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A characterization of Hilbert-Schmidt operators

by

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In this note we substantiate a conjecture of Pietsch [4] by proving that in a Hilbert space the class of all Hilbert-Schmidt operators coincides with the class of all absolutely p-summing operators for arbitrary fixed p with $1 \leqslant p < +\infty$.

Let H_1 and H_2 be Hilbert spaces. A linear operator $T: H_1 \to H_2$ is called a *Hilbert-Schmidt operator* provided $\sum_{\iota \in I} \|Te_\iota\|^2 < +\infty$ for some (equivalently, for each) orthonormal basis $(e_\iota)_{\iota \in I}$ in H_1 (cf. [1], p. 138). If $1 \leqslant p < +\infty$, then a linear operator $T: H_1 \to H_2$ is *absolutely p-summing* (cf. [4]) provided there is a positive constant C such that for arbitrary sequence (x_n) in H_1

$$(1) \qquad \Big(\sum_{n=1}^{\infty} \|Tx_n\|^p\Big)^{1/p} \leqslant C \sup_{\|a\|=1} \Big(\sum_{n=1}^{\infty} |\langle x_n, a\rangle|^p\Big)^{1/p}.$$

The class of all Hilbert-Schmidt operators from H_1 into H_2 [resp. of all absolutely p-summing operators] will be denoted by $\mathfrak{S}_2(H_1, H_2)$ [resp. $\Pi_p(H_1, H_2)$]

THEOREM. $\mathfrak{S}_2(H_1, H_2) = \Pi_p(H_1, H_2)$ *for* $1 \leqslant p < +\infty$.

The case $p = 1$ was probably first explicitly stated in the literature by Pietsch (cf. e.g. [5], p. 42, and the references in [5]). However, it seems to be known to Grothendieck (cf. [2], p. 55, Théorème 6). The Theorem in the case where $1 < p < 2$ has been recently established by Pietsch [4], Satz 5. Combining this result with [4], Satz 11, we get the inclusion $\mathfrak{S}_2(H_1, H_2) \subset \Pi_p(H_1, H_2)$ for $1 \leqslant p < +\infty$. The new result obtained in this note is the inclusion $\Pi_p(H_1, H_2) \subset \mathfrak{S}_2(H_1, H_2)$. However, we present here for completeness a direct proof of the general case.

1. Preliminary remarks. It is well known that every Hilbert-Schmidt operator is compact. Similarly

1.1. *If* $T \in \Pi_p(H_1, H_2)$, *then* T *is compact.*

Proof. We shall show first that if $(e_n)_{n=1}^{\infty}$ is an orthonormal sequence in H_1, then $\lim_{n=1} Te_n = 0$. Indeed, if it was not true, then there would exist $\eta > 0$ and an orthonormal sequence $(e_n)_{n=1}^{\infty}$ such that $\|Te_n\| > \eta$ for $n = 1, 2, \ldots$ Therefore

$$(2) \qquad \Big(\sum_{n=1}^{N} \|Te_n\|^p\Big)^{1/p} > N^{1/p}\eta \quad \text{for} \quad N = 1, 2, \ldots$$

On the other hand, the orthonormality of the sequence (e_n) implies

$$\Big(\sum_{n=1}^{N} |\langle e_n, a\rangle|^2\Big)^{1/2} \leqslant \|a\| \quad \text{for} \quad a \in H_1.$$

Thus for $p \geqslant 2$

$$(3) \qquad \sup_{\|a\|=1} \Big(\sum_{n=1}^{N} |\langle e_n, a\rangle|^p\Big)^{1/p} \leqslant \sup_{\|a\|=1} \Big(\sum_{n=1}^{N} |\langle e_n, a\rangle|^2\Big)^{1/2} \leqslant 1,$$

and for $1 \leqslant p < 2$, by Hölder inequality,

$$(4) \qquad \sup_{\|a\|=1} \Big(\sum_{n=1}^{N} |\langle e_n, a\rangle|^p\Big)^{1/p} \leqslant N^{1/p-1/2} \sup_{\|a\|=1} \Big(\sum_{n=1}^{N} |\langle e_n, a\rangle|^2\Big)^{1/2} \leqslant N^{1/p-1/2}.$$

Clearly inequalities (2), (3) and (4) contradict (1).

To complete the proof of the compactness of T it is enough to show that if $T: H_1 \to H_2$ is a non-compact linear operator, then $\underline{\lim}_n \|Te_n\| > 0$ for some orthonormal sequence $(e_n)_{n=1}^{\infty}$ in H_1.

