[3] Г. Е. Шилов, *О расширении максимальных идеалов*, Докл. АН СССР 29 (1940), p. 83-85.

[4] W. Żelazko, *Metric generalizations of Banach algebras*, Rozprawy Matematyczne 47 (1965).

[5] — *A note on topological divisors of zero in p-normed algebras*, Coll. Math (in print).

[6] — *On generalized topological divisors of zero in real m-convex algebras*, Studia Math. (to appear).

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A generalized function calculus based on the Laplace transform*

by

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1. Introduction

In this paper the classical Laplace transform is extended in a very natural way to a space of generalized functions. This extension is carried out by utilizing a method for constructing generalized functions suggested by Mikusiński [5]. This method has been used to construct Schwartz's space D' [8] and to extend the Fourier transform to a space of generalized functions [4].

The usual operations of translation, addition, etc. are defined for the g. f.'s (g. f. = generalized function), and the classical formulas pertaining to such operations are extended to the g. f.'s. Differentiation, integration, and convergence are defined and the usual limit interchanges in distribution theory are justified. In particular, the convergence defined is the "weak" sequential convergence suggested by Mikusiński, and it is shown that the g. f. space is "complete" with respect to this sequential convergence. Using this completeness property, it is shown that the Laplace transform maps the g. f. space onto the class of all functions which are analytic in some half plane $Rz > a \geqslant 0$. We also give several characterizations of the g. f.'s which are distributional derivatives of continuous functions of exponential order, and an inversion formula for such g. f.'s is presented.

Convolution is defined for the g. f.'s, and we give conditions under which the convolution equation $A*X = B$ has a solution X. Multiplication of a g. f. by a suitably well-bebaved function is also defined. In the final section, we consider g. f.'s depending on a parameter and establish some of the limit interchanges that have been used formally in the operational calculus.

2. Preliminaries

We will denote by A the class of all functions $f(z)$ which are analytic in some half plane $Rz > a \geqslant 0$. The half plane may depend on the function. A sequence $\{f_n(z)\}$ of functions in A *converges* to $f(z)$ in A if there exists

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a half plane in which the $f_n(z)$ and $f(z)$ are analytic and $f_n(z)$ converges to $f(z)$ uniformly on compact subsets of this half plane. This convergence will be denoted by $\lim f_n(z) = f(z)$.

The *support* of a continuous function g is the closure of the set $\{t: g(t) \neq 0\}$. A complex valued function $p(t)$ defined on $(-\infty, \infty)$ will be called *perfect* if:

(i) p has support in $[0, \infty)$,

(ii) p is infinitely differentiable,

(iii) $p^{(k)}(0) = 0$ for all $k \geqslant 0$,

(iv) $p^{(k)}$ is of exponential order $k \geqslant 0$.

Every perfect function p has a Laplace transform

$$\int_0^\infty p(t)\exp(-zt)\,dt$$

which determines an element in A [9].

Definition 2.1. A sequence $\{p_n\}$ of perfect functions is *fundamental* if the sequence $\{L\{p_n\}(z)\}$ converges in A, where $L\{p_n\}$ denotes the Laplace transform of p_n.

Two fundamental sequences $\{p_n\}$ and $\{q_n\}$ are *equivalent* if $\lim L\{p_n\}(z) = \lim L\{q_n\}(z)$. It is easy to see that this relation is an equivalence relation and the equivalence classes determined by it are called *generalized functions*. The space of all generalized functions will be denoted by G. If $\{p_n\}$ is a fundamental sequence, the generalized function determined by $\{p_n\}$ will be denoted by $[p_n(t)]$.

Throughout the remainder of this paper ordinary functions will be denoted by small letters whereas, with the single exception of the Dirac delta "function" $\delta(t)$, elements of G will be denoted by capital letters.

Finally if $X(t) = [x_n(t)]$ is an element of G, the *Laplace transform* $L\{X(t)\}$ of $X(t)$ is defined by

(i) $$L\{X(t)\} = \lim L\{x_n\}.$$

Clearly this definition is independent of the sequence representing $X(t)$. Note that $L\{X(t)\}$ is an element in A. We have used the same symbol L to denote the classical Laplace transform and the transform defined on G, but this should cause no difficulty considering our convention of capital letters.

3. Elementary operations

Let $X(t) = [x_n(t)]$, $Y(t) = [y_n(t)]$ belong to G, and let c be a complex number.

Definition 3.1. The *sum* and *scalar product* are defined by $X(t) + Y(t) = [x_n(t) + y_n(t)]$, $cX(t) = [cx_n(t)]$.

