_k \subset \Gamma$, $k = 1, 2, \ldots, n$, *where* $\lambda_n = 0$.

By the term "almost everywhere" in part (3) of this theorem we mean almost everywhere with respect to the measure on $\Gamma$ defined by arc length. We shall also show that this factorization is unique.

If $\{a_i\}$, $i = 1, 2, \ldots$, are the zeros of a function $F \in \mathfrak{N}(\mathscr{D})$ (counted with their multiplicity) and $a_i'$ satisfies $|a_i - a_i'| = \min_{\zeta\in\Gamma} |a_i - \zeta|$, we shall also show that

$$\sum_{i=1}^{\infty} |a_i - a_i'| < \infty. \tag{1.3}$$

(We remark that $a_i'$ exists and, except for at most a finite number of indices $i$, it is uniquely determined by $a_i$.)

In § 8 we use this result to give the following representation of the function B:

THEOREM III. *There exist integers* $m_1, \ldots, m_{n-1}$ *and points* $b_k \in \gamma_k$, $k = 1, 2, \ldots, n-1$, *such that*

$$B(z) = \left\{\prod_{k=1}^{n-1} [B(z, b_k)]^{m_k}\right\} \left\{\prod_{i=1}^{\infty} \frac{B(z, a_i)}{B(z, a_i')}\right\}.$$

*The infinite product on the right is absolutely and uniformly convergent whenever* $z$ *is restricted to a compact subset of* $\mathscr{D}$.

When $\mathscr{D}$ is the unit circle, the above product is the Blaschke product ([12], Chapter VII). Other authors have obtained related results. In general, they restricted their attention to annular regions. In some cases multiple-valued factors were introduced in the place of the functions $B(z, a)$; moreover, in this connection use was made of the universal covering surface of the domain (see [3], [8] and [10]). We shall discuss in more detail the relation between these works and ours in § 9. There we shall also indicate how to extend our results to other domains. In particular, we shall discuss briefly the case when the boundary of $\mathscr{D}$ contains isolated points.

We begin by obtaining a basic relation between harmonic functions and real parts of analytic (single-valued) functions on $\mathscr{D}$.

**§ 2.** We fix a bounded multiply connected domain, $\mathscr{D}$, whose boundary, $\Gamma$, consists of $n$ analytic Jordan arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$, as described in § 1. We recall that $\gamma_n$ denotes the outer curve. Let $\Delta_k$, $k = 1, 2, \ldots, n$, be a doubly connected subdomain of $\mathscr{D}$ whose boundary consists of $\gamma_k$ and an analytic Jordan curve $\bar{\gamma}_k$ lying in $\mathscr{D}$. We can find such subdomains and an annulus $\mathscr{R} = \{z; R < |z| < 1\}$ such that (i) $\bar{\mathscr{R}}$ is conformally equivalent to $\bar{\Delta}_k$ [3] and (ii) the domains $\Delta_1, \Delta_2, \ldots, \Delta_n$ are mutually disjoint. In order to satisfy (i) we must select the doubly connected regions $\mathscr{R}, \Delta_1, \Delta_2, \ldots, \Delta_n$ so that they have the same modulus (see [5], p. 334). We can certainly do this, obtaining property (ii) at the same time, if we choose $R$ close to 1 and $\bar{\gamma}_k$ close to $\gamma_k$, $k = 1, 2, \ldots, n$.

Property (i) asserts that there exists a conformal map, $\varphi_k$, of $\bar{\mathscr{R}}$ onto $\bar{\Delta}_k$; by composing this mapping with the correspondence $z \to R/z$, if necessary, we can assume that the circumference $|z| = 1$ corresponds to $\gamma_k$. Whenever $R < \varrho < 1$ the circumference $|z| = \varrho$ is mapped by $\varphi_k$ onto an analytic Jordan curve $\gamma_{k,\varrho}$ (lying "between" $\bar{\gamma}_k$ and $\gamma_k$). We let $\mathscr{D}_\varrho$ be the subdomain of $\mathscr{D}$ whose boundary, $\Gamma_\varrho$, consists of the curves $\gamma_{1,\varrho}, \gamma_{2,\varrho}, \ldots, \gamma_{n,\varrho}$.

(3) By this we mean that there exists a conformal map of a domain containing $\bar{\mathscr{R}}$ onto a domain containing $\bar{\Delta}_k$.



We assume as known that there exists a *Green's function* for the domain $\mathscr{D}$; that is, there exists a function $g$ of the form $g(z, \zeta) = h(z, \zeta) - \log|z-\zeta|$, where, for $\zeta \in \mathscr{D}$, $h(z, \zeta)$ is the continuous function of $z \in \bar{\mathscr{D}}$ agreeing with $\log|z-\zeta|$ when $z \in \Gamma$ and harmonic for $z \in \mathscr{D}$. It follows that, for $\zeta \in \mathscr{D}$ fixed, $g(z, \zeta)$ is harmonic when $z$ is restricted to $\mathscr{D} - \{\zeta\}$. Furthermore, we also have the relation $g(z, \zeta) = g(\zeta, z)$.

Let

$$P(z, \zeta) = \frac{-1}{2\pi} \frac{\partial g}{\partial n_\zeta}(z, \zeta)$$

for $z \in \mathscr{D}$ and $\zeta \in \Gamma$, where $\partial/\partial n_\zeta$ denotes the derivative in the direction of the normal pointing towards the exterior of $\gamma_k$ if $\zeta \in \gamma_k$. If $f$ is a continuous function on $\Gamma$, then

$$u(z) = \int_\Gamma P(z, \zeta) f(\zeta)\, ds(\zeta), \tag{2.1}$$

for $z \in \mathscr{D}$, defines the harmonic function having boundary values $f(\zeta)$ (the solution of the Dirichlet problem); that is, defining $u$ by (2.1) when $z \in \mathscr{D}$ and letting $u(z) = f(z)$ for $z \in \Gamma$ the function $u$ is then continuous on $\bar{\mathscr{D}}$, harmonic in $\mathscr{D}$ and equal to $f$ on $\bar{\mathscr{D}} - \mathscr{D} = \Gamma$ (see [5]).

We denote by $\omega_k$, $k = 1, 2, \ldots, n-1$, the harmonic measure corresponding to the component $\gamma_k$ of $\Gamma$; that is, $\omega_k$ is the continuous function on $\bar{\mathscr{D}}$, harmonic in $\mathscr{D}$ and having values identically zero on $\gamma_j$, $j \neq k$, and 1 on $\gamma_k$. Adopting the usual conventions concerning the positive and negative directions of traversing the components of the boundary $\Gamma$ we have, symbolically,

$$\int_\Gamma = \int_{\gamma_n} - \sum_{j=1}^{n-1} \int_{\gamma_j}$$

and, thus, by (2.1)

$$\omega_k(z) = -\int_{\gamma_k} P(z, \zeta)\, ds(\zeta) \quad \text{for} \quad 1 \leqslant k \leqslant n-1, \tag{2.2}$$
$$\omega_n(z) = \int_{\gamma_n} P(z, \zeta)\, ds(\zeta) = 1 - \sum_{k=1}^{n-1} \omega_k(z).$$

Suppose $u$ is a harmonic function on $\mathscr{D}$. Let

$$f = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}.$$

Then $f$ is analytic (in fact, if $u$ were the real part of an analytic function, $F$, on $\mathscr{D}$ we would then have $F' = f$). Let $C$ be a differentiable curve in $\mathscr{D}$ parametrized by arc length. Then, for $z(s)$ on $C$, $z(s) = x(s) + iy(s)$

$$f(z(s))z'(s) = \left\{\frac{\partial u}{\partial x} x'(s) + \frac{\partial u}{\partial y} y'(s)\right\} + i\left\{\frac{\partial u}{\partial x} y'(s) - \frac{\partial u}{\partial y} x'(s)\right\}.$$

The last term is $\partial u(z(s))/\partial n$, the derivative of $u$ in the direction of the normal to $C$ at $z(s)$. If $C$ is a simple closed curve and is traversed in the positive (counter-clockwise) direction as $s$ increases, then the normal is directed towards the exterior of $C$; moreover,

$$\int_C \left(\frac{\partial u}{\partial x} x'(s) + \frac{\partial u}{\partial y} y'(s)\right) ds = 0$$

in case $C$ is closed. Thus

$$\int_C f(z)\,dz = i \int_C \frac{\partial u}{\partial n}\,ds \tag{2.3}$$

for every simple closed differentiable curve $C$ in $\mathscr{D}$.

Let

$$\omega = \sum_{j=1}^{n-1} \lambda_j \omega_j$$

be the harmonic function on $\mathscr{D}$ having real boundary values $\lambda_k$ on $\gamma_k$, $k < n$, and $0$ on $\gamma_n$. Let

$$W = \frac{\partial \omega}{\partial x} - i\frac{\partial \omega}{\partial y} \quad \text{and} \quad W_j = \frac{\partial \omega_j}{\partial x} - i\frac{\partial \omega_j}{\partial y}, \quad j = 1, 2, \ldots, n-1.$$

Then from (2.3) we have

$$\int_{\gamma_k} W(z)\,dz = i\int_{\gamma_k} \frac{\partial \omega}{\partial n}\,ds \quad \text{and} \quad \int_{\gamma_k} W_j(z)\,dz = i\int_{\gamma_k} \frac{\partial \omega_j}{\partial n}\,ds$$

for $k = 1, 2, \ldots, n$. If we define

$$a_{kj} = \frac{1}{2\pi}\int_{\gamma_k} \frac{\partial \omega_j}{\partial n}\,ds, \quad 1 \leqslant j,\ k \leqslant n-1,$$

then Green's theorem and the fact that $\omega$ is 0 on $\gamma_n$ give us

$$\iint |\nabla \omega|^2\,dx\,dy = \int_\Gamma \omega \frac{\partial \omega}{\partial n}\,ds = \sum_{k=1}^{n-1} -\lambda_k \int_{\gamma_k} \frac{\partial \omega}{\partial n}\,ds$$

$$= \sum_{k=1}^{n-1} -\lambda_k \int_{\gamma_k} \sum_{j=1}^{n-1} \lambda_j \frac{\partial \omega_j}{\partial n}\,ds = -2\pi \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} \lambda_k \lambda_j a_{kj}.$$

Thus, if there exist $\lambda_1, \ldots, \lambda_{n-1}$ such that

$$\sum_{j=1}^{n-1} a_{kj}\lambda_j = 0$$



for $k = 1, 2, \ldots, n-1$, then $\omega$ must be constant (since, in this case, $\nabla\omega = 0$). Since $\omega$ is 0 on $\gamma_n$, this would imply that $\omega$ is identically zero on $\mathscr{D}$, which can only happen if $\lambda_1 = \ldots = \lambda_{n-1} = 0$. This shows that the matrix $(a_{kj})$ is non-singular (we have actually shown that $(-a_{kj})$ is positive definite).

The following result is an easy consequence of these considerations:

LEMMA (2.4). *Suppose $u$ is a real-valued harmonic function on $\mathscr{D}$; then there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ such that*

$$v = u - \sum_{j=1}^{n-1} \lambda_j \omega_j$$

*is the real part of an analytic function on $\mathscr{D}$. The constants $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are unique.*

Proof. Let

$$f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

and $\gamma_k' = \gamma_{k,p}$, $k = 1, \ldots, n-1$, Jordan arcs of the type described at the beginning of this section. Then, since $W_j$ is analytic on $\overline{\mathscr{D}}$,

$$\int_{\gamma_k'} W_j(z)\,dz = \int_{\gamma_k} W_j(z)\,dz.$$

By (2.3) we have seen that this last integral equals

$$i\int_{\gamma_k} \frac{\partial \omega_j}{\partial n}\,ds$$

while the former equals

$$i\int_{\gamma_k'} \frac{\partial \omega_j}{\partial n}\,ds.$$

Thus,

$$2\pi a_{kj} = \int_{\gamma_k'} \frac{\partial \omega_j}{\partial n}\,ds.$$

Since the matrix $(a_{kj})$ is non-singular we can find $\lambda_1, \ldots, \lambda_{n-1}$ such that

$$2\pi i \sum_{j=1}^{n-1} \lambda_j a_{kj} = \int_{\gamma_k'} f(z)\,dz = i\int_{\gamma_k'} \frac{\partial u}{\partial n}\,ds \tag{2.5}$$

for $k = 1, 2, \ldots, n-1$ (the last equality follows from (2.3)). Define

$$g(z) = f(z) - \sum_{j=1}^{n-1} \lambda_j W_j(z);$$

then, from (2.5) and the identities established immediately before (2.5),

$$\int_{\gamma_{k,\varrho}} g(z)\,dz = \int_{\gamma'_k} g(z)\,dz = 0$$

for $R < \varrho < 1$ and $k = 1, 2, \ldots, n-1$. But this is equivalent to the fact that

$$\int_C g(z)\,dz = 0$$

for all simple closed curves $C$ in $\mathscr{D}$. Thus, $g$ has a primitive $G = U+iV$ in $\mathscr{D}$. Therefore,

$$\frac{\partial u}{\partial x} - \sum_{j=1}^{n-1} \lambda_j \frac{\partial \omega_j}{\partial x} - i\left(\frac{\partial u}{\partial y} - \sum_{j=1}^{n-1} \frac{\partial \omega_j}{\partial y}\right)$$

$$= g = G' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y}.$$

Comparing real and imaginary parts we obtain the fact that $U$ and

$$u - \sum_{j=1}^{n-1} \lambda_j \omega_j$$

differ by a constant. Since the former is the real part of an analytic function, this establishes the first part of our lemma.