We shall define such a sequence (e_n) inductively. Since T is not compact, T is not a limit in the operator norm $|||\cdot|||$ of a sequence of operators of finite dimensional ranges. Hence there is $\eta > 0$ such that $|||T-P||| > \eta$ for each linear operator $P: H_1 \to H_2$ of a finite-dimensional range. In particular, $|||T||| > \eta$. Hence there is e_1 in H_1 with $\|e_1\| = 1$ such that $\|Te_1\| > \eta$. Let us suppose that for some $m \geqslant 1$ the elements $e_1, e_2, \ldots, e_m$ have already been defined in such a way that $\|Te_i\| > \eta$ and $\langle e_i, e_j\rangle = \delta_i^j$ for $i, j = 1, 2, \ldots, m$. Let Q denote the orthogonal projection from H_1 onto the subspace E_m spanned by elements $e_1, e_2, \ldots, e_m$. Let $P = TQ$. Then P has a finite-dimensional range. Therefore $|||T-P||| > \eta$. Choose e in H_1 with $\|e\| = 1$ such that $\|(T-P)e\| > \eta$. Obviously $Te \neq Pe$. Thus $e - Qe \neq 0$. Let us set

$$e_{m+1} = \|(e-Qe)\|^{-1}(e-Qe).$$

Clearly $\|e_{m+1}\| = 1$ and $Te_{m+1} = \|e-Qe\|^{-1}(T-P)e$. Since $\|e\| = 1$ and since Q is an orthogonal projection, $\|e-Qe\| \leqslant \|e\| = 1$. Therefore $\|Te_{m+1}\| \geqslant \|(T-P)e\| > \eta$. Finally, since $Qe_{m+1} = 0$, the vector e_{m+1} is orthogonal to each element in the range of Q. In particular, $\langle e_{m+1}, e_i\rangle = 0$ for $i = 1, 2, \ldots, m$. This completes the inductive step and the proof of 1.1.

The countable character of (1) implies that $T \in \Pi_p(H_1, H_2)$ if and only if the restriction of T to each separable subspace of H_1 is absolutely p-summing. The same is true for Hilbert-Schmidt operators. Therefore without loss of generality we shall assume in the sequel that H_1 is separable.

1.2. *If $T: H_1 \to H_2$ is compact, then there is an orthonormal basis (f_n) in H_1 such that $Tf_n = \lambda_n g_n$ where (g_n) is an orthonormal sequence in H_2, λ_n are non-negative and $\lim_n \lambda_n = 0$.*

Proof. Let us consider in H_1 the bilinear form $S(x, y) = \langle Tx, Ty\rangle$. Clearly S is symmetric, non-negative and compact. Thus there is an orthonormal basis (f_n) in H_1 in which S has the diagonal representation, i.e. $\langle Tf_n, Tf_m\rangle = 0$ for $n \neq m$, and $\lim_n \langle Tf_n, Tf_n\rangle = 0$ (cf. [6], § 93). We put $\lambda_n = Tf_n$ for $n = 1, 2, \ldots$, and we define (g_n) as an arbitrary orthonormal sequence in H_2 such that

$$g_n = \|Tf_n\|^{-1} Tf_n \quad \text{for} \quad Tf_n \neq 0.$$

2. Proof of the Theorem.

2.1. $\Pi_p(H_1, H_2) \subset \mathfrak{S}_2(H_1, H_2)$.

Let $T \in \Pi_p(H_1, H_2)$. Then, by 1.1, T is compact. Let (f_n), (g_n) and (λ_n) be as in 1.2. Let r_n denote the n-th Rademacher function, i.e.

$$r_n(t) = \begin{cases} (-1)^k & \text{for} \quad 2^{-n}k < t < 2^{-n}(k+1), \\ 0 & \text{for} \quad t = 2^{-n}k \text{ and for } t = 1 \end{cases}$$
$$(k = 0, 1, \ldots, 2^n-1;\ n = 1, 2, \ldots).$$

Fix a positive integer N and set for $k = 0, 1, \ldots, 2^N-1$

$$x_k = \sum_{n=1}^{N} r_n^k f_n,$$

where

$$r_n^k = r_n\big((2k+1)2^{-(N+1)}\big) \quad (k = 0, 1, \ldots, 2^N-1;\ n = 1, 2, \ldots, N).$$

Then

$$\|Tx_k\| = \Big\|\sum_{n=1}^{N} r_n^k Tf_n\Big\| = \Big\|\sum_{n=1}^{N} r_n^k \lambda_n g_n\Big\| = \Big(\sum_{n=1}^{N} \lambda_n^2\Big)^{1/2}.$$

Therefore

$$(5) \qquad \Big(\sum_{k=0}^{2^N-1} \|Tx_k\|^p\Big)^{1/p} = 2^{N/p}\Big(\sum_{n=1}^{N} \lambda_n^2\Big)^{1/2}.$$

On the other hand, if $a = \sum_{i=1}^{\infty} \langle a, f_i \rangle f_i$, then

$$|\langle x_k, a\rangle|^p = \Big|\sum_{n=1}^{N} r_n^k \overline{\langle a, f_n\rangle}\Big|^p = 2^N \int_{k2^{-N}}^{(k+1)2^{-N}} \Big|\sum_{n=1}^{N} r_n(t) \overline{\langle a, f_n\rangle}\Big|^p dt.$$