It is clear that these definitions are independent of the sequences $\{x_n\}$ and $\{y_n\}$. Moreover,

$$L\{X(t)+Y(t)\} = L\{X(t)\}+L\{Y(t)\} \quad \text{and} \quad L\{cX(t)\} = cL\{X(t)\}.$$

Proceeding *via* sequences as above, we obtain:

$$L\{X(ct)\}(z) = (1/c)f(z/c), \quad c \neq 0,$$
$$L\{X(t-d)\}(z) = \exp(-dz)f(z), \quad d \geqslant 0,$$
$$L\{\exp(ct)X(t)\}(z) = f(z-c),$$
$$L\{(-t)^n X(t)\}(z) = f^{(n)}(z), \quad n \geqslant 0,$$

where $L\{X(t)\} = f$.

4. The delta function

Definition 4.1. A sequence $\{\delta_n\}$ of perfect functions will be called a *δ-sequence* if:

(i) $\delta_n(t) \geqslant 0$,

(ii) the support of δ_n is in $[0, 1/n]$,

(iii) $\int_0^\infty \delta_n(t)\,dt = 1$.

Example 4.1. Let

$$\delta_n(t) = \begin{cases} c_n \exp\big(-1/t + 1/(t-1/n)\big), & 0 \leqslant t < 1/n, \\ 0, & t \geqslant 1/n, \end{cases}$$

where c_n is such that

$$\int_0^\infty \delta_n(t)\,dt = 1.$$

Then $\{\delta_n\}$ is a δ-sequence.

THEOREM 4.1. *If $\{\delta_n\}$ is a δ-sequence, then* $\lim L\{\delta_n\} = 1$.

Proof. Let $\varepsilon > 0$. Each δ_n is bounded so that $L\{\delta_n\}$ is analytic in $\mathrm{R}z > 0$. Let K be a compact subset of $\mathrm{R}z > 0$, and let $M = \sup\{|z|: z \text{ in } K\}$. Then

$$|\exp(-zt) - 1| \leqslant \sum_{j=1}^\infty |zt|^j/j! \leqslant \sum_{j=1}^\infty (Mt)^j/j! = h(t).$$

Since $h(t)$ is continuous and $h(0) = 0$, there exists $d > 0$ such that $|h(t)| < \varepsilon$, whenever $0 \leqslant t \leqslant d$. Choose $N > 0$ such that $1/N < d$. Then for $n \geqslant N$, z in K,

$$\left|\int_0^\infty \delta_n(t)\exp(-zt)\,dt - 1\right| \leqslant \int_0^{1/n} \delta_n(t)h(t)\,dt < \varepsilon.$$

Thus $\lim L\{\delta_n\} = 1$.

From Theorem 4.1 it follows that all δ-sequences are equivalent and therefore determine a generalized function which will be denoted by $\delta(t)$. Also from Theorem 4.1, $L\{\delta(t)\} = 1$, a property usually attributed to the Dirac delta function.

We will have need for the following:

THEOREM 4.2. *Let $\{\delta_n\}$ be a δ-sequence and $f(t)$ a continuous function on $[0, \infty)$. Then*

$$\lim f*\delta_n(t) = \lim \int_0^t f(t-s)\,\delta_n(s)\,ds = f(t),$$

where the convergence is uniform on compact subsets of $[0, \infty)$.

Proof. Let $\varepsilon > 0$ and $b > 0$. Since f is uniformly continuous on $[0, b]$, there exists $d > 0$ such that $|f(s-t)-f(s)| < \varepsilon$ whenever $|t| < d$ and $s, s-t$ are in $[0, b]$. Now choose $N > 0$ such that $1/N < d$; then for $n \geqslant N$ and $0 \leqslant s \leqslant b$,

$$|f*\delta_n(s)-f(s)| \leqslant \int_0^{1/n} \delta_n(t)\,|f(s-t)-f(s)|\,dt < \varepsilon.$$

5. Derivatives, integrals, and convergence

Let the kth derivative or iterated integral of a function $x(t)$ be denoted by $x^{(k)}(t)$ and $x^{(-k)}(t)$, respectively.

THEOREM 5.1. *If $\{x_n\}$ is a fundamental sequence such that $\lim L\{x_n\} = g$, then $\{x_n^{(k)}\}$ is a fundamental sequence and $\lim L\{x_n^{(k)}\}(z) = z^k g(z)$ for $k = 0, \pm 1, \ldots$*

Proof. The functions $x_n^{(k)}$ are perfect, and since $L\{x_n^{(k)}\}(z) = z^k L\{x_n\}(z)$, the conclusion follows.

Definition 5.1. Let $X(t) = [x_n]$ belong to G. The *distributional derivative* of $X(t)$, $DX(t)$, is defined by $DX(t) = [x_n^{(1)}(t)]$, and the integral, $D^{-1}X(t)$, is defined by $D^{-1}X(t) = [x_n^{(-1)}(t)]$. Higher order derivatives and integrals are defined as usual.