We now need to show that the constants $\lambda_1, \ldots, \lambda_{n-1}$ are unique. But this is an easy consequence of the fact that the matrix $(a_{kj})$ is non-singular: For, if

$$v = u - \sum_{j=1}^{n-1} \lambda_j \omega_j$$

is the real part of an analytic function $G$, then, for each simple closed and differentiable curve $C$ in $\mathscr{D}$,

$$0 = \int_C G'(z)\,dz = \int_C (v_x - iv_y)(z)\,dz = i\int_C \frac{\partial v}{\partial n}\,ds,$$

the last equality being a consequence of (2.3). Letting $C = \gamma'_k$, we then have

$$\sum_{j=1}^{n-1} \lambda_j a_{kj} = \frac{1}{2\pi}\sum_{j=1}^{n-1} \lambda_j \int_{\gamma_k} \frac{\partial \omega_j}{\partial n}\,ds = \frac{1}{2\pi}\int_{\gamma'_k} \frac{\partial u}{\partial n}\,ds, \qquad k = 1, 2, \ldots, n-1.$$



Letting $\mathfrak{M} = (\pi_{jk})$ denote the inverse to the matrix $(a_{kj})$, we then must have, for $j = 1, 2, \ldots, n-1$,

$$\lambda_j = \frac{1}{2\pi}\sum_{k=1}^{n-1} \pi_{jk} \int_{\gamma'_k} \frac{\partial u}{\partial n}\,ds. \tag{2.6}$$

This proves Lemma (2.4).

Letting $\lambda$ be the (column) vector with components $\lambda_j$, $j = 1, \ldots, n$, and $\nu = \nu(u)$ the (column) vector with components

$$\frac{1}{2\pi}\int_{\gamma_k} \frac{\partial u}{\partial n}\,ds, \qquad k = 1, \ldots, n,$$

(2.6) can then be written in the simple form

$$\lambda = \mathfrak{M}\nu. \tag{2.6'}$$

**§ 3.** We now pass to the proof of Theorem I. For $\zeta \in \Gamma$ fixed let us consider $P(z, \zeta)$ as a function of $z \in \mathscr{D}$. By Lemma (2.4) we can find (in a unique way) $\Lambda_j = \Lambda_j(\zeta)$, $j = 1, 2, \ldots, n-1$, such that

$$p(z, \zeta) = P(z, \zeta) - \sum_{j=1}^{n-1} \Lambda_j(\zeta)\omega_j(z)$$

is the real part of an analytic function of $z$. Using (2.6) we can easily deduce that the coefficients $\Lambda_j(\zeta)$, as functions of $\zeta \in \Gamma$, are uniformly continuous. Moreover, $P(z, \zeta)$ and its derivatives with respect to $z$ are jointly continuous in $\mathscr{D}\times\Gamma$. Thus, it follows that $\partial p/\partial x = p_x$ and $\partial p/\partial y = p_y$ $(z = x+iy)$ are jointly continuous in $\mathscr{D}\times\Gamma$ and so is

$$\mathscr{P}(z, \zeta) = \int_{z_0}^{z} \{p_x(w, \zeta) - ip_y(w, \zeta)\}\,dw + p(z_0, \zeta).$$

This integral is independent of the path joining $z_0$ to $z$ since $p(z, \zeta)$ is the real part of an analytic function of $z$. Thus, we have shown property (1) of Theorem I. We remark that the imaginary part of $\mathscr{P}(z_0, \zeta)$ is zero and that $\mathrm{Re}\{\mathscr{P}(z, \zeta)\} = p(z, \zeta)$.

Let $\bar{u}$ be a real-valued continuous function on $\Gamma$ and define, for $z \in \mathscr{D}$,

$$v(z) = \int_\Gamma p(z, \zeta)\bar{u}(\zeta)\,ds(\zeta) = \int_\Gamma \{P(z, \zeta) - \sum_{j=1}^{n-1} \Lambda_j(\zeta)\,\omega_j(z)\}\bar{u}(\zeta)\,ds(\zeta).$$

Since $p(z, \zeta)$ is the real part of an analytic function (of $z$), so is $v(z)$. Applying the uniqueness part of Lemma (2.4) to

$$u(z) = \int_\Gamma P(z, \zeta)\bar{u}(\zeta)\,ds(\zeta)$$

and, then, formula (2.6) we obtain

$$(3.1)\qquad \int_{\Gamma} \Lambda_j(\zeta)\bar{u}(\zeta)\,ds(\zeta) = \lambda_j = \frac{1}{2\pi}\sum_{k=1}^{n-1}\pi_{jk}\int_{\gamma_k}\frac{\partial u}{\partial n}\,ds.$$

From (2.1) we thus have $v(z)\to\bar{u}(\eta)$ when $z\to\eta\in\gamma_n$ and $v(z)\to\bar{u}(\eta)-\lambda_j$ when $\eta\in\gamma_j$, $j=1,2,\ldots,n-1$.

In particular, when

$$(3.2)\qquad \bar{u}(\zeta) = \begin{cases} 1 & \text{for } \zeta\in\gamma_k,\\ 0 & \text{for } \zeta\notin\gamma_k,\end{cases}$$

$k=1,\ldots,n-1$,

$$v(z) = \operatorname{Re}\Big\{\int_{\Gamma}\mathscr{P}(z,\zeta)\bar{u}(\zeta)\,ds(\zeta)\Big\} = \omega_k(z)-1\cdot\omega_k(z) = 0$$

(for $1\cdot\omega_k$ is clearly the (unique) linear combination of $\omega_1,\ldots,\omega_{n-1}$ that, upon subtraction from $\omega_k$, yields the real part of an analytic function in $\mathscr{D}$). Similarly, when $\bar{u}$ is defined as in (3.2) with $k=n$, then

$$v(z) = \operatorname{Re}\Big\{\int_{\Gamma}\mathscr{P}(z,\zeta)\bar{u}(\zeta)\,ds(\zeta)\Big\} = \omega_n(z)-\sum_{k=1}^{n-1}(-1)\,\omega_k(z) = 1.$$

Property (2) now follows since the analytic functions

$$\int_{\gamma_k}\mathscr{P}(z,\zeta)\,ds(\zeta),$$

having constant real values, must have constant imaginary values. The latter, however, must be zero since they clearly are so at $z_0$.

Now suppose $F=u+iv$ is continuous on $\overline{\mathscr{D}}$ and analytic in $\mathscr{D}$. Since $u$ is already the real part of an analytic function $\lambda_j=0$, $1\leqslant j\leqslant n-1$, and

$$u(z) = \int_{\Gamma}p(z,\zeta)u(\zeta)\,ds(\zeta).$$

Hence, the analytic function

$$F(z)-\int_{\Gamma}\mathscr{P}(z,\zeta)u(\zeta)\,ds(\zeta)$$

has real part zero and must, therefore, be a pure imaginary constant $i\alpha = i\alpha_F$. Evaluating at $z_0$ we obtain $\alpha_F = v(z_0)$ and we thus obtain property (3).

If $\mathscr{P}_1(z,\zeta)$ is a kernel satisfying the three properties of Theorem I, it is clear that the third property holds for an $F$ analytic in $\mathscr{D}$ having a real part that is the restriction of a continuous function on $\overline{\mathscr{D}}$ (for we



can approximate $F$ uniformly on compact subsets of $\mathscr{D}$ by rational functions whose poles are outside $\overline{\mathscr{D}}$). In order to show that such a kernel is equal to $\mathscr{P}(z,\zeta)$ it suffices to show that for each $z\in\mathscr{D}$

$$(3.3)\qquad \int_{\Gamma}\{\mathscr{P}(z,\zeta)-\mathscr{P}_1(z,\zeta)\}u_0(\zeta)\,ds(\zeta) = 0$$

for all continuous functions $u_0$ on $\Gamma$. By Lemma (2.4) there exist coefficients $\lambda_1,\ldots,\lambda_{n-1}$ and an analytic function $F$ on $\mathscr{D}$ whose real part is

$$u-\sum_{k=1}^{n-1}\lambda_k\omega_k.$$

By property (3)

$$\begin{aligned}0 &= F(z)-F(z)\\ &= \int_{\Gamma}\{\mathscr{P}(z,\zeta)-\mathscr{P}_1(z,\zeta)\}u_0(\zeta)\,ds(\zeta)-\sum_{k=1}^{n-1}\lambda_k\int_{\gamma_k}\{\mathscr{P}(z,\zeta)-\mathscr{P}_1(z,\zeta)\}\,ds(\zeta).\end{aligned}$$

But the last term is 0, by property (2). Thus, we obtain (3.3) and the uniqueness of our kernel is established.

**§ 4.** We shall now show that the conformal mappings described in the introduction are given by formulae (1.1) and (1.2). More precisely we shall show:

THEOREM IV. *The function defined, for $a\in\mathscr{D}$, by*

$$B(z,a) = (z-a)\exp\Big\{-\int_{\Gamma}\mathscr{P}(z,\zeta)\log|\zeta-a|\,ds(\zeta)\Big\}$$

*is a conformal mapping of $\mathscr{D}$ onto the unit disc $\{w;|w|<1\}$ with $n-1$ circular slits removed. These slits are located on circles about the origin of radii*

$$\varrho_j = \varrho_j(a) = \exp\Big\{\sum_{k=1}^{n-1}\pi_{jk}\omega_k(a)\Big\},\qquad j=1,2,\ldots,n-1.$$

*Furthermore, $B(a,a)=0$.*

*If $a'\in\gamma_k\subset\Gamma$, the function defined by*

$$B(z,a') = (z-a')\exp\Big\{-\int_{\Gamma}\mathscr{P}(z,\zeta)\log|\zeta-a'|\,ds(\zeta)\Big\}$$

*is a conformal mapping of $\mathscr{D}$ onto the annular region $\{w;\exp\pi_{kk}<|w|<1\}$ with $n-2$ circular slits removed. These slits are located on circles about the origin of radii $\varrho_j=\exp\{\pi_{jk}\}$.*

In the course of this proof we will show that *the coefficients of the matrix* $\mathfrak{M} = (\pi_{jk})$ *satisfy* $\pi_{kk} < \pi_{jk} < 0$ *for* $j, k = 1, \ldots, n-1$, $j \neq k$ [4].

We fix $a \in \mathscr{D}$ and $\gamma_k' = \gamma_{k,\varrho}$, $k = 1, \ldots, n-1$, the Jordan arcs introduced in the proof of Lemma (2.4). We choose $\varrho$ close enough to 1 so that $a$ is exterior to each $\gamma_k' = \gamma_{k,\varrho}$. Thus, $\log|z-a|$ is the real part of an analytic function inside and on $\gamma_k'$. Hence, by (2.3) and Cauchy's theorem,

$$\int_{\gamma_k'} \frac{\partial}{\partial n_\zeta} \log|\zeta - a|\, ds(\zeta) = 0. \tag{4.1}$$

Recall that $h(z, \zeta) = g(z, \zeta) + \log|z-\zeta|$ denoted the solution of the Dirichlet problem having boundary value $\log|z-\zeta|$. Thus, by (4.1) and (3.1), with $\bar{u}(\zeta) = \log|\zeta - a|$ and $u(z) = h(z, a)$,

$$\int_\Gamma \Lambda_j(\zeta)\log|\zeta-a|\, ds(\zeta) = \frac{1}{2\pi}\sum_{k=1}^{n-1} \pi_{jk} \int_{\gamma_k'} \frac{\partial h(\zeta, a)}{\partial n_\zeta}\, ds(\zeta)$$
$$= \frac{1}{2\pi}\sum_{k=1}^{n-1} \pi_{jk} \int_{\gamma_k'} \frac{\partial g(\zeta, a)}{\partial n_\zeta}\, ds(\zeta).$$

Because of the symmetry of the Green function, the integrals in the last expression tend to

$$-2\pi \int_{\gamma_k} P(a, \zeta)\, ds(\zeta) = 2\pi\omega_k(a)$$

as $\varrho$ tends to 1 (see (2.2)). Thus,

$$\int_\Gamma \Lambda_j(\zeta)\log|\zeta-a|\, ds(\zeta) = \sum_{k=1}^{n-1} \pi_{jk}\,\omega_k(a). \tag{4.2}$$

Let

$$L(z, a) = \int_\Gamma \mathscr{P}(z, \zeta)\log|\zeta - a|\, ds(\zeta).$$

Then, by (4.2),

$$\mathrm{Re}\{L(z, a)\} = \int_\Gamma \Big\{P(z, \zeta) - \sum_{j=1}^{n-1} \Lambda_j(\zeta)\,\omega_j(z)\Big\} \log|\zeta - a|\, ds(\zeta) \tag{4.3}$$
$$= h(z, a) - \sum_{j=1}^{n-1} \omega_j(z) \sum_{k=1}^{n-1} \pi_{jk}\,\omega_k(a) = \varphi(z, a).$$

[4] We remark that $\mathfrak{M}$ is a symmetric matrix. This follows from the fact that the same is true for its inverse $\mathfrak{A} = (a_{jk})$. This last fact is an immediate consequence of the symmetry of the Green function and the definition of the coefficients

$$a_{jk} = \frac{1}{2\pi}\int_{\gamma_j} \frac{\partial \omega_k(z)}{\partial n_z}\, ds(z) = -\frac{1}{4\pi^2}\int_{\gamma_j}\int_{\gamma_k} \frac{\partial g(z, \zeta)}{\partial n_z \partial n_\zeta}\, ds(z)\, ds(\zeta) \quad \text{(see (2.2))}.$$



We now claim that, *if $z$ is restricted to a compact subset $C \subset \mathscr{D}$, $L(z, a)$ has a harmonic extension, as a function of $a$, to a domain $\mathscr{D}_C \supset \overline{\mathscr{D}}$.* Because of the analyticity of the boundary curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ and the fact that $g(z, a) = g(a, z) = 0$ when $a \in \Gamma$, we can apply the Schwarz reflection principle to obtain such a harmonic extension of $g(z, a)$ as a function of $a$; the same is true for the harmonic measures $\omega_1(a), \ldots, \omega_{n-1}(a)$. Thus, our claim is established for the function $\varphi(z, a) = \mathrm{Re}\{L(z, a)\}$ since it is a linear combination of $h(z, a) = g(z, a) + \log|z - a|$ and $\omega_1(a), \ldots, \omega_{n-1}(a)$ (see (4.3)). By the Cauchy-Riemann equations,

$$L'(z, a) = \frac{\partial}{\partial z} L(z, a) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\varphi(z, a).$$

Thus, $L'(z, a)$ also has an extension (as a function of $a$) that is harmonic on such a domain $\mathscr{D}_C \supset \mathscr{D}$. The same, therefore, can be said of $L(z, a)$ [5].