From Theorem 5.1 it follows that the derivative and integral are well-defined and $L\{D^k X(t)\}(z) = z^k L\{X(t)\}(z)$. Furthermore,

THEOREM 5.2. *Every element of G is infinitely differentiable.*

Example 5.1. The identity $L\{D^k \delta(t)\}(z) = z^k$ follows from Theorem 5.1 and Theorem 4.1.

Convergence in G is introduced as suggested by Mikusiński [5], that is, the convergence is defined as a "weak" convergence with respect to the spaces involved.

Definition 5.2. Let $X_n(t)$ belong to G. The sequence $\{X_n(t)\}$ *converges* to $X(t)$ in G if $\lim L\{X_n(t)\} = L\{X(t)\}$. We write $\lim X_n(t) = X(t)$.

THEOREM 5.3. *If $\lim X_n(t) = X(t)$, $\lim Y_n(t) = Y(t)$, and c is a complex number, then $\lim cX_n(t) = cX(t)$, $\lim\big(X_n(t)+Y_n(t)\big) = X(t)+Y(t)$.*

The following theorem plays a central role in distribution theory:

THEOREM 5.4. *If $\lim X_n(t) = X(t)$, then $\lim D^k X_n(t) = D^k X(t)$ for $k = 0, \pm 1, \ldots$*

Proof. $\lim L\{D^k X_n(t)\} = z^k \lim L\{X_n(t)\} = z^k L\{X(t)\} = L\{D^k X(t)\}$.

THEOREM 5.5. *If the $X_n(t)$ in G are such that $\lim X_n(t)$ exists in A, then there exists $X(t)$ in G such that $\lim X_n(t) = X(t)$.*

Proof. Let $X_n(t) = [x_{nm}(t)]$. Then

$$\lim_m L\{x_{nm}\} = L\{X_n(t)\}.$$

By hypothesis the sequence $L\{X_n(t)\}$ converges to a function g which is analytic in a half plane $\mathrm{R}z > a \geqslant 0$. The double sequence $\{L\{x_{n,m}\}\}$ has a subseqence which converges to g, uniformly on compact subsets of $\mathrm{R}z > a$. Let this subsequence be denoted by $\{L\{y_k\}\}$. Then $X(t) = [y_k]$ is such that $L\{X(t)\} = g$.

This Theorem could be interpreted as a "completeness" theorem. Using this result we characterize $L\{G\}$, the image of G under L.

THEOREM 5.6. *Let $f(z)$ be analytic in some half plane $\mathrm{R}z > a \geqslant 0$. Then there is an element $X(t)$ in G such that $L\{X(t)\} = f$.*

Proof. First note that any polynomial is the transform of an element of G so that from Theorem 5.5 it is enough to show that f is the limit (in the sense of convergence in A) of a sequence of polynomials.

Let D_n be an expanding sequence of closed, bounded rectangles such that the union of the D_n is the half plane $\mathrm{R}z > a$. Using a result in [2], p. 303, Th. 16.6.4, for every $n \geqslant 0$ there is a sequence $\{p_{nk}(z)\}$ of polynomials such that

$$\lim_k \max_{D_n} |p_{kn}(z)-f(z)| = 0.$$

Thus for every n there exists $k(n)$ such that

$$\max_{D_n} |p_{nk(n)}(z)-f(z)| < 1/n.$$

The sequence $\{p_{nk(n)}\}$ is then such that

$$\lim_n p_{nk(n)} = f.$$

Finally using the convergence defined, another interpretation of the distributional derivative may be given.

THEOREM 5.7. *Let $h > 0$ and $X(t)$ belong to G. Then*

$$\lim_{h\to 0} \big(X(t-h)-X(t)\big)/(-h) = DX(t).$$

Proof. We have

$$L\{(X(t-h)-X(t))/(-h)-DX(t)\}(z) = L\{X(t)\}(z)\cdot\big((-1/h)\{\exp(-zh)-1\}-z\big) = L\{X(t)\}(z)\sum_{n=2}^{\infty}(-z)^n h^{n-1}/n!$$

and the series in the last term converges to 0 in A.

This formula might be interpreted as a left-hand derivative in the classical sense.

6. Ordinary functions

We now show that the usual calculus of functions having Laplace transforms can be imbedded in G. Let E denote the set of functions defined on $[0,\infty)$ which are locally integrable and of exponential order. If $x(t)$ is in E and $\{\delta_n\}$ is a δ-sequence, the function

$$x_n(t) = x*\delta_n(t) = \int_0^t x(t-s)\,\delta_n(s)\,ds$$

is a perfect function (cf. [6]) provided it is agreed that $x_n(t)=0$ if $t<0$. Moreover, $L\{x_n\} = L\{x\}\cdot L\{\delta_n\}$ so that $\lim L\{x_n\} = L\{x\}$. Thus, $\{x_n\}$ is a fundamental sequence with $L\{[x_n]\} = L\{x\}$.

THEOREM 6.1. *The correspondence $x\to[x_n]$ defines a mapping from E into G which preserves sums, scalar products, and the Laplace transforms in the two spaces agree.*

The element $[x_n]$ in G will be denoted by x^*.

If x belongs to E is such that $x^{(1)}$ belongs to E, then classically $L\{x^{(1)}\}(z) = zL\{x\}(z)-x(0)$. However, in the case of generalized functions

(1) $$L\{Dx^*\} = \lim L\{x_n^{(1)}\} = zL\{x\}$$

and

(2) $$L\{(x^{(1)})^*\} = L\{x^{(1)}\} = zL\{x^*\}-x(0).$$

From (1), (2) and Theorem 4.1,

(3) $$Dx^* = (x^{(1)})^*+x(0)\cdot\delta,$$

the usual formula connecting the classical and distributional derivatives [3].

Locally integrable functions which are not in E but which do have Laplace transforms can also be imbedded in G. That is, the imbedding above will be extended to include these functions. A locally integrable function x such that $L\{x\}$ converges in $\mathrm{R}z\geqslant a$ has the property that $x^{(-1)}(t) = 0(\exp(at))$ (cf. [1]). $(x^{(-1)})^*$ is therefore well defined from the above considerations, and also $L\{(x^{(-1)})^*\}(z) = (1/z)L\{x\}(z)$.

The imbedding may be extended by defining $x\to D(x^{(-1)})^*$ since if x is of exponential order, we have $D(x^{(-1)})^* = x^*+x^{(-1)}(0)\cdot\delta = x^*$ from (3). This extended correspondence will again be denoted by $x\to x^*$ and the extension is easily seen to preserve sums and scalar products. The Laplace transforms in the two spaces coincide. Thus the usual calculus of functions having Laplace transforms in the classical sense is included in G.

In this context Definition 2.1 can be given the following interpretation:

THEOREM 6.2. *If $X=[x_n]$, then $\lim x_n^* = X$.*

Concerning convergence in E and G, we have the following

THEOREM 6.3. *Let $f_n(t)$ be defined for $t\geqslant 0$ and suppose there is some integer $r\geqslant 0$ such that*

(i) *$f_n^{(-r)}$ is continuous,*

(ii) *$|f_n^{(-r)}(t)|\leqslant M\exp(ct)$, $c>0$,*

(iii) *$f_n^{(-r)}$ converges uniformly on compact subsets to $h(t)$.*

Then $\lim L\{f_n^{(-r)}\} = L\{h\}$.

Proof. Let $\varepsilon>0$. Let K be a compact subset of $\mathrm{R}z>c$ and set $b=\inf\{x\colon z=x+iy$ is in $K\}$. Then $c<b$ and for $z=x+iy$ in K,

(1) $$|L\{h-f_n^{(-r)}\}(z)| \leqslant \int_0^\infty |h(t)-f_n^{(-r)}(t)|\exp(-bt)\,dt \leqslant \int_0^B |h(t)-f_n^{(-r)}(t)|\exp(-bt)\,dt+2M\int_B^\infty \exp(c-b)t\,dt.$$

Now B can be chosen so large that the last term in (1) is $<\varepsilon/2$ and then with such a B fixed the first term can be made $<\varepsilon/2$ by virtue of (iii). Thus, $\lim L\{f_n^{(-r)}\} = L\{h\}$.

COROLLARY 6.1. *If the f_n are perfect and are as in Theorem 6.3, then $\{f_n\}$ is fundamental, $h^* = [f_n^{(-r)}]$, and $L\{[f_n]\}(z) = z^rL\{h\}(z)$.*

COROLLARY 6.2. *If the f_n are in E and are as in Theorem 6.3, then $f_n^* = D^rh^*$.*

Proof. From Theorem 6.1, $\lim(f_n^{(-r)})^* = h^*$. Differentiating r times yields the result.

7. Generalized functions of finite order

Definition 7.1. An element X of G is of *finite order* if there is a continuous function h of exponential order such that $X = D^rh^*$ for some $r\geqslant 0$. The smallest such r is called the *order* of X.

We give a characterization of elements of finite order by means of their transforms.

THEOREM 7.1. *If f is analytic in $\mathrm{R}z \geqslant a \geqslant 0$ and $f(z) = O(z^n)$, then f is the Laplace transform of an element of G of order $\leqslant n+2$.*

Proof. Let $g(z) = z^{-n-2}f(z)$. Then since $g(z) = O(z^{-2})$,

$$h(t) = (1/2\pi)\exp(at)\int_{-\infty}^{\infty} g(a+iy)\exp(iyt)\,dy$$

is continuous, exponential order, and $L\{h\} = g$ (see [1]). Thus $L\{D^{n+2}h^*\} = z^{n+2}g(z) = f(z)$.