Let $B(z, a) = (z-a)e^{-L(z,a)}$. Then, for $z \in \gamma_j$, $j = 1, 2, \ldots, n-1$, $h(z, a) = \log|z - a|$ and (see (4.3))

$$|B(z, a)| = \exp\Big\{\sum_{k=1}^{n-1} \pi_{jk}\,\omega_k(a)\Big\} = \varrho_j(a) \tag{4.4}$$

while, for $z \in \gamma_n$,

$$|B(z, a)| = 1. \tag{4.4'}$$

Thus, the images of the curves $\gamma_j$ are contained in circles $C_j$ centered at zero with radii $\varrho_j(a)$, $j = 1, \ldots, n$. Moreover $\varrho_n = 1$.

Suppose that $w$ is a complex number not on any such circle; we show that

$$\frac{1}{2\pi i}\int_\Gamma \frac{(B(z, a) - w)'}{B(z, a) - w}\, dz = \begin{cases} 1 & \text{if } |w| < 1, \\ 0 & \text{if } |w| > 1. \end{cases} \tag{4.5}$$

Thus, $B(z, a)$ assumes each such value, $w$, of modulus less than 1, precisely once and maps $\mathscr{D}$ into the unit disc. This clearly implies that $B(z, a)$ is a conformal mapping of $\mathscr{D}$ onto the unit disc with slits on the circles $C_j$, $j = 1, 2, \ldots, n-1$, removed.

[5] Recall that $\mathscr{P}(z_0, \zeta)$ has imaginary part 0; thus, the same is true of $L(z_0, a)$. Hence,

$$L(z, a) = \int_{z_0}^{z} L'(w, a)\, dw + \varphi(z_0, a).$$

If we assume that the path of integration joining $z_0$ to $z$ can always be chosen in $C$, our assertion about $L(z, a)$ is obviously true. If this is not the case, we may include $C$ in a larger, arcwise connected, compact subset of $\mathscr{D}$ in which such paths can be found.

Let $\Delta_C \arg F$ denote the increment of the argument of the function $F$ along the curve $C$. It is well known that

$$\frac{1}{2\pi i}\int_{\gamma_k}\frac{\big(B(z,a)-w\big)'}{B(z,a)-w}\,dz=\frac{1}{2\pi}\Delta_{\gamma_k}\arg\{B(z,a)-w\}. \tag{4.6}$$

We recall that the increment of the argument, along a closed curve $C$, of the sum of two continuous functions, one of which has modulus strictly larger than the other on $C$, equals the increment of the argument of the former. Thus, if $|w|>|B(z,a)|$ for $z\in\gamma_k$, then

$$\Delta_{\gamma_k}\arg\{B(z,a)-w\}=\Delta_{\gamma_k}\arg w=0.$$

On the other hand, if $|w|<|B(z,a)|$ for $z\in\gamma_k$, then

$$\Delta_{\gamma_k}\arg\{B(z,a)-w\}=\Delta_{\gamma_k}\arg\{B(z,a)\}.$$

But $B(z,a)$ is the product of an exponential, whose contribution to this increment is clearly 0, and $z-a$. Thus, in this case,

$$\frac{1}{2\pi}\Delta_{\gamma_k}\arg B(z,a)=\frac{1}{2\pi}\Delta_{\gamma_k}(z-a)=\delta_{kn}$$

(the Kronecker $\delta$), since $a\in\mathscr{D}$ implies that it is interior to $\gamma_n$, but exterior to $\gamma_1,\ldots,\gamma_{n-1}$. This, together with (4.6), implies (4.5).

We remark that we have shown that $\varrho_j(a)<1$, $j=1,\ldots,n-1$; that is,

$$\sum_{k=1}^{n-1}\pi_{jk}\,\omega_k(a)<0\qquad\text{for all } a\in\mathscr{D}.$$

Letting $a$ tend to one of the boundary curves $\gamma_k$ $(1\leqslant k<n)$ we obtain $\pi_{jk}\leqslant 0$. The sharper result announced immediately following Theorem IV will be apparent at the end of our study of the function

$$B(z,a')=(z-a')\exp\Big\{-\int_\Gamma \mathscr{P}(z,\zeta)\log|\zeta-a'|\,ds(\zeta)\Big\},$$

where $a'\in\gamma_k\subset\Gamma$. This study is very similar to the one just completed; consequently, we can omit some of the details.

From (4.3), by taking the limit as $a\to a'$, we obtain

$$|B(z,a')|=\exp\Big\{\sum_{j=1}^{n-1}\pi_{jk}\,\omega_j(z)\Big\}. \tag{4.7}$$

Thus, for $z\in\gamma_j$, $j=1,2,\ldots,n-1$,

$$|B(z,a')|=e^{\pi_{jk}}=\varrho_j=\varrho_j^{(k)} \tag{4.8}$$

while, for $z\in\gamma_n$,

$$|B(z,a')|=1. \tag{4.8'}$$



Thus the images of the curves $\gamma_j$ are contained in circles $C_j$ centered at zero with radii $\varrho_j=\varrho_j^{(k)}$, $j=1,\ldots,n$, with $\varrho_n=1$. As in the preceding case, the fact that $B(z,a')$ is a conformal mapping follows from an evaluation of the integrals of the logarithmic derivatives of $B(z,a')-w$ along the boundary curve $\Gamma$. In order to obtain these evaluations we first show that

$$\frac{1}{2\pi i}\int_{\gamma_j}\frac{B'(z,a')}{B(z,a')}\,dz=\frac{1}{2\pi}\Delta_{\gamma_j}\arg B(z,a')=\begin{cases}1 & \text{if } j=k,n,\\ 0 & \text{if } j\neq k,n.\end{cases} \tag{4.9}$$

By (2.3) and (4.7) we have, for $j<n$ (recall that $\mathfrak{M}^{-1}=(\pi_{jk})^{-1}=(a_{jk})=\mathfrak{A}$),

$$\frac{1}{2\pi i}\int_{\gamma_j}\frac{B'(z,a')}{B(z,a')}\,dz=\frac{1}{2\pi}\int_{\gamma_j}\frac{\partial}{\partial n_z}\big(\log|B(z,a')|\big)\,ds(z)$$
$$=\frac{1}{2\pi}\int_{\gamma_j}\sum_{m=1}^{n-1}\pi_{mk}\frac{\partial}{\partial n_z}\omega_m(z)\,ds(z)=\sum_{m=1}^{n-1}\pi_{mk}\,a_{jm}=\delta_{jk}.$$

On the other hand, since $B'(z,a')/B(z,a')$ is analytic in $\mathscr{D}$ (the denominator is never zero)

$$0=\frac{1}{2\pi i}\int_\Gamma\frac{B'(z,a')}{B(z,a')}\,dz=\frac{1}{2\pi i}\Big(\int_{\gamma_n}-\sum_{j=1}^{n-1}\int_{\gamma_j}\Big)\frac{B'(z,a')}{B(z,a')}\,dz$$
$$=\frac{1}{2\pi}\int_{\gamma_n}\frac{\partial}{\partial n_z}\log|B(z,a')|\,ds(z)-\frac{1}{2\pi}\sum_{j=1}^{n-1}\int_{\gamma_j}\frac{\partial\log|B(z,a')|}{\partial n_z}\,ds(z).$$

Thus,

$$\frac{1}{2\pi i}\int_{\gamma_n}\frac{B'(z,a')}{B(z,a')}\,dz=1,$$

and (4.9) is established.

Since $-\mathfrak{M}$ is a positive definite matrix [6], it follows that $\pi_{kk}<0$ $(k=1,2,\ldots,n-1)$. Thus, the ring $\{w;\,e^{\pi_{kk}}<|w|<1\}$ is non-empty. If $w$ is in this ring, but is not on any of the circles $C_j$, then from (4.9) and an argument similar to the previous one we obtain

$$\frac{1}{2\pi}\Delta_\Gamma\arg\{B(z,a')-w\}=1.$$

[6] See the discussion immediately preceding Lemma (2.4), where we show that $-\mathfrak{M}^{-1}=-\mathfrak{A}$ is positive definite.

On the other hand, for $|w|>1$ or $|w|<e^{\pi_{kk}}$ $(w\notin C_j)$ we have (again using (4.9))

$$\frac{1}{2\pi}\Delta_\Gamma \arg\{B(z,a')-w\}=0.$$

We thus obtain the analog to (4.5) (see (4.6))

$$\frac{1}{2\pi i}\int_\Gamma \frac{(B(z,a')-w)'}{B(z,a')-w}\,dz=\begin{cases}1 & \text{if } e^{\pi_{kk}}<|w|<1,\\ 0 & \text{if } |w|<e^{\pi_{kk}} \text{ or } |w|>1\end{cases}$$

as long as $w\notin C_j$, $j=1,2,\ldots,n$, and the second part of Theorem IV is established.

It is now easy to show by similar arguments, using the identities (4.4), (4.4′), (4.8) and (4.8′), that the following result is also true:

THEOREM. *Let $a\in\mathcal{D}$ and $a'\in\gamma_k\subset\Gamma$; then the function $B(z,a)/B(z,a')$ is a conformal mapping of $\mathcal{D}$ onto the interior of the disc*

$$\Big\{w;\ |w|<\exp\Big(\sum_{j=1}^{n-1}\pi_{kj}\,\omega_j(a)-\pi_{kk}\Big)=\tau_k=\tau_k(a)\Big\}$$

*with concentric circular slits (centered at 0) removed. The curve $\gamma_k$ is mapped into the circle about 0 of radius $\tau_k(a)$ and the point $a$ is mapped onto 0.*

We omit the proof, as well as the calculation of the radii of the circles containing these slits. It is obvious, however, that $\gamma_n$ corresponds to a slit on $|w|=1$.

We can characterize the analytic functions on $\overline{\mathcal{D}}$ with the help of the conformal mappings of Theorem IV. We first prove the following result concerning non-zero analytic functions:

THEOREM (4.10). *Suppose $F$ is analytic and never zero on $\overline{\mathcal{D}}$; then there exist integers $m_1, m_2, \ldots, m_{n-1}$ such that for any $n-1$ points $a_j'\in\gamma_j$, $j=1,\ldots,n-1$,*

$$F(z)=\gamma\prod_{j=1}^{n-1}\{B(z,a_j')\}^{m_j}\cdot\exp\Big\{\int_\Gamma \mathscr{P}(z,\zeta)\log|F(\zeta)|\,ds(\zeta)\Big\},$$

*where $\gamma$ is a constant of absolute value 1.*

Proof. Let (see (2.3))

$$m_k=\frac{1}{2\pi}\int_{\gamma_k}\frac{\partial\log|F|}{\partial n}\,ds=\frac{1}{2\pi i}\int_{\gamma_k}\frac{F'(z)}{F(z)}\,dz.$$

Since $F'/F$ is analytic in $\overline{\mathcal{D}}$, $m_1,\ldots,m_{n-1}$ are integers. Choose any $n-1$ points $a_1',\ldots,a_{n-1}'$ on the curves $\gamma_1,\ldots,\gamma_{n-1}$. From (4.7) and the fact that $\mathfrak{A}=(a_{jk})=(\pi_{jk})^{-1}=\mathfrak{M}^{-1}$ we have

$$\omega_j(z)=\sum_{m=1}^{n-1}a_{jm}\log|B(z,a_m')|. \tag{4.11}$$



Thus, from (3.1) with $u=\log|F|$ and (4.11),

$$\begin{aligned}\operatorname{Re}\Big\{\int_\Gamma \mathscr{P}(z,\zeta)\log|F(\zeta)|\,ds(\zeta)\Big\}&=\log|F(z)|-\sum_{j=1}^{n-1}\lambda_j\omega_j(z)\\&=\log|F(z)|-\sum_{j=1}^{n-1}\Big(\sum_{k=1}^{n-1}\pi_{jk}m_k\Big)\Big\{\sum_{m=1}^{n-1}a_{jm}\log|B(z,a_m')|\Big\}\\&=\log|F(z)|-\sum_{k,m=1}^{n-1}\delta_{km}\log|B(z,a_m')|^{m_k}\\&=\log|F(z)|-\sum_{j=1}^{n-1}\log|B(z,a_j')|^{m_j}.\end{aligned}$$

This shows that the analytic functions $F(z)$ and

$$\prod_{j=1}^{n-1}\{B(z,a_j')\}^{m_j}\cdot\exp\Big\{\int_\Gamma \mathscr{P}(z,\zeta)\log|F(\zeta)|\,ds(\zeta)\Big\}$$

have the same absolute value. Thus one must be a constant multiple of absolute value 1 times the other; this, however, is the assertion in our theorem.