THEOREM 7.2. *If X in G is of finite order n, then $L\{X\}(z) = O(z^{n-1})$.*

Proof. If $X = D^n h^*$, then $L\{X\}(z) = z^n L\{h\}$ which implies $L\{X\}(z) = O(z^{n-1})$.

The following example shows that not all elements of G are of finite order.

Example 7.1. The series

$$\sum_{0}^{\infty} D^n\delta/n!$$

converges in G, and its Laplace transform is $\exp(z)$. Thus from Theorem 7.2, this element cannot be of finite order.

Remark. Notice that this series fails to converge in Schwartz's D' [7].

THEOREM 7.3. *Let $X = D^r h^*$, where h is continuous and $|h(t)| \leqslant M\exp(ct)$. Let $L\{h\} = f$. Then the sequence*

$$f_n(t) = (1/2\pi)\exp(ct)\int_{-n}^{n} f(c+iy)\cdot\exp(iyt)\,dy$$

is such that $\lim D^r f_n^* = X$.

Proof. The integral

$$(1/2\pi)\exp(ct)\int_{-\infty}^{\infty} f(c+iy)\exp(iyt)\,dy$$

converges absolutely and uniformly on compact subintervals of $t \geqslant 0$ to $h(t)$ (cf. [1]). Thus the sequence f_n converges to h, uniformly on compact subsets. Also,

$$|f_n(t)| \leqslant (1/2\pi)\exp(ct)\int_{-\infty}^{\infty}|f(c+iy)|\,dy = M\exp(ct).$$

Thus, from Theorem 6.3, $\lim L\{f_n\} = L\{h\}$ so that $\lim f_n^* = h^*$. Differentiating r times gives $\lim D^r f_n^* = X$.

Theorem 7.3 gives an inversion formula for X.

THEOREM 7.4. *Let g be analytic in $\mathrm{R}z > a$ and $g(z) = O(z^n)$. If $r = n+2$, the function*

$$f(t) = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} z^{-r}g(z)\exp(zt)\,dz \qquad (c > a)$$

is such that $L\{D^r f^*\} = g$.

Proof. f is continuous, exponential order, and $L\{f\}(z) = z^{-r}g(z)$. Therefore, $L\{D^r f^*\} = g$.

We include here some results of Korevar [3], with slight restatements, which we will use in a later section.

Definition 7.2. A fundamental sequence $\{f_n\}$ is of *exponential type* (N) if $|f_n^{(-N)}(t)| \leqslant M\exp(ct)$ for all n, some $c > 0$. An element X of G is of *exponential type* (N) if $X = [f_n]$, where $\{f_n\}$ is of exponential type (N).

Remark. If $\{f_n\}$ is of exponential type (N), then it is of exponential type $(N+p)$ for all $p \geqslant 0$.

THEOREM 7.5. *If $\{f_n\}$ is of exponential type (N), then $f_n^{(-N-1)}$ converges uniformly on compact subsets of $t \geqslant 0$ to a continuous function $h(t)$ in E.*

Proof. See [3].

COROLLARY 7.1. *If $\{f_n\}$ is of exponential type (N), then $[f_n]$ is of finite order $\leqslant N+1$.*

Proof. From Theorem 7.5 and Theorem 6.3, $\lim L\{f_n^{(-N-1)}\} = L\{h\}$ so that $\lim L\{f_n\} = z^{N+1}L\{h\}$. Hence $[f_n] = D^{N+1}h^*$.

THEOREM 7.6. *If X is of finite order r, then X is of exponential type (r).*

Proof. Suppose $X = D^r h^*$, $|h(t)| \leqslant M\exp(ct)$. Then $|h*\delta_n(t)| \leqslant M\exp(ct)$. Therefore, since $X = D^r h^* = [(h*\delta_n)^{(r)}]$, X is of exponential type (r).

Theorems 7.5 and 7.6 thus characterize g. f.'s of finite order. This characterization will be used in a later section.

8. Convolution

Definition 8.1. Let $X = [x_n]$, $Y = [y_n]$ belong to G. The *convolution* of X and Y, $X*Y$, is defined by

$$[x_n * y_n] = \Big[\int_0^t x_n(t-s)\,y_n(s)\,ds\Big].$$

Since each $x_n * y_n$ is perfect and $\lim L\{x_n * y_n\} = \lim L\{x_n\}\cdot L\{y_n\} = L\{X\}\cdot L\{Y\}$, $\{x_n * y_n\}$ is fundamental and $L\{X*Y\} = L\{X\}\cdot L\{Y\}$.

From Definition 8.1 and the usual properties of the convolution of functions, it is easily seen that G is a commutative algebra over the complex field with identity (δ) and no zero divisors. Even such formula as $D^k(X*Y) = D^k X*Y$ hold for arbitrary X, Y in G.

Concerning convergence, the following result holds:

THEOREM 8.1. *If* $\lim X_n = X$, $\lim Y_n = Y$, *then* $\lim X_n*Y_n = X*Y$.

Proof. Since $\lim L\{X_n*Y_n\} = \lim L\{X_n\}\cdot L\{Y_n\} = L\{X\}L\{Y\} = L\{X*Y\}$, the conclusion follows.

Some applications of the convolution product will now be considered. We consider the convolution equation

$$A*X = B, \tag{1}$$

where A and B are arbitrary elements of G. We show that (1) has a solution X under appropriate conditions.

THEOREM 8.2. *If* $1/L\{A\}$ *is analytic in some half plane* $\mathrm{R}z > a$, *then* (1) *has a unique solution.*

Proof. The function $f = L\{B\}/L\{A\}$ is analytic in some half plane. From Theorem 5.6, there is X in G such that $L\{X\} = f$. This is the desired solution of (1). That X is unique follows from the fact that G has no zero-divisors.

Now let P be a linear differential operator with constant coefficients. The equation

$$P\{X\} = B \tag{2}$$

can be interpreted as a convolution equation of the form (1) by setting

$$A = \sum_{j=0}^{n} a_j D^j \delta \quad \text{where} \quad P(x) = \sum_{j=0}^{n} a_j x^j.$$

Thus, since $1/L\{A\}$ is a rational function, it follows from Theorem 8.2 that (2) always has a solution for any B in G. In particular, any such differential operator has an elementary solution, that is, the equation $P\{X\} = \delta$ always has a solution. We also note that if B is of order m, from Theorem 7.2 $L\{B\}(z) = O(z^{m-1})$. Hence $L\{B\}/L\{A\} = O(z^{m-n-1})$ and from Theorem 7.1, X is of order $\leqslant (m-n+1)$.

Using this same idea we can consider differential operators of "infinite order",

$$P = \sum_{j=0}^{\infty} a_j D^j.$$

That is, the equation

$$P\{X\} = B \tag{3}$$

would correspond to the convolution equation (1) with

$$A = \sum_{j=0}^{\infty} a_j D^j \delta,$$

provided this series converges. We note that from Theorem 8.1, equation (1) actually has meaning in this case, and from Theorem 8.2, (3) will have a solution X if $1/L\{A\}$ is analytic in some half plane. In particular, we note that (3) has a solution if $a_j = 1/j!$. This operator would correspond to the differential operator $\exp(D)$ which often appears in the Heaviside operational calculus.

9. Multiplication by a smooth function

In this section multiplication of a g. f. of finite order by a suitably well-behaved function will be defined. This multiplication corresponds roughly to the convolution of two Schwartz distributions since the Laplace transform of the product of two functions is given by a complex convolution formula [1]. For this reason no attempt at a general definition of such a product will be made and only g. f.'s of finite order will be considered. The functions considered will be infinitely differentiable and such that every derivative is of exponential order. Any such function will be called a *smooth function.*

Several lemmas will be needed for the definition.

LEMMA 9.1. *Let* f_n *be continuous and* $|f_n(t)| \leqslant M\exp(at)$ *for* $t \geqslant 0$ *and suppose that* f_n *converges uniformly on compact subsets of* $t \geqslant 0$ *to* f. *Let* g *be continuous and* $|g(t)| \leqslant M\exp(bt)$. *Then* $\lim L\{f_n g\} = L\{fg\}$.

Proof. Since $|f_n(t)g(t)| \leqslant M^2\exp\big((a+b)t\big)$ and $f_n g$ converges to fg, uniformly on compact subsets, the lemma follows from Theorem 6.3.

LEMMA 9.2. *If* $\{f_n\}$ *is of exponential type* (N) *and* $a(t)$ *is a smooth function,* $\{a\cdot f_n\}$ *is fundamental.*

Proof. The proof goes by induction on N. For $N=0$, from Theorem 7.5, $f_n^{(-1)}$ converges to a function f, uniformly on compact subsets. Since

$$af_n = (af_n^{(-1)})^1 - a^1 f_n^{(-1)}, \tag{1}$$

it follows from Lemma 9.1 that $\{a\cdot f_n\}$ is fundamental.

Suppose the lemma holds for $N < k$ and let $\{f_n\}$ be of exponential type (k). $f_n^{(-1)}$ is then of exponential type $(k-1.)$ From the induction hypothesis $\{af_n^{(-1)}\}$ and $\{a^1 f_n^{(-1)}\}$ are fundamental, so that from (1) it follows that $\{a\cdot f_n\}$ is fundamental.

LEMMA 9.3. *If* $X = [x_n] = [y_n]$, *where* $\{x_n\}$ *and* $\{y_n\}$ *are of exponential type, and if* $a(t)$ *is smooth, then* $\lim L\{ax_n\} = \lim L\{ay_n\}$.