In this proof we can replace $\log|F|$ by a function $u$ that is harmonic on $\overline{\mathcal{D}}$. We then obtain numbers $m_1,\ldots,m_{n-1}$ (not necessarily integers) such that

$$u(z)=\operatorname{Re}\Big\{\int_\Gamma \mathscr{P}(z,\zeta)u(\zeta)\,ds(\zeta)\Big\}+\sum_{j=1}^{n-1}\log|B(z,a_j')|^{m_j}, \tag{4.12}$$

for $z\in\mathcal{D}$.

If $F$ is a general analytic function on $\overline{\mathcal{D}}$, then it has at most a finite number of zeros, $\{a_j\}$, in $\mathcal{D}$ (we assume $F$ is not identically zero and that it is not zero on $\Gamma$). Thus

$$G(z)=F(z)/\prod_{a_j\in\mathcal{D}}B(z,a_j)$$

satisfies the hypotheses of Theorem (4.10). Applying this theorem to $G$ we obtain

$$G(z)=\gamma\prod_{j=1}^{n-1}\{B(z,a_j')\}^{m_j}\cdot\exp\Big\{\int_\Gamma \mathscr{P}(z,\zeta)\log|G(\zeta)|\,ds(\zeta)\Big\}.$$

But

$$\int_\Gamma \mathscr{P}(z,\zeta)\log|G(\zeta)|\,ds(\zeta)=\int_\Gamma \mathscr{P}(z,\zeta)\log|F(\zeta)|\,ds(\zeta)-\int_\Gamma\sum_{a_j\in\mathcal{D}}\mathscr{P}(z,\zeta)\log|B(\zeta,a_j')|\,ds(\zeta).$$

Since $|B(\zeta, a_j')|$ is constant on each of the curves $\gamma_1, \ldots, \gamma_{n-1}$ and is 1 on $\gamma_n$ it follows from (2) of Theorem I that the last term is 0. Thus

$$(4.13)\quad F(z) = \gamma \prod_{j=1}^{n-1} \{B(z, a_j')\}^{m_j} \prod_{a_j \in \mathscr{D}} B(z, a_j) \exp\Big\{\int_{\Gamma} \mathscr{P}(z, \zeta) \log|F(\zeta)|\, ds(\zeta)\Big\}.$$

We remark that if we take for $F(z)$ the function $z-a$, with $a \in \mathscr{D}$, (4.13) gives us the formula for $B(z, a)$ of Theorem IV.

Formula (4.13) shows how such an analytic function can be obtained from its zeros and its absolute value on $\Gamma$. Theorems II and III give us such a result when $F$ is considerably more general.

**§ 5.** We shall now show that the kernel we have introduced varies continuously with the domain. In order to make this statement precise we need to introduce some notation. Consider the domains $\mathscr{D}_\varrho$, $R < \varrho \leqslant 1$, introduced in § 2 ($\mathscr{D}_1 = \mathscr{D}$) by means of the conformal maps $\varphi_k$ ($k = 1, 2, \ldots, n$). If $\zeta = \varphi_k(e^{i\theta}) \in \gamma_k \subset \Gamma$, we let $\zeta_\varrho = \varphi_k(\varrho e^{i\theta})$ for $R < \varrho \leqslant 1$. Let $z_0 \in \mathscr{D}_\varrho$ for $R \leqslant \varrho_0 < \varrho \leqslant 1$ and $\mathscr{P}_\varrho(z, \zeta_\varrho)$ the kernel of Theorem I associated with the domain $\mathscr{D}_\varrho$ and the point $z_0$. We shall derive the following result:

THEOREM (5.1). *$\mathscr{P}_\varrho(z, \zeta_\varrho)$ tends to $\mathscr{P}(z, \zeta)$ as $\varrho < 1$ tends to 1, the convergence being uniform in $\theta$, as $\zeta_\varrho = \varphi_k(\varrho e^{i\theta})$ approaches $\varphi_k(e^{i\theta}) = \zeta \in \gamma_k \subset \Gamma$, and in $z$, if the latter is restricted to a compact subset of $\mathscr{D}_\varrho$, $\varrho > \varrho_0$.*

We prove this theorem by establishing a sequence of lemmas. In the sequel we let $g_\varrho(z, \zeta)$ denote the Green function associated with the domain $\mathscr{D}_\varrho$. Thus, $g_1(z, \zeta) = g(z, \zeta)$.

LEMMA (5.2). *Suppose $\zeta \in \overline{\mathscr{D}}_{\varrho_0}$; then there exists a number $\varrho_1$, $\varrho_0 < \varrho_1 < 1$, and a domain $\mathscr{D}^* \supset \overline{\mathscr{D}}$ such that $g_\varrho(z, \zeta)$, $\varrho_1 < \varrho \leqslant 1$, has a harmonic extension, as a function of $z$, to the domain $\mathscr{D}^* - \{\zeta\}$.*

Proof. We first observe that if $u_0$ is continuous in the ring $\{z; R_1 < |z| \leqslant R_2\}$, harmonic in its interior and identically zero on the boundary circle $\{z; |z| = R_2\}$, then $u_0$ has a harmonic extension to a harmonic function, $u$, on the ring $\{z; R_1 < |z| < R_2^2/R_1\}$. In fact, the harmonic extension, $u$, can be defined by letting

$$u(z) = \begin{cases} -u_0\left(\dfrac{R_2^2}{\bar z}\right) & \text{for} \quad R_2 \leqslant |z| < R_2^2/R_1, \\ u_0(z) & \text{for} \quad R_1 < |z| \leqslant R_2. \end{cases}$$

The function $u$ so defined is clearly continuous in the larger ring and is, locally, the imaginary part of an analytic function that is real-valued when $|z| = R_2$ (to see this, we can use the fact that in a simply connected subset of $\{z; R_1 < |z| \leqslant R_2\}$, whose boundary contains an arc of $|z| = R_2$, a harmonic conjugate, $v$, of $u_0$ converges uniformly as we tend to the points of a compact subset of this arc. This reduces our problem to the well known Schwarz reflection principle for analytic functions).



Now let $u_\varrho(z) = g_\varrho(\varphi_k(z), \zeta)$. Thus, $u_\varrho$ is harmonic in the ring $\{z; \varrho_0 < |z| < \varrho\}$, is continuous on $\{z; \varrho_0 < |z| \leqslant \varrho\}$ and zero when $|z| = \varrho$. Let $\varrho_1^2 > \varrho_0$ be such that the ring $\mathscr{R}_1 = \{z; \varrho_0 < |z| < \varrho_1^2/\varrho_0\}$ is contained in the domain of $\varphi_k$ (recall that the $\varphi_k$'s, introduced in § 2, were defined on a domain containing $\overline{\mathscr{R}}$). Thus, for $\varrho_1 < \varrho < 1$, $u_\varrho(z)$ has a harmonic extension to $\{z; \varrho_0 < |z| < \varrho^2/\varrho_0\} \supset \mathscr{R}_1$. Letting $u_\varrho$ also denote this extension to $\mathscr{R}_1$ we obtain the desired extension by putting $g_\varrho(z, \zeta) = u_\varrho(\varphi_k^{-1}(z))$ for $z$ in the images of $\mathscr{R}_1$ under $\varphi_k$ ($k = 1, 2, \ldots, n$). This certainly agrees with $g_\varrho(z, \zeta)$ for $z \in \mathscr{D}_\varrho$ and we obtain an extension of $g_\varrho$ to the domain $\mathscr{D}^*$ bounded by the curves that are the images, under $\varphi_k$, of the circles $\{z; |z| = \varrho_1^2/\varrho_0\}$.

An immediate consequence is the following corollary:

COROLLARY (5.3). *The functions $h_\varrho(z, \zeta) = g_\varrho(z, \zeta) + \log|z - \zeta|$ have harmonic extensions, as functions of $z$, to $\mathscr{D}^*$, provided $\varrho_1 < \varrho \leqslant 1$.*

We let $h_\varrho$ also denote this extension.

LEMMA (5.4). $\lim\limits_{\varrho \to 1-0} h_\varrho(z, \zeta) = h(z, \zeta)$, *the convergence being uniform in $z \in \mathscr{D}^*$ and $\zeta \in \overline{\mathscr{D}}_{\varrho_0}$.*

Proof. $h_\varrho(z, \zeta)$ is a harmonic function of $z$ having boundary values $\log|z - \zeta|$ for $z \in \Gamma_\varrho$. Thus, from the maximum principle and the joint continuity of $h$,

$$|h_\varrho(z, \zeta) - h(z, \zeta)| \leqslant \max_{w \in \Gamma_\varrho} \big|\log|w - \zeta| - h(w, \zeta)\big| \to 0 \quad \text{as} \quad \varrho \to 1$$

uniformly in $\zeta \in \overline{\mathscr{D}}_{\varrho_0}$.

Since the values $h_\varrho$ assumes in $\mathscr{D}^* - \overline{\mathscr{D}}_\varrho$ are obtained by "reflecting" the values of $g_\varrho$ in $\mathscr{D}_\varrho - \overline{\mathscr{D}}_{\varrho_0}$ (see the proof of (5.2)), the lemma follows immediately from the relation $h_\varrho(z, \zeta) = g_\varrho(z, \zeta) + \log|z - \zeta|$.

This lemma implies that the derivatives of $h_\varrho(z, \zeta)$, with respect to $x$ and $y$, where $x + iy = z$, converge uniformly when $z$ is restricted to a compact subset of $\mathscr{D}^*$ and $\zeta \in \overline{\mathscr{D}}_{\varrho_0}$. Since the functions $h_\varrho$ and $h$ are symmetric, if we let $z = \varphi_k(e^{i\theta})$ and $z_\varrho = \varphi_k(\varrho e^{i\theta})$, we thus have, for $k = 1, 2, \ldots, n-1$,

$$(5.5)\quad \frac{\partial h_\varrho(\zeta, z_\varrho)}{\partial n_{z_\varrho}} \to \frac{\partial h(\zeta, z)}{\partial n_z}$$

as $\varrho \to 1-0$, uniformly in $\theta$ and $\zeta \in \overline{\mathscr{D}}_{\varrho_0}$. Letting

$$P_\varrho(\zeta, z_\varrho) = \frac{-1}{2\pi} \frac{\partial g(\zeta, z_\varrho)}{\partial n_{z_\varrho}}$$

be the Poisson kernel of the domain $\mathscr{D}_\varrho$, (5.5) clearly implies (changing the notation in (5.6) so that it is consistent with that used in the announcement of Theorem (5.1))

$$P_\varrho(z, \zeta_\varrho) \to P(z, \zeta) \tag{5.6}$$

as $\varrho \to 1-0$, uniformly in $\theta$ and $z \in \overline{\mathscr{D}}_{\varrho_0}$. Hence (see discussion immediately preceding (3.1)), since the imaginary part of all the $\mathscr{P}_\varrho(z_0, \zeta)$ is 0,

$$\mathscr{P}_\varrho(z, \zeta_\varrho) = \int_{\Gamma_{\varrho_0}} \mathscr{P}_{\varrho_0}(z, w) P_\varrho(w, \zeta_\varrho)\, ds(w)$$

$$\to \int_{\Gamma_{\varrho_0}} \mathscr{P}_{\varrho_0}(z, w) P(w, \zeta)\, ds(w) = \mathscr{P}(z, \zeta), \quad \text{as} \quad \varrho \to 1-0,$$

uniformly in $\theta$ and $z$ restricted to a compact subset of $\mathscr{D}_{\varrho_0}$. Since $\varrho_0$ could have been chosen so close to 1 that a given compact subset of $\mathscr{D}$ is contained in $\mathscr{D}_{\varrho_0}$, Theorem (5.1) is proved.

**§ 6.** We now consider the example obtained when $\mathscr{D}$ is the annular region $\{z \in C;\, R_1 < |z| < R_2\}$, whose boundary consists of the circle $\gamma_1 = \{z \in C;\, |z| = R_1\}$ and the circle $\gamma_2 = \{z \in C;\, |z| = R_2\}$.

Let

$$\varphi = \frac{1}{2\pi|\zeta|}\left\{\frac{\zeta+z}{\zeta-z} + 2\sum_{k=1}^{\infty} \frac{R_1^{2k}}{R_2^{2k}-R_1^{2k}}\left[\left(\frac{z}{\zeta}\right)^k - \left(\frac{z}{\zeta}\right)^{-k}\right] + \frac{\log|\zeta| - \log R_2}{\log R_1 - \log R_2}\right\}.$$

It is not hard to show that $\mathscr{P}(z, \zeta) = \varphi(z, \zeta) - i\,\mathrm{Im}\{\varphi(z_0, \zeta)\}$ satisfies properties (1), (2), (3) of Theorem I. Indeed, if $(z, \zeta) \in \mathscr{D} \times \Gamma$, then, letting $z = re^{i\theta}$,

$$\frac{R_1^{2k}}{R_2^{2k}-R_1^{2k}}\left|\left(\frac{z}{\zeta}\right)^k - \left(\frac{z}{\zeta}\right)^{-k}\right| \leqslant \frac{R_1^{2k}}{R_2^{2k}-R_1^{2k}}\left[\left(\frac{r}{R_1}\right)^k + \left(\frac{R_2}{r}\right)^k\right]$$

$$= \frac{(R_1/R_2)^k}{1-(R_1/R_2)^k}\left(\frac{r}{R_2}\right)^k + \frac{(R_1/R_2)^k}{1-(R_1/R_2)^k}\left(\frac{R_1}{r}\right)^k,$$

from which we obtain that the series occurring in the expression for $\varphi(z, \zeta)$ is absolutely convergent; when $z$ is restricted to a compact subset of $\mathscr{D}$ it is uniformly convergent. Thus, property (1) holds.

If we let $\zeta = \varrho e^{i\Phi}$, we have

$$\varphi(re^{i\theta}, \varrho e^{i\Phi}) = \frac{1}{2\pi\varrho}\left\{\frac{\varrho^2 - r^2}{\varrho^2 - 2\varrho r\cos(\theta-\Phi) + r^2} + \right.$$
$$\left. +2\sum_{k=1}^{\infty} \frac{R_1^{2k}}{R_2^{2k}-R_1^{2k}}\,\frac{r^{2k}-\varrho^{2k}}{(r\varrho)^k}\cos k(\theta-\Phi) + \frac{\log\varrho - \log R_2}{\log R_1 - \log R_2}\right\} +$$
$$+\frac{i}{2\pi\varrho}\left\{\frac{2\varrho r\sin(\theta-\Phi)}{\varrho^2 - 2\varrho r\cos(\theta-\Phi) + r^2} + 2\sum_{k=1}^{\infty}\frac{R_1^{2k}}{R_2^{2k}-R_1^{2k}}\,\frac{r^{2k}+\varrho^{2k}}{(r\varrho)^k}\sin k(\theta-\Phi)\right\}. \tag{6.1}$$



Since

$$\frac{1}{2\pi\varrho}\int_0^{2\pi} \frac{\varrho^2 - r^2}{\varrho^2 - 2\varrho r\cos(\theta-\Phi) + r^2}\,\varrho\, d\theta$$

is 1 when $\varrho = R_2$ and equals $-1$ when $\varrho = R_1$ (for this is the integral of the Poisson kernel for the interior and exterior of a circle), it follows that

$$\int_0^{2\pi} \varphi(re^{i\theta}, R_2 e^{i\Phi}) R_2\, d\Phi = 1 \quad \text{and} \quad \int_0^{2\pi} \varphi(re^{i\theta}, R_1 e^{i\Phi}) R_1\, d\Phi = 0.$$

But the first integral is

$$\int_{\gamma_2} \varphi(z, \zeta)\, ds(\zeta) = 1$$

while the second is

$$\int_{\gamma_1} \varphi(z, \zeta)\, ds(\zeta) = 0.$$

Clearly, the same is true for $\mathscr{P}(z, \zeta)$. Thus, property (2) is satisfied.

In order to see that property (3) holds we note that it follows immediately from (6.1) that $\mathrm{Re}\{\mathscr{P}(R_2 e^{i\theta}, R_1 e^{i\Phi})\} = 0$, while, if $F = u + iv$ is a continuous function on $\overline{\mathscr{D}}$ that is analytic in $\mathscr{D}$,

$$\int_{\gamma_2} \mathrm{Re}\{\mathscr{P}(z, \zeta)\} u(\zeta)\, ds(\zeta) = \frac{1}{2\pi R_2}\int_{\gamma_2} \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} u(\zeta)\, ds(\zeta) + o(1)$$

as $z \to \eta = R_2 e^{i\theta} \in \gamma_2$. But the last integral (the *Poisson integral* of $u$) defines the solution of the Dirichlet problem for the interior of the circle $\gamma_2$ (see [12]). Thus,

$$\int_\Gamma \mathrm{Re}\{\mathscr{P}(z, \zeta)\} u(\zeta)\, ds(\zeta) = \left(\int_{\gamma_2} - \int_{\gamma_1}\right) \mathrm{Re}\{\mathscr{P}(z, \zeta)\} u(\zeta)\, ds(\zeta) \to u(\eta)$$

as $z \to \eta = R_2 e^{i\theta} \in \gamma_2$.

On the other hand, again using (6.1), we see that

$$\mathrm{Re}\{\mathscr{P}(R_1 e^{i\theta}, R_2 e^{i\Phi})\} = \frac{1}{2\pi R_2},$$

while

$$\int_{\gamma_1} \mathrm{Re}\{\mathscr{P}(z, \zeta)\} u(\zeta)\, ds(\zeta)$$

$$= \frac{1}{2\pi R_1}\int_{\gamma_1} \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} u(\zeta)\, ds(\zeta) + \frac{1}{2\pi R_1}\int_{\gamma_1} u(\zeta)\, ds(\zeta) + o(1)$$

as $z \to \eta = R_1 e^{i\theta} \epsilon \gamma_1$. Consequently, as in the case just considered,

$$\int_\Gamma \mathrm{Re}\{\mathscr{P}(z,\zeta)\} u(\zeta)\, ds(\zeta) = \Big(\int_{\gamma_2} - \int_{\gamma_1}\Big) \mathrm{Re}\{\mathscr{P}(z,\zeta)\} u(\zeta)\, ds(\zeta) \tag{6.2}$$

$$\to \frac{1}{2\pi}\Big(\int_{\gamma_2} \frac{u(\zeta)}{|\zeta|} ds(\zeta) - \int_{\gamma_1} \frac{u(\zeta)}{|\zeta|} ds(\zeta)\Big) + u(\eta) = u(\eta)$$

as $z \to \eta = R_1 e^{i\theta} \epsilon \gamma_1$, since $u$ is the real part of an analytic function $F$.

Since

$$\mathrm{Re} \int_\Gamma \{\mathscr{P}(z,\zeta)\} u(\zeta)\, ds(\zeta)$$

is a harmonic function, it follows from the above considerations and the uniqueness of solutions of the Dirichlet problem that

$$\mathrm{Re} \int_\Gamma \mathscr{P}(z,\zeta) u(\zeta)\, ds(\zeta) = u(z).$$

Moreover, the function

$$F(z) - iv(z_0) - \int_\Gamma \mathscr{P}(z,\zeta) u(\zeta)\, ds(\zeta)$$

is an analytic function having real part zero, and imaginary part 0 at $z_0$; it follows that it is identically 0, which proves property (3).

Another simple consequence of the above considerations is the following result:

Lemma (6.3). *Suppose $u$ is harmonic in the annular region $\{z; r_1 < |z| < r_2\}$. If $r_1 < R_1 < r < R_2 < r_2$, then*

$$\int_0^{2\pi} u(re^{i\theta})\, d\theta = \frac{\log r - \log R_1}{\log R_2 - \log R_1} \int_0^{2\pi} u(R_2 e^{i\theta})\, d\theta + \frac{\log R_2 - \log r}{\log R_2 - \log R_1} \int_0^{2\pi} u(R_1 e^{i\theta})\, d\theta;$$

*that is, the means*

$$\int_0^{2\pi} u(re^{i\theta})\, d\theta$$

*depend linearly on $\log r$. In particular, when $u \geqslant 0$,*

$$\sup_{r_1<r<r_2} \int_0^{2\pi} u(re^{i\theta})\, d\theta < \infty.$$

Proof. By Lemma (2.4) we know that there exists a number $\lambda_1$ such that $u(z) - \lambda_1 \omega_1(z) = v(z)$ is the real part of an analytic function. Thus, by Cauchy's theorem,

$$\int_0^{2\pi} v(re^{i\theta})\, d\theta$$



is independent of $r$. In the special case when $\mathscr{D} = \{z; R_1 \leqslant |z| \leqslant R_2\}$ we have seen that (see (6.2) and the fact, stated after (3.1), that $v(z) \to \bar{u}(\eta) - \lambda_1$)

$$\lambda_1 = -\frac{1}{2\pi}\int_\Gamma \frac{u(\zeta)}{|\zeta|} ds(\zeta) \quad \text{and} \quad \omega_1(z) = \frac{\log|z|/R_2}{\log R_1/R_2}.$$

Thus,

$$v(z) = u(z) + \Big(\frac{1}{2\pi}\int_\Gamma \frac{u(\zeta)}{|\zeta|} ds(\zeta)\Big) \frac{\log|z| - \log R_2}{\log R_1 - \log R_2}.$$

Since

$$\int_0^{2\pi} v(re^{i\theta})\, d\theta$$

is independent of $r$,

$$\int_0^{2\pi} u(R_2 e^{i\theta})\, d\theta = \int_0^{2\pi} v(R_2 e^{i\theta})\, d\theta = \int_0^{2\pi} v(re^{i\theta})\, d\theta$$

$$= \int_0^{2\pi} u(re^{i\theta})\, d\theta + \frac{\log r - \log R_2}{\log R_1 - \log R_2}\Big\{\int_0^{2\pi} u(R_2 e^{i\theta})\, d\theta - \int_0^{2\pi} u(R_1 e^{i\theta})\, d\theta\Big\}.$$

But this clearly reduces to the relation of our lemma.

An almost immediate corollary of Lemma (6.3) is the following version of Jensen's formula:

Lemma (6.4). *Suppose $F$ is analytic in the ring $\{z; R \leqslant |z| < 1\} = \mathscr{R}$ and is not zero in the closed subring $\{z; R \leqslant |z| \leqslant R_1 < 1\}$. If $\{a_i\}$ are the zeros of $F$ (counted with their multiplicity), then for $R_1 < \varrho < 1$*

$$2\pi \sum_{R_1 < |a_i| \leqslant \varrho} \log \frac{\varrho}{|a_i|} = \int_0^{2\pi} \log|F(\varrho e^{i\theta})|\, d\theta - \frac{\log \varrho/R}{\log R_1/R} \int_0^{2\pi} \log|F(R_1 e^{i\theta})|\, d\theta +$$

$$+ \frac{\log \varrho/R_1}{\log R_1/R} \int_0^{2\pi} \log|F(Re^{i\theta})|\, d\theta.$$

*In particular, we see that the mean*

$$\int_0^{2\pi} \log|F(\varrho e^{i\theta})|\, d\theta$$

*is a convex function of $\log \varrho$.*

Proof. Using the fact that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|a-\varrho e^{i\theta}|\,d\theta$$

equals $\log|a|$ if $|a|>\varrho$ and equals $\log\varrho$ if $|a|<\varrho$, Lemma (6.3) applied to the function

$$u(z) = \log|F(z)/\prod_{R_1<|a_i|\leqslant\varrho}(z-a_i)|$$

implies

$$\int_0^{2\pi}\log|F(R_1e^{i\theta})|\,d\theta - \sum_{R_1<|a_i|\leqslant\varrho}\log|a_i|$$

$$= \frac{\log R_1-\log R}{\log\varrho-\log R}\Big\{\int_0^{2\pi}\log|F(\varrho e^{i\theta})|\,d\theta - 2\pi\sum_{R_1<|a_i|\leqslant\varrho}\log\varrho\Big\}+$$

$$+\frac{\log\varrho-\log R_1}{\log\varrho-\log R}\Big\{\int_0^{2\pi}\log|F(Re^{i\theta})|\,d\theta-2\pi\sum_{R_1<|a_i|\leqslant\varrho}\log|a_i|\Big\},$$

but this clearly reduces to the equality in the lemma.

From the last part of Lemma (6.4) we obtain

COROLLARY (6.5). *Suppose $F$ is an analytic function in the ring $\{z: R_1<|z|<R_2\}$ satisfying*

$$M = \sup_{R_1<r<R_2}\int_0^{2\pi}\log^+|F(re^{i\theta})|\,d\theta<\infty,$$

*then*

$$\sup_{R_1<r<R_2}\int_0^{2\pi}\big|\log|F(re^{i\theta})|\big|\,d\theta<\infty.$$

Proof. Since

$$\big|\log|F(re^{i\theta})|\big| = 2\log^+|F(re^{i\theta})|-\log|F(re^{i\theta})|,$$

$$\int_0^{2\pi}\big|\log|F(re^{i\theta})|\big|\,d\theta\leqslant 2M+\sup_{R_1<r<R_2}\Big\{-\int_0^{2\pi}\log|F(re^{i\theta})|\,d\theta\Big\}.$$

On the other hand, the fact that

$$\int_0^{2\pi}\log|F(re^{i\theta})|\,d\theta$$

is a convex function of $\log r$ implies that it is bounded from below on the interval $(R_1, R_2)$ and the result follows.