Proof. By the remark following Definition 7.2, it may be assumed that $\{x_n\}$ and $\{y_n\}$ are both of exponential type (N). The proof again goes by induction on N.

Let $f_n = x_n - y_n$. For $N=0$, from Theorem 7.5, $f_n^{(-1)}$ converges to 0 uniformly on compact subsets and $|f_n^{(-1)}(t)| \leqslant M\exp(bt)$. From Lemma 9.1, $\lim L\{af_n^{(-1)}\} = 0$ and $\lim L\{a^1 f_n^{(-1)}\} = 0$. Hence from (1), $\lim L\{a\cdot f_n\} = 0$.

Suppose the result holds for $N < k$ and assume $\{f_n\}$ is of exponential type (k). Then $f_n^{(-1)}$ is of exponential type $(k-1)$. From the induction hypothesis, $\lim L\{af_n^{(-1)}\} = 0$ and $\lim L\{a^1 f_n^{(-1)}\} = 0$, so that from (1) $\lim L\{af_n\} = 0$.

The following definition can now be made:

Definition 9.1. Let $X = [x_n]$ belong to G, where $\{x_n\}$ is of exponential type (N), and let $a(t)$ be smooth. The product $a\cdot X$ is defined as $aX = [ax_n]$.

From Lemma 9.2, the sequence $\{ax_n\}$ is fundamental and from Lemma 9.3 the product is independent of the particular exponential type sequence representing X.

Remark. If $X = h^*$, where h is continuous and exponential order, the classical Laplace transform of $a\cdot h$ coincides with the generalized Laplace transform $L\{a\cdot X\}$. Indeed, $h^* = [h*\delta_n]$ and $h*\delta_n$ converges to h uniformly on bounded intervals. Therefore $a\cdot(h*\delta_n)$ converges to $a\cdot h$ in the same manner. From Theorem 6.3, $\lim L\{a(h*\delta_n)\} = L\{ah\} = L\{ah^*\}$.

Some of the usual identities will now be established.

THEOREM 9.1. *Let a be smooth and X of finite order. Then* $D(aX) = a^1X + aDX$.

Proof. Let $X = [x_n]$, where $\{x_n\}$ is of exponential type. Then

$$D(aX) = [(ax_n)^1] = [a^1 x_n] + [ax_n^1] = a^1X + aDX.$$

A generalization of this formula is

$$aD^kX = \sum_{j=0}^{k} (-1)^j \binom{k}{j} D^{k-j}(a^jX). \tag{2}$$

THEOREM 9.2. *If a is smooth, then* $a\cdot\delta = a(0)\cdot\delta$.

Proof. Let $\{\delta_n\}$ be a δ-sequence. Then $\lim L\{(a(t)-a(0))\delta_n(t)\}(z) = 0$ since

$$|L\{(a(t)-a(0))\delta_n(t)\}(z)| \leqslant \int_0^\infty |a(t)-a(0)|\exp(-xt)\delta_n(t)\,dt$$
$$\leqslant \max_{0\leqslant t\leqslant 1/n} |a(t)-a(0)|.$$

In a similar fashion it can be shown that $a(t)\cdot\delta(t-n) = a(n)$. This identity can be used to illustrate that general limit interchanges are not possible with respect to multiplication. For $\lim\delta(t-n) = 0$, but if $a(t) = t$ the sequence $a(t)\delta(t-n) = n$ will not converge. The following does hold however:

THEOREM 9.3. *Let $X_n = D^k h_n^*$ for $n \geqslant 0$ and let*

$$\lim_{n\to\infty} h_n(t) = h_0(t),$$

where the convergence is uniform on compact subsets, and $|h_n(t)| \leqslant M\exp(at)$. Let a_n be smooth and such that for $0 \leqslant j \leqslant k$, $\lim a_n^j = a_0^j$, where the convergence is uniform on compact subsets, and $|a_n^j(t)| \leqslant M\exp(at)$. Then $\lim a_nX_n = a_0X_0$.

Proof. From (2), for $n \geqslant 0$

$$a_nX_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} D^{k-j}(a_n^j h_n^*). \tag{3}$$

From Theorem 6.3, $\lim a_n^j h_n^* = a_0^j h_0^*$ for $0 \leqslant j \leqslant k$. Therefore, the result follows from (3).

10. Generalized functions depending on a parameter

We now consider g. f.'s depending upon a parameter and establish some of the limit interchanges which have been used in the operational calculus. Throughout this section, we let S be a subset of the complex numbers with c a limit point of S. Suppose that for each b in S there is associated a g.f., denoted by $X(b,t)$. We denote the distributional derivative of $X(b,t)$ by $D_2X(b,t)$. We say $\lim_{b\to c} X(b,t) = X(c,t)$ if $\lim_{b\to c} L\{X(b,t)\} = L\{X(c,t)\}$.