**§ 7.** We now pass to the proof of Theorem II. Suppose $F\in\mathfrak{N}(\mathscr{D})$ and $F$ is not identically 0. Thus, there exists a harmonic function, $h$, such that $\log^+|F(z)|\leqslant h(z)$ for all $z\in\mathscr{D}$. Letting $u_k(\zeta) = h(\varphi_k(\zeta))$ and $f_k(\zeta) = F(\varphi_k(\zeta))$ for $\zeta\in\mathscr{R}$, we therefore have

$$\log^+|f_k(\zeta)|\leqslant u_k(\zeta)$$

for $k = 1, 2, \ldots, n$. But, if $u\geqslant 0$ is harmonic in $\mathscr{R}$, then

$$\sup_{R<\varrho<1}\int_0^{2\pi}u(\varrho e^{i\theta})\,d\theta<\infty$$

(see (6.3)). Consequently, there exists a constant $K<\infty$, independent of $k = 1, 2, \ldots, n$ such that

(7.1) $$\int_0^{2\pi}\log^+|F(\varphi_k(\varrho e^{i\theta}))|\,d\theta\leqslant K$$

for $R<\varrho<1$ [7]. On the other hand, this condition on $F$ is equivalent (see (6.5)) to the following apparently stronger one:

(7.2) $$\int_0^{2\pi}\big|\log|F(\varphi_k(\varrho e^{i\theta}))|\big|\,d\theta\leqslant M<\infty.$$

This inequality asserts that the $L_1$-norms of the family of functions $\{g_\varrho(\theta)\} = \{\log|F(\varphi_1(\varrho e^{i\theta}))|\}$ are bounded. Thus, by Helly's theorem, we can find a sequence $\varrho_m\to 1$ such that $\{g_{\varrho_m}(\theta)\}$ converges weakly to a bounded Borel measure $\mu_1$ on $[0, 2\pi]$. This can clearly be done in such a way that $F(\varphi_1(\varrho_m e^{i\theta}))$ is never 0. Applying Helly's theorem and (7.2) for $k = 2$ and $\varrho = \varrho_m$ $(m = 1, 2, 3, \ldots)$, we can extract a subsequence from $\{\log|F(\varphi_2(\varrho_m e^{i\theta}))|\}$ that converges weakly to a similar measure $\mu_2$ on $[0, 2\pi]$. Continuing this process we obtain bounded Borel measures $\mu_1, \mu_2, \ldots, \mu_n$ and a sequence $\{\varrho_j\}$, tending to 1, such that for each continuous function $f$

(7.3) $$\lim_{j\to\infty}\int_0^{2\pi}f(\theta)\log|F(\varphi_k(\varrho_j e^{i\theta}))|\,d\theta = \int_0^{2\pi}f(\theta)\,d\mu_k(\theta),$$

$k = 1, 2, \ldots, n$.

Let $\nu$ be the measure on $\Gamma$, the boundary of $\mathscr{D}$, obtained by letting $d\nu(\theta) = |\varphi_k'(e^{i\theta})|\,d\mu_k(\theta)$ on the component $\gamma_k$. Then, from (7.3) with $f(\theta) = \mathscr{P}(z, \varphi_k(e^{i\theta}))|\varphi_k'(e^{i\theta})|$, for $z\in\mathscr{D}$ fixed, we obtain

(7.4) $$\lim_{j\to\infty}\int_0^{2\pi}\mathscr{P}(z, \varphi_k(e^{i\theta}))|\varphi_k'(e^{i\theta})|\log|F(\varphi_k(\varrho_j e^{i\theta}))|\,d\theta = \int_{\gamma_k}\mathscr{P}(z,\zeta)\,d\nu(\zeta),$$

$k = 1, 2, \ldots, n$.

---

(7) It is not hard to show that (7.1) is equivalent to the condition that $F$ belong to $\mathfrak{N}(\mathscr{D})$.

We shall say that a sequence of functions, defined on a domain $\mathscr{D}$, *converges almost uniformly* to a function on $\mathscr{D}$ if it converges uniformly to this function on each compact subset of $\mathscr{D}$. By taking an appropriate subsequence and relabeling, if necessary, we can assume that the convergence (7.4) is almost uniform (for it follows from (7.2) and the boundedness of $\mathscr{P}(z,\zeta)$, when $z$ is restricted to a compact subset of $\mathscr{D}$ and $\zeta\in\Gamma$, that the functions of $z$ defined by the integrals in (7.4) are uniformly bounded on compact subsets of $\mathscr{D}$).

We now let $ds_j$ be the element of arc length on $\Gamma_j=\Gamma_{\varrho_j}$, the boundary of $\mathscr{D}_j=\mathscr{D}_{\varrho_j}$; that is, $ds_j=ds_j(\theta)=\varrho_j|\varphi_k'(\varrho_j e^{i\theta})|\,d\theta$ on the component $\gamma_{k,j}=\gamma_{k,\varrho_j}$ of $\Gamma_j$. We now claim that it follows easily from (7.4) that if we put $\mathscr{P}_j(z,\zeta_j)=\mathscr{P}_{\varrho_j}(z,\zeta_j)$, where $\zeta_j\in\Gamma_j$, then

$$\lim_{j\to\infty}\int_{\Gamma_j}\mathscr{P}_j(z,\zeta_j)\log|F(\zeta_j)|\,ds_j=\int_{\Gamma}\mathscr{P}(z,\zeta)\,d\nu(\zeta), \tag{7.5}$$

the convergence being almost uniform in $\mathscr{D}$.

In order to see this it suffices to show that

$$\int_{\gamma_{k,j}}\mathscr{P}_j(z,\zeta_j)\log|F(\zeta_j)|\,ds_j\to\int_{\gamma_k}\mathscr{P}(z,\zeta)\,d\nu(\zeta)\quad\text{ as }\quad j\to\infty,$$

for $k=1,2,\ldots,n$, the convergence being almost uniform. But

$$\int_{\gamma_{k,j}}\mathscr{P}_j(z,\zeta_j)\log|F(\zeta_j)|\,ds_j$$

$$=\int_0^{2\pi}\mathscr{P}_j\big(z,\varphi_k(\varrho_i e^{i\theta})\big)\big\{\log\big|F\big(\varphi_k(\varrho_j e^{i\theta})\big)\big|\big\}\varrho_j|\varphi_k'(\varrho_j e^{i\theta})|\,d\theta$$

$$=\int_0^{2\pi}\big\{\mathscr{P}_j\big(z,\varphi_k(\varrho_j e^{i\theta})\big)\varrho_j|\varphi_k'(\varrho_j e^{i\theta})|-\mathscr{P}\big(z,\varphi_k(e^{i\theta})\big)|\varphi_k'(e^{i\theta})|\big\}\log\big|F\big(\varphi_k(\varrho_j e^{i\theta})\big)\big|\,d\theta+$$

$$+\int_0^{2\pi}\mathscr{P}\big(z,\varphi_k(e^{i\theta})\big)|\varphi_k'(e^{i\theta})|\log\big|F\big(\varphi_k(\varrho_j e^{i\theta})\big)\big|\,d\theta.$$

By Theorem (5.1) and (7.2) we see that the first term of this sum tends to 0 almost uniformly in $z$ as $j$ tends to $\infty$, while (7.4) asserts that the last term tends to

$$\int_{\gamma_k}\mathscr{P}(z,\zeta)\,d\nu(\zeta)$$

almost uniformly. Thus, (7.5) is established.

We define, for $j=1,2,3,\ldots$,

$$G_j(z)=\exp\Big\{\int_{\Gamma_j}\mathscr{P}_j(z,\zeta_j)\log|F(\zeta_j)|\,ds_j\Big\}.$$



Then, by the convergence described after equality (3.1),

$$|G_j(\zeta)|=\lim_{\substack{z\to\zeta\in\Gamma_j\\ z\in\mathscr{D}_j}}|G_j(z)|$$

exists and satisfies

$$|G_j(\zeta)|=|F(\zeta)|\,e^{-\lambda_{k,j}}, \tag{7.6}$$

for $\zeta\in\gamma_{k,j}$. The constants $\lambda_{k,j}$, $k=1,2,\ldots,n-1$, are associated with the domain $\mathscr{D}_j$ as well as the function $|F|$ and are given by (3.1); we also have $\lambda_{n,j}=0$. Moreover $\lambda_{k,j}\to\lambda_k$ as $j\to\infty$ since $\mathscr{P}_j\to\mathscr{P}$ and $P_j\to P$ (see Theorem (5.1) and (5.6)).

From (7.5) we see that the sequence $\{G_j\}$ converges almost uniformly to the exponential function

$$G(z)=\exp\int_{\Gamma}\mathscr{P}(z,\zeta)\,d\nu(\zeta).$$

Let $B_j(z)=F(z)/G_j(z)$ for $z\in\mathscr{D}_j$ and $B(z)=F(z)/G(z)=\lim_{j\to\infty}B_j(z)$. By (7.6), $|B_j(z)|$ is defined and equals $e^{\lambda_{k,j}}$ when $z\in\gamma_{k,j}$. Since the numbers $\lambda_{k,j}$ are uniformly bounded, it follows from the maximum modulus principle that so are the functions $B_j$; consequently $B$ is bounded in $\mathscr{D}$. It thus follows from Fatou's theorem that for almost every [8] $\zeta\in\Gamma$

$$\lim_{z\to\zeta}B(z)=B(\zeta)$$

exists if $z$ tends to $\zeta$ non-tangentially. We shall now show that

$$|B(\zeta)|=e^{\lambda_k}\quad\text{ for a.e. }\zeta\in\gamma_k. \tag{7.7}$$

Toward this end we first show that

$$\overline{\lim_{z\to\zeta\in\gamma_k}}\,|B(z)|\leqslant e^{\lambda_k}.$$

Let $\omega_{k,j}$ be the harmonic measure corresponding to the component $\gamma_{k,j}$ of $\Gamma_j$ and put $h_j(z)=|B_j(z)|+M\big(1-\omega_{k,j}(z)\big)$. We can always choose $M$ so that $h_j(z)\leqslant 0$ for $z\in\gamma_{l,j}$, $l\neq k$ (for the $B_j$'s are uniformly bounded; thus an appropriate negative constant $M$ will have this property). When $z\in\gamma_{k,j}$ we have $h_j(z)=e^{\lambda_{k,j}}$. Thus $h_j(z)\leqslant e^{\lambda_{k,j}}$ for $z\in\mathscr{D}_j$. Since $\omega_{k,j}(z)\to\to\omega_k(z)$ (= the harmonic measure corresponding to the component $\gamma_k$ of $\Gamma$) and $\lambda_{k,j}\to\lambda_k$ as $j\to\infty$ (see (5.6)), we therefore obtain, for $z\in\mathscr{D}_j$,

$$h(z)=|B(z)|+M\big(1-\omega_k(z)\big)\leqslant e^{\lambda_k}.$$

[8] "Almost every" means almost every with respect to the arc length measure on $\Gamma$.

Since $\omega_k(z) \to 1$ as $z \to \zeta \in \gamma_k$, it follows that

$$\overline{\lim_{r \to \zeta \in \gamma_k}} |B(z)| \leqslant e^{\lambda_k}.$$

Consequently, $|B(\zeta)| \leqslant e^{\lambda_k}$ for a.e. $\zeta \in \gamma_k$. In order to show that equality holds almost everywhere on $\gamma_k$ it is, therefore, sufficient to show that

$$(7.8) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left|B\big(\varphi_k(e^{i\theta})\big)\right| d\theta \geqslant e^{\lambda_k}.$$

We note that

$$|B(\zeta_j)| = \left|\frac{F(\zeta_j)}{G(\zeta_j)}\right| = e^{\lambda_{k,j}}\left|\frac{G_j(\zeta_j)}{G(\zeta_j)}\right|,$$

when $\zeta_j \in \gamma_{k,j}$. Let $H_j(z) = G_j\big(\varphi_k(z)\big)/G\big(\varphi_k(z)\big)$ for $R \leqslant |z| \leqslant 1$. Then

$$\int_0^{2\pi} H_j(\varrho e^{i\theta})\, d\theta$$

is independent of $\varrho$, $R \leqslant \varrho \leqslant 1$ (this follows from the Cauchy integral theorem). Fixing $\varrho$ strictly between $R$ and 1, we thus have

$$e^{\lambda_{k,j}}\left|\int_0^{2\pi} H_j(\varrho e^{i\theta})\, d\theta\right| \leqslant \int_0^{2\pi} \left|B\big(\varphi_k(\varrho_j e^{i\theta})\big)\right| d\theta.$$

But $H_j(\varrho e^{i\theta}) \to 1$ (uniformly in $\theta$) and $B\big(\varphi_k(\varrho_j e^{i\theta})\big) \to B\big(\varphi_k(e^{i\theta})\big)$ as $j \to \infty$. Hence, by the last inequality,

$$2\pi e^{\lambda_k} = \lim_{j\to\infty} e^{\lambda_{k,j}}\left|\int_0^{2\pi} H_j(\varrho e^{i\theta})\, d\theta\right| \leqslant \lim_{j\to\infty}\int_0^{2\pi} \left|B\big(\varphi_k(\varrho_j e^{i\theta})\big)\right| d\theta$$
$$= \int_0^{2\pi} \left|B\big(\varphi_k(e^{i\theta})\big)\right| d\theta.$$

This establishes (7.8) and Theorem II is thus proved.

**§ 8.** We now study the structure of the function $B$. Our main result will be Theorem III. Its proof will be based on inequality (1.3) which we now restate in the form of a theorem:

THEOREM (8.1). *Let $F \in \mathfrak{N}(\mathscr{D})$, $F$ not identically zero, and $a_1, a_2, a_3, \ldots$ the zeros of $F$. If $a_i'$ satisfies*

$$|a_i' - a_i| = \min_{\zeta \in \Gamma} |\zeta - a_i|,$$

*then*

$$\sum_{i=1}^{\infty} |a_i' - a_i| < \infty.$$



Proof. In case $F$ is analytic in the ring $\{z;\, R \leqslant |z| < 1\}$ this theorem is an easy consequence of Lemma (6.4). In fact, we can assume that $R$ is so chosen that no zero lies on the circumference $|z| = R$ and we order the zeros $\{a_i\}$ so that $R < |a_1| \leqslant |a_2| \leqslant |a_3| \leqslant \ldots < 1$; then Lemma (6.4) and (7.1) (we take for $\varphi_k$ the identity mapping) imply that there exists $A > 0$ such that

$$\sum_{i=1}^{n} \log\frac{\varrho}{|a_i|} \leqslant \sum_{R<|a_i|\leqslant\varrho} \log\frac{\varrho}{|a_i|} \leqslant A < \infty,$$

whenever $\varrho$ is close enough to 1 so that the first $n$ of these zeros satisfy $|a_i| \leqslant \varrho$. Letting $\varrho \to 1$ we obtain

$$-\sum_{i=1}^{n} \log|a_i| \leqslant A < \infty$$

for all $n = 1, 2, 3, \ldots$ If we let $a_i' = a_i/|a_i|$ denote the point on the outer circumference, $|z| = 1$, that is closest to $a_i$, then $|a_i' - a_i| = 1 - |a_i| < -\log|a_i|$. Thus,

$$(8.2) \qquad \sum_{i=1}^{\infty} 1 - |a_i| = \sum_{i=1}^{\infty} |a_i' - a_i| \leqslant A < \infty.$$

The general case can be reduced to this one. We do this by considering separately the zeros belonging to the subdomains $\Delta_k$ ($k = 1, 2, \ldots, n-1$) introduced in the beginning of the second section. Let $\varphi_k$ be the conformal map of $\overline{\mathscr{R}} = \{z;\, R \leqslant |z| \leqslant 1\}$ onto $\overline{\Delta}_k$ that was introduced at that time. We order these zeros so that the moduli of their inverse images $\alpha_i = \varphi_k^{-1}(a_i)$ tend monotonically to 1 (provided there are infinitely many such zeros; if not, there is nothing to prove). There exists a constant $M$ such that $|\varphi_k(w_1) - \varphi_k(w_2)| \leqslant M|w_1 - w_2|$ whenever $w_1$ and $w_2$ belong to $\overline{\mathscr{R}}$. Thus, using the notation of (8.2) we obtain, by that inequality applied to the zeros of the function $F\big(\varphi_k(w)\big)$,

$$\sum_{a_i \in \Delta_k} |a_i' - a_i| \leqslant \sum_{a_i \in \Delta_k} |\varphi_k(\alpha_i') - a_i| \leqslant M \sum_{i=1}^{\infty} |\alpha_i' - \alpha_i| < \infty.$$

The theorem now follows since the number of zeros in $\mathscr{D} - \bigcup_{k=1}^{n} \Delta_k$ must be finite.

COROLLARY (8.3). *Suppose $F$ satisfies the conditions of Theorem (8.1); then the product*

$$B_0(z) = \prod_{i=1}^{\infty} \frac{B(z, a_i)}{B(z, a_i')}$$

*converges absolutely and uniformly in $z$ when the latter is restricted to a compact subset $C \subset \mathscr{D}$.*

Proof. In the fourth section we showed that, when $z$ is restricted to a compact subset $C \subset \mathscr{D}$, $L(z, a) = -\log B(z, a)/(z-a)$ has a harmonic extension, as a function of $a$, to a domain $\mathscr{D}_C \supset \overline{\mathscr{D}}$ (see discussion immediately following (4.3)). Thus, $B(z, a)$ is real-analytic in $a$ and it follows easily that there exists a constant $M_C$ such that for all $z \in C$

$$|B(z, a') - B(z, a)| \leqslant M_C |a' - a|,$$

for $a, a' \in \overline{\mathscr{D}}$. Moreover, in the course of the proof of Theorem IV (see § 4) we have shown that $|B(z, a')| > \exp\{\pi_{kk}\}$. Thus, if

$$m = \min_{1 \leqslant k \leqslant n-1} e^{\pi_{kk}},$$

we have

$$\left|1 - \frac{B(z, a)}{B(z, a')}\right| \leqslant \frac{|B(z, a') - B(z, a)|}{m} \leqslant \frac{M_C}{m} |a' - a|.$$

Consequently,

$$\sum_{i=1}^{\infty} \left|1 - \frac{B(z, a_i)}{B(z, a_i')}\right| \leqslant \frac{M_C}{m} \sum_{i=1}^{\infty} |a_i' - a_i| < \infty$$

which implies the absolute and uniform convergence of the product $B_0(z)$ announced above.

In case $\mathscr{D}$ is the unit disc, the function $B_0$ is the classical Blaschke product. It is well known that such products are characterized by the property

$$\lim_{\varrho \to 1-} \int_0^{2\pi} \left|\log |B_0(\varrho e^{i\theta})|\right| d\theta = 0$$

(see [12], Chapter VII). The following lemma is an extension of this property:

LEMMA (8.4). *Let*

$$c_k = \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{n-1} \pi_{jk} \big(\omega_j(a_i) - \omega_j(a_i')\big)\Big),$$

*for* $k = 1, 2, \ldots, n-1$ *and* $c_n = 0$; *then*

$$\lim_{\varrho \to 1-} \int_0^{2\pi} \left|\log \left|B_0\big(\varphi_k(\varrho e^{i\theta})\big)\right| - c_k\right| d\theta = 0.$$

Proof. It follows from (4.3) and (4.7) that

$$\log \left|\frac{B(z, a_i)}{B(z, a_i')}\right| = \sum_{j,m}^{n-1} \pi_{jm} \big(\omega_j(a_i) - \omega_j(a_i')\big) \omega_m(z) - g(z, a_i).$$



Since each harmonic measure $\omega_j$ has an extension to a domain that includes $\overline{\mathscr{D}}$ (see discussion following (4.3)), it follows that there exists a constant $M$ such that $|\omega_j(a_i) - \omega_j(a_i')| \leqslant M |a_i - a_i'|$. Thus, from Theorem (8.1) we can conclude that the series

$$\sum_{i=1}^{\infty} \Big(\sum_{j=1}^{n-1} \pi_{jm} \big(\omega_j(a_i) - \omega_j(a_i')\big)\Big)$$

converge to finite sums $c_m$ ($m = 1, \ldots, n-1$). Since $g(z, a_i) = g(z, a_i) - g(z, a_i')$, we can conclude, from the same reasoning, that the series

$$\sum_{i=1}^{\infty} g(z, a_i)$$

converges uniformly when $z$ is restricted to a compact subset of $\mathscr{D}$ not containing any of the $a_i$'s. Hence,

$$\log |B_0(z)| = \sum_{m=1}^{n-1} c_m \omega_m(z) - \sum_{i=1}^{\infty} g(z, a_i).$$

Since $g(z, a_i) \geqslant 0$, it follows that the lemma is established if we can show that

$$\lim_{\varrho \to 1} \int_0^{2\pi} \sum_{i=1}^{\infty} g\big(\varphi_k(\varrho e^{i\theta}), a_i\big) d\theta = 0 \tag{8.5}$$

for each $k = 1, 2, \ldots, n-1$. Toward this end, we fix $k$ and let

$$\alpha_i = \begin{cases} \varphi_k^{-1}(a_i) & \text{when} \quad a_i \in \Delta_k, \\ 1 & \text{when} \quad a_i \notin \Delta_k. \end{cases}$$

Then, from (8.2), we have $\sum (1 - |\alpha_i|) < \infty$.

We define a sequence $\{u_n\}$ of harmonic functions on the ring $\mathscr{R} = \{z; R < |z| < 1\}$ by letting

$$u_n(z) = \sum_{i=1}^{n} \left\{ g\big(\varphi_k(z), a_i\big) + \log \left|\frac{z - \alpha_i}{1 - \bar{\alpha}_i z}\right| \right\}.$$

(Note that, for $a_i \notin \Delta_k$, $\alpha_i = 1$ and, thus, the logarithmic term is zero while $g\big(\varphi_k(z), a_i\big)$ is harmonic. On the other hand, when $a_i \in \Delta_k$,

$$g\big(\varphi_k(z), a_i\big) + \log \left|\frac{z - \alpha_i}{1 - \bar{\alpha}_i z}\right| = h\big(\varphi_k(z), a_i\big) - \log \left|\frac{\varphi_k(z) - \varphi_k(\alpha_i)}{z - \alpha_i}\right| - \log |1 - \bar{\alpha}_i z|$$

is a sum of harmonic functions on the ring $\mathscr{R}$).

Since

$$\log\left|\frac{z-a}{1-\bar{a}z}\right| = 0$$

when $|a| = 1$, the convergence of the series $\sum(1-|a_i|) = \sum|a_i'-a_i|$ implies, by the same reasoning as that employed at the beginning of this proof, the uniform convergence of the series

(8.6) $$u(z) = \sum_{i=1}^{\infty}\left\{g(\varphi_k(z), a_i)+\log\left|\frac{z-a_i}{1-\bar{a}_i z}\right|\right\} = \lim_{n\to\infty} u_n(z)$$

for $z$ in a compact subset of $\mathscr{R}$.

Let $R < R_1 < R_2 < 1$. If $v$ is any harmonic function on $\mathscr{R}$, it follows from Lemma (6.3) that

(8.7) $$\lim_{\varrho\to 1}\int_0^{2\pi} v(\varrho e^{i\theta})\,d\theta = \frac{\log R_1}{\log R_1/R_2}\int_0^{2\pi} v(R_2 e^{i\theta})d\theta - \frac{\log R_2}{\log R_1/R_2}\int_0^{2\pi} v(R_1 e^{i\theta})\,d\theta.$$

Since, for $n$ fixed,

$$\lim_{\varrho\to 1}\int_0^{2\pi} u_n(\varrho e^{i\theta})\,d\theta = 0,$$

equality (8.7) implies

(8.8) $$\frac{\log R_1\int_0^{2\pi} u_n(R_2 e^{i\theta})\,d\theta - \log R_2\int_0^{2\pi} u_n(R_1 e^{i\theta})\,d\theta}{\log R_1 - \log R_2} = 0.$$

Now, because of the uniform convergence (8.6) on the compact set $\{z\in\mathscr{R};\ |z| = R_1 \text{ or } |z| = R_2\}$, (8.7) and (8.8) imply

(8.9) $$\lim_{\varrho\to 1}\int_0^{2\pi} u(\varrho e^{i\theta})\,d\theta = \frac{\log R_1\int_0^{2\pi} u(R_2 e^{i\theta})\,d\theta - \log R_2\int_0^{2\pi} u(R_1 e^{i\theta})\,d\theta}{\log R_1 - \log R_2} = 0.$$

Since

$$u(z) = \sum_{i=1}^{\infty} g(\varphi_k(z), a_i) + \sum_{i=1}^{\infty}\log\left|\frac{z-a_i}{1-\bar{a}_i z}\right|,$$

(8.5) and, therefore, the lemma will be established if we show

(8.10) $$\lim_{\varrho\to 1}\int_0^{2\pi}\sum_{i=1}^{\infty}\log\left|\frac{\varrho e^{i\theta}-a_i}{1-\bar{a}_i\varrho e^{i\theta}}\right|\,d\theta = 0.$$



But this is easily shown: Since $(z-a)/(1-\bar{a}z)$ $(|a| < 1)$ maps the unit circle (conformally) into itself each of the terms of the series being integrated is non-positive; furthermore,

$$\frac{-1}{2\pi}\int_0^{2\pi}\log\left|\frac{\varrho e^{i\theta}-a}{1-\bar{a}\varrho e^{i\theta}}\right|\,d\theta = -\log\max\{|a|, \varrho\} = \log\min\{|a|^{-1}, \varrho^{-1}\}.$$

We know that

$$-\sum_{i=1}^{\infty}\log|a_i| < \infty$$

(see the proof of (8.1)). Thus, given $\varepsilon > 0$, let $N$ be such that

$$-\sum_{i=N}^{\infty}\log|a_i| < \varepsilon.$$

We then have

$$-\frac{1}{2\pi}\int_0^{2\pi}\sum_{i=N}^{\infty}\log\left|\frac{\varrho e^{i\theta}-a_i}{1-\bar{a}_i\varrho e^{i\theta}}\right|\,d\theta \leqslant \sum_{i=N}^{\infty}\log\frac{1}{|a_i|} < \varepsilon,$$

independently of $\varrho < 1$. Since it is clear that

$$\lim_{\varrho\to 1}\int_0^{2\pi}\sum_{i=1}^{N-1}\log\left|\frac{\varrho e^{i\theta}-a_i}{1-\bar{a}_i\varrho e^{i\theta}}\right|\,d\theta = 0,$$

(8.10) and the lemma are proved.

We are now in a position to prove Theorem III. We first observe that by Theorem (5.1) and Lemma (8.4) (we again use the notation introduced in § 5)

$$\lim_{\varrho\to 1}\int_0^{2\pi}\mathscr{P}_\varrho(z, \varphi_k(\varrho e^{i\theta}))\,|\varphi_k'(\varrho e^{i\theta})|\,\varrho\log\left|B_0(\varphi_k(\varrho e^{i\theta}))\right|\,d\theta$$

$$= \int_0^{2\pi}\mathscr{P}(z, \varphi_k(e^{i\theta}))\,|\varphi_k'(e^{i\theta})|\,c_k\,d\theta = \int_{\gamma_k}\mathscr{P}(z, \zeta)\,c_k\,ds(\zeta) = 0,$$

the convergence being uniform for $z$ restricted to a compact subset of $\mathscr{D}$ (the last equality is a consequence of property (2) of Theorem I, when $1 \leqslant k \leqslant n-1$, and the fact that $c_n = 0$ — see Lemma (8.4)). Consequently,

(8.11) $$\lim_{\varrho\to 1}\int_{\Gamma_\varrho}\mathscr{P}_\varrho(z, \zeta_\varrho)\log|B_0(\zeta_\varrho)|\,ds(\zeta_\varrho) = 0.$$

Let us now define $G_0$ by letting $G_0(z) = F(z)/B_0(z)$. Since $B_0$ has the same zeros as $F$ (and with the same multiplicity) the function $G_0$ is never zero. Hence, by Theorem (4.10) applied to the domain $\mathscr{D}_\varrho$, there exist integers $m_1, m_2, \ldots, m_{n-1}$ [9] such that for any $n-1$ points $b_{k,\varrho} \in \gamma_{k,\varrho}$ $(k = 1, \ldots, n-1)$,

$$(8.12) \qquad G_0(z) = a_\varrho \prod_{k=1}^{n-1} [B_\varrho(z, b_{k,\varrho})]^{m_k} \exp\Big\{\int_{\Gamma_\varrho} \mathscr{P}_\varrho(z, \zeta_\varrho) \log|G_0(\zeta)|\, ds(\zeta_\varrho)\Big\},$$

where $|a_\varrho| = 1$. In this expression we have (see Theorem IV)

$$B_\varrho(z, b_{k,\varrho}) = (z - b_{k,\varrho}) \exp\Big\{-\int_{\Gamma_\varrho} \mathscr{P}_\varrho(z, \zeta_\varrho) \log|\zeta_\varrho - b_{k,\varrho}|\, ds(\zeta_\varrho)\Big\}.$$

We can clearly choose the numbers $b_{k,\varrho}$ so that the limits

$$b_k = \lim_{\varrho \to 1} b_{k,\varrho}$$

exist. It then will follow from (5.1) that

$$(8.13) \qquad \lim_{\varrho \to 1} B_\varrho(z, b_{k,\varrho}) = B(z, b_k),$$

the convergence being uniform for $z$ in a compact subset of $\mathscr{D}$. With these choices of $b_{k,\varrho}$, equality (8.12) implies that the limit

$$\lim_{\varrho \to 1} a_\varrho = a$$

exists.