Definition 10.1. The *derivative of $X(b,t)$ with respect to the parameter b at $c = b$* is defined by

$$\lim_{b\to c}\big(X(b,t)-X(c,t)\big)/(b-c) = D_1X(c,t),$$

provided the limit exists.

THEOREM 10.1. *If $D_1X(c,t)$ exists, then $D_{21}X(c,t)$ exists and* $D_{21}X(c,t) = D_{12}X(c,t)$.

Proof. Since

$$\lim_{b\to c} L\{(D_2X(b,t)-D_2X(c,t))/(b-c)\}(z)$$
$$= zL\{D_1X(c,t)\}(z) = L\{D_{12}X(c,t)\}(z),$$

the result follows.

THEOREM 10.2. *If $D_1X(c,t)$ exists, then*

$$\frac{\partial}{\partial b}L\{X(c,t)\}(z) = L\{D_1X(c,t)\}(z).$$

Proof. We have

$$L\{D_1 X(c,t)\}(z) = \lim_{b\to c} L\{(X(b,t)-X(c,t))/(b-c)\}(z)$$
$$= \lim_{b\to c}(L\{X(b,t)\}(z)-L\{X(c,t)\}(z))/(b-c)$$
$$= \frac{\partial}{\partial b} L\{X(c,t)\}(z);$$

hence the result.

These limit interchanges have been used formally to solve partial differential equations by Laplace transform methods.

Finally, we have the following

THEOREM 10.3. *If $D_1 X(c,t)$ exists, then for each $Y(t)$ in G*

$$D_1(X(c,t)*Y(t)) = Y(t)*D_1(c,t).$$

Proof. Since

$$\lim_{b\to c} L\{(X(b,t)*Y(t)-X(c,t)*Y(t))\}/(b-c)$$
$$= L\{Y(t)\}\cdot L\{D_1 X(c,t)\} = L\{Y(t)*D_1 X(c,t)\},$$

$D_1(X(c,t)*Y(t))$ exists and is equal to $Y(t)*D_1 X(c,t)$.

Bibliography

[1] G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Berlin 1937.

[2] E. Hille, *Analytic function theory II*, New York 1959.

[3] J. Korevaar, *Distributions defined from the point of view of applied mathematics*, Proc. Kon. Ned. Ak. Wet. Amsterdam A 58 (1955), p. 483-503.

[4] M. J. Lighthill, *Introduction to Fourier analysis and generalized functions*, Cambridge 1958.

[5] J. Mikusiński, *Sur la méthode de généralisation de Laurent-Schwartz et sur la convergence faible*, Fund. Math. 35 (1948), p. 235-239.

[6] — and C. Ryll-Nardzewski, *Sur le produit de composition*, Studia Math. 12 (1951), p. 51-57.

[7] L. Schwartz, *Théorie des distributions I et II*, Paris 1950-1951.

[8] G. Temple, *Generalized functions*, J. London Math. Soc. 28 (1953), p. 134-148.

[9] J. Weston, *Operational calculus and generalized functions*, Proc. Royal Soc. Lond., Ser. A 250 (1959), p. 460.

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A kernel associated with certain multiply connected domains and its applications to factorization theorems

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§ 1. Introduction. In this paper we introduce a natural extension of the kernel

$$\mathscr{P}(z,\zeta) = \frac{1}{2\pi}\frac{\zeta+z}{\zeta-z}, \quad |z|<1,\ |\zeta|=1,$$

associated with the unit disc. It is well known that many properties of analytic functions on this domain can be derived by making use of $\mathscr{P}(z,\zeta)$, whose real part is the Poisson kernel and whose imaginary part is the conjugate Poisson kernel. The theory of classical H^p spaces, for example, can be easily developed by making use of its basic properties (see [11]). We construct similar kernels associated with multiply connected domains of connectivity n. We shall show how they can be used in order to obtain some basic conformal mappings of such domains onto canonical "slit" domains. Our main result is a generalization of the classical factorization theorem for functions in the Nevanlinna class of the unit disc.

Let $\mathscr{D}$ be a bounded domain whose boundary Γ consists of n disjoint simple closed analytic curves $\gamma_1,\dots,\gamma_n$ (we shall indicate later in § 9 how to include more general domains in these considerations). We shall always let γ_n denote the curve whose interior contains $\mathscr{D}$ (thus $\gamma_1,\dots,\gamma_{n-1}$ are contained in the interior of γ_n). The existence and basic properties of the kernel associated with such domains will be the subject of § 3 where we shall prove the following theorem:

THEOREM I. *Let $z_0\in\mathscr{D}$ be fixed, then there exists a unique jointly continuous function, $\mathscr{P}$, defined on $\mathscr{D}\times\Gamma$ satisfying*

(1) *for $\zeta\in\Gamma$ fixed $\mathscr{P}(z,\zeta)$ defines an analytic function of $z\in\mathscr{D}$;*

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