In order to obtain the relation between $B$ and $B_0$ recall that (see § 7)

$$G_j(z) = \exp\Big\{\int_{\Gamma_j} \mathscr{P}_j(z, \zeta_j) \log|F(\zeta_j)|\, ds(\zeta_i)\Big\}.$$

Thus, since $F = G_0 B_0$, it follows from (8.12) that

$$G_j(z) = \exp\Big\{\int_{\Gamma_j} \mathscr{P}_j(z, \zeta_j) \log|G_0(\zeta_j)|\, ds(\zeta_j)\Big\} \exp\Big\{\int_{\Gamma_j} \mathscr{P}_j(z, \zeta_j) \log|B_0(\zeta_j)|\, ds(\zeta_j)\Big\}$$

$$= G_0(z)\, a_{\varrho_j}^{-1} \prod_{k=1}^{n-1} [B_{\varrho_j}(z, b_{k,\varrho_j})]^{-m_k} \exp\Big\{\int_{\Gamma_j} \mathscr{P}_j(z, \zeta_j) \log|B_0(\zeta_j)|\, ds(\zeta_j)\Big\}.$$

[9] Note that these integers are equal to the integrals

$$\frac{1}{2\pi i} \int_{\gamma_{k,\varrho}} (G_0'(z)/G_0(z))\, dz$$

(see the proof of (4.10)) and, thus, are independent of $\varrho$.



From (8.11), (8.13), recalling that $G_j(z) \to G(z)$ and $a_j \to a$, we thus obtain

$$G(z) = G_0(z)\, a^{-1} \prod_{k=1}^{n-1} [B(z, b_k)]^{-m_k}.$$

Consequently,

$$B(z) = F(z)/G(z) = B_0(z) G_0(z)/G(z) = a \prod_{k=1}^{n-1} [B(z, b_k)]^{m_k} B_0(z)$$

and Theorem III is proved.

The following theorem completes the result announced in Lemma (8.4) and extends the characterization of the classical Blaschke product discussed immediately before this lemma.

THEOREM (8.14). *A necessary and sufficient condition for a function, $B$, analytic in $\mathscr{D}$ to be represented as a product of the form*

$$B(z) = a \prod_{k=1}^{n-1} [B(z, b_k)]^{m_k} \prod_{i=1}^{\infty} \frac{B(z, a_i)}{B(z, a_i')},$$

*where $a_1, a_2, \ldots$ are the zeros of $B$,*

$$\sum_{i=1}^{\infty} |a_i - a_i'| < \infty,$$

*$b_k \in \gamma_k$, $m_k$ is an integer, $k = 1, 2, \ldots, n-1$ and $|a| = 1$, is that there exist constants $c_1, c_2, \ldots, c_{n-1}, c_n$, with $c_n = 0$ such that*

$$(8.15) \qquad \lim_{\varrho \to 1} \int_0^{2\pi} \Big|\log\big|B(\varphi_k(\varrho e^{i\theta}))\big| - c_k\Big|\, d\theta = 0,$$

$k = 1, 2, \ldots, n-1$.

Proof. The necessity was proved in (8.4), where we based our argument on the fact that

$$\sum_{i=1}^{\infty} |a_i - a_i'| < \infty.$$

Conversely, we first remark that an easy consequence of (8.15) is that $B$ belongs to $\mathfrak{N}(\mathscr{D})$. Thus, by Theorems II and III, $B = B_1 G$ where $B_1$ is a product of the type considered and

$$G(z) = \lim_{j \to \infty} \exp\Big\{\int_{\Gamma_j} \mathscr{P}_j(z, \zeta_j) \log|B(\zeta_j)|\, ds(\zeta_j)\Big\} = 1$$

(see the argument used in order to establish (8.11)).

**§ 9.** In this final section we shall make several observations in connection with this material and compare it with existing results of the same general nature.

We have excluded, in our discussion, domains having a finite number of isolated boundary points. We shall sketch how this can be reduced to the one considered here.

Let $\mathscr{D}_0 = \{z; 0 < |z| < 1\}$ and, as before, we say that an analytic function, $F$, on $\mathscr{D}_0$ belongs to the Nevanlinna class $\mathfrak{N}(\mathscr{D}_0)$ if there exists a harmonic function $h$ on $\mathscr{D}_0$ such that $\log^+|F(z)| \leqslant h(z)$.

LEMMA (9.1). *If $F \in \mathfrak{N}(\mathscr{D}_0)$, then the singularity of $F$ at 0 is a pole.*

Proof. If follows from Lemma (2.4) that there exists a constant $\lambda$ and an analytic function $f$ on $\mathscr{D}_0$ whose real part is $h(z) - \lambda \log|z|$. Since $h \geqslant 0$, $|\exp\{-f(z)\}| = e^{-h(z)}|z|^\lambda \leqslant |z|^\lambda$. Thus, 0 is a pole of $e^{-f(z)}$; therefore, there exists an integer $N$ such that $|z|^N e^{-h(z)}$ tends to a positive number, $a$, as $z \to 0$. Thus, for $|z|$ small,

$$N\log|z| - h(z) > A = \log\frac{a}{2}.$$

Consequently, $\log|F(z)| \leqslant \log^+|F(z)| \leqslant h(z) \leqslant N\log|z| - A$. This shows that $|F(z)| \leqslant e^{-A}|z|^N$; therefore, 0 must be a pole of $F$.

From this lemma we can easily deduce the fact if $\mathscr{D}$ is the unit disc and $F \in \mathfrak{N}(\mathscr{D}_0)$, then there exists an integer $k$ such that $G(z) = z^k F(z)$ can be defined for $z = 0$ in such a way that $G \in \mathfrak{N}(\mathscr{D})$.

More generally, let $\mathscr{D}_0$ be a bounded domain whose boundary consists of $n$ analytic disjoint Jordan arcs and $m$ isolated points $b_1, \ldots, b_m$, and let $\mathscr{D} = \mathscr{D}_0 \cup \{b_1, \ldots, b_m\}$. We then have the following result:

THEOREM (9.2). *Let $F \in \mathfrak{N}(\mathscr{D}_0)$; then there exist integers $k_1, \ldots, k_m$ such that*

$$G(z) = \prod_{i=1}^{m} [B(z, b_i)]^{k_i} F(z)$$

*can be defined for $z = b_1, \ldots, b_m$ in such a way that $G \in \mathfrak{N}(\mathscr{D})$. That is, $\mathfrak{N}(\mathscr{D}_0)$ consists of the restrictions to $\mathscr{D}_0$ of functions of the form*

$$F(z) = G(z) / \prod_{i=1}^{m} [B(z, b_i)]^{k_i}$$

*with $G \in \mathfrak{N}(\mathscr{D})$.*

It is well known (see [2], p. 169) that bounded domains of finite connectivity having no isolated boundary points can be mapped onto bounded domains whose boundary curves are circles. Using these mappings we obtain the following factorization for $F \in \mathfrak{N}(\mathscr{D})$, where $\mathscr{D}$ is such a domain of connectivity $n$: $F = BG$, *where*



(1) *$G$ is an exponential function belonging to $\mathfrak{N}(\mathscr{D})$,*

(2) *$B$ is bounded and has the form*

$$B(z) = \prod_{k=1}^{n-1} [B_k(z)]^{m_k} \prod_{i=1}^{\infty} A_k(z, a_i)$$

*where $B_k$ is a conformal mapping of $\mathscr{D}$ onto a "slit ring" domain of the type introduced earlier, $m_1, \ldots, m_{n-1}$ are integers, $\{a_i\}$ is the sequence of zeros of $F$, $A_k(z, a_i)$ is a conformal mapping of $\mathscr{D}$ onto a "slit disc" centered at 0 such that $A_k(a_i, a_i) = 0$ and such that as $z$ approaches the $k$-th boundary component $A_k(z, a_i)$ approaches the unit circle.*

It is clear that we can generalize this factorization still further by using Theorem (9.2). We shall not, however, develop this subject further.

We now comment briefly on the connection between this work and that of Sarason [8] and Voichick [10]. The former has considered similar problems when the domain is an annulus and has introduced a type of Blaschke product. The latter has not developed such a product but has considered more general multiply connected regions. Both authors defined an *interior function* on $\mathscr{D}$ to be a bounded, possibly *multiple-valued*, analytic function having single-valued and continuous absolute value on $\mathscr{D}$ with constant (a.e.) boundary values on the components of $\Gamma$. We first observe that if $\{a_j\}$ are the zeros of such a function, $F$, the product, $B_0$, of Corollary (8.3) converges absolutely and uniformly when $z$ is restricted to a compact subset of $\mathscr{D}$ (the proof is the same). Applying (4.12) to the harmonic function $u = \log F/B_0$ we see that there exist numbers $m_1, \ldots, m_{n-1}$ such that

$$\prod_{j=1}^{n-1} [B(z, a_j')]^{-m_j} F(z) \tag{9.3}$$

is a (single-valued) analytic function on $\mathscr{D}$. In fact it is the product of $B_0$ and an exponential function (the factorization of Theorem II) and is bounded with non-tangential boundary values (a.e.) that are constant on the components of $\Gamma$. This identifies the interior functions introduced by the above authors in terms of the functions introduced here. In particular, we see that it is not necessary to consider multiple-valued interior functions; if we substitute them by these single-valued functions of the form of (9.3) the results in [10] remain valid.

We close with the remark that very little was said about (non-tangential) boundary values in this paper. Most of the classical results on $H^p$ spaces, however, extend to these domains and can be derived easily from the results concerning the class $\mathfrak{N}(\mathscr{D})$ obtained here.

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# Derivatives of Fourier series and integrals

by

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**I. Notation and definitions.** Throughout this paper we shall be dealing with $n$-dimensional Euclidean space $E^n$, $n \geqslant 2$. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ denote points of $E^n$, we use the standard notation $x+y = (x_1+y_1, \ldots, x_n+y_n)$, $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$ for $\lambda$ real, $(x\cdot y) = x_1y_1+ \ldots +x_ny_n$, $|x| = (x\cdot x)^{1/2}$, $a = (a_1, \ldots, a_n)$ where the $a_j$ are non-negative integers, $a! = a_1! \ldots a_n!$, $|a| = a_1+\ldots+a_n$, $x^a = x_1^{a_1} \ldots x_n^{a_n}$, $D^a = (\partial/\partial x)^a = (\partial/\partial x_1)^{a_1} \ldots (\partial/\partial x_n)^{a_n}$, $\mu = (\mu_1, \ldots, \mu_n)$ where the $\mu_j$ are positive or negative integers, $Q = \{x \mid -\pi < x_j \leqslant \pi, j = 1, \ldots, n\}$, $Q_\mu = Q$ translated by $2\pi\mu$. A real-valued function $f$ on $E^n$ will be called *periodic* if it is periodic $2\pi$ in each variable. If $f \epsilon L(Q)$,

$$S[f] = \sum c_\mu e^{i(\mu\cdot x)} \quad \text{where} \quad c_\mu = \frac{1}{(2\pi)^n}\int_Q f(x)\, e^{-i(\mu\cdot x)}\, dx.$$

If $f \epsilon L(E^n)$ is any integrable function,

$$\hat{f}(x) = \frac{1}{(2\pi)^n}\int_{E^n} f(y)\, e^{-i(x\cdot y)}\, dy.$$

We say $S[f]$ is *Bochner-Riesz $a$-summable of order $\gamma$ at $x$ to sum $s$* if

$$\sigma_a^{(\gamma)}(x, \varepsilon) = \left(\frac{\partial}{\partial x}\right)^a \sum_{\varepsilon|\mu|\leqslant 1} c_\mu e^{i(\mu\cdot x)} (1-\varepsilon^2|\mu|^2)^\gamma$$

tends to $s$ as $\varepsilon \to 0$. We say $S[f]$ is *Abel $a$-summable at $x$ to sum $s$* if

$$f_a(x, \varepsilon) = \left(\frac{\partial}{\partial x}\right)^a \sum c_\mu e^{i(\mu\cdot x)} e^{-\varepsilon|\mu|}$$

tends to $s$ as $\varepsilon \to 0$. Alternately, we say

$$\int_{E^n} \hat{f}(y)\, e^{i(x\cdot y)}\, dy$$