Hence

$$\Big(\sum_{k=0}^{2^N-1} |\langle x_k, a\rangle|^p\Big)^{1/p} = \Big(2^N \sum_{k=0}^{2^N-1} \int_{k2^{-N}}^{(k+1)2^{-N}} \Big|\sum_{n=1}^{N} r_n(t) \overline{\langle a, f_n\rangle}\Big|^p dt\Big)^{1/p}$$

$$= 2^{N/p} \Big(\int_0^1 \Big|\sum_{n=1}^{N} r_n(t) \overline{\langle a, f_n\rangle}\Big|^p dt\Big)^{1/p}.$$

Therefore, by Khinchin inequality ([7], Chap. V, Theorem 8.4)

$$(6) \qquad \Big(\sum_{k=0}^{2^N-1} |\langle x_k, a\rangle|^p\Big)^{1/p} \leqslant 2^{N/p} B_p \Big(\sum_{n=1}^{N} |\langle a, f_n\rangle|^2\Big)^{1/2} \leqslant 2^{N/p} B_p \|a\|,$$

where B_p is a constant depending only on p.

Comparing (5) and (6) with (1) we obtain

$$2^{N/p} \Big(\sum_{n=1}^{N} \lambda_n^2\Big)^{1/2} \leqslant C \sup_{\|a\|=1} \Big(\sum_{k=1}^{2^N-1} |\langle x_k, a\rangle|^p\Big)^{1/p} \leqslant C B_p 2^{N/p}.$$

Thus

$$\Big(\sum_{n=1}^{N} \lambda_n^2\Big)^{1/2} \leqslant C B_p \quad \text{for} \quad N = 1, 2, \ldots$$

Since $\|Tf_n\| = \lambda_n$ for $n = 1, 2, \ldots$, the last inequality implies

$$\sum_{n=1}^{\infty} \|Tf_n\|^2 \leqslant (CB_p)^2 < +\infty.$$

This shows that $T \in \mathfrak{S}_2(H_1, H_2)$.

2.2. $\mathfrak{S}_2(H_1, H_2) \subset \Pi_p(H_1, H_2)$.

Let (f_n), (g_n) and (λ_n) be as in 1.2. If $T \in \mathfrak{S}_2(H_1, H_2)$, then

$$A = \Big(\sum_{n=1}^{\infty} \lambda_n^2\Big)^{1/2} < +\infty.$$

Let $(x_m)_{m=1}^{\infty}$ be an arbitrary sequence in H_1. Let for $0 \leqslant t \leqslant 1$

$$b(t) = A^{-1} \sum_{n=1}^{\infty} r_n(t) \lambda_n f_n,$$

where r_n denotes as in 2.1 the n-th Rademacher function. Then (cf. [3]) using the Khinchin inequality ([7], Chap. V, Theorem 8.4; [5], p. 39) we get

$$\sup_{\|a\|=1} \sum_{m=1}^{\infty} |\langle x_m, a\rangle| \geqslant A^{-1} \sup_{0\leqslant t\leqslant 1} \sum_{m=1}^{\infty} |\langle x_m, b(t)\rangle|$$

$$\geqslant A^{-1} \int_0^1 \sum_{m=1}^{\infty} \Big|\sum_{n=1}^{\infty} r_n(t) \lambda_n \langle x_m, f_n\rangle\Big| dt$$

$$= A^{-1} \sum_{m=1}^{\infty} \int_0^1 \Big|\sum_{n=1}^{\infty} r_n(t) \lambda_n \langle x_m, f_n\rangle\Big| dt$$

$$\geqslant (\sqrt{3}A)^{-1} \sum_{m=1}^{\infty} \Big(\sum_{n=1}^{\infty} \lambda_n^2 |\langle x_m, f_n\rangle|^2\Big)^{1/2}$$

$$= (\sqrt{3}A)^{-1} \sum_{m=1}^{\infty} \|Tx_m\|.$$

This proves the inclusion $\mathfrak{S}_2(H_1, H_2) \subset \Pi_1(H_1, H_2)$.

Finally, let $p > 1$ and let $q = p(p-1)^{-1}$. Then

$$\Big(\sum_{m=1}^{\infty} |c_m|^p\Big)^{1/p} = \sup_{\sum_{m=1}^{\infty} |t_m|^q = 1} \sum_{m=1}^{\infty} |t_m c_m|.$$

Since inequality (1) is established already for $p = 1$, we get

$$\Big(\sum_{m=1}^{\infty} \|Tx_m\|^p\Big)^{1/p} = \sup_{\sum_{m=1}^{\infty} |t_m|^q = 1} \sum_{m=1}^{\infty} \|Tt_m x_m\|$$

$$\leqslant \sqrt{3}A \sup_{\sum_{m=1}^{\infty} |t_m|^q = 1} \sup_{\|a\|=1} \sum_{m=1}^{\infty} |\langle t_m x_m, a\rangle|$$

$$= \sqrt{3}A \sup_{\|a\|=1} \sup_{\sum_{m=1}^{\infty} |t_m|^q = 1} \sum_{m=1}^{\infty} |t_m \langle x_m, a\rangle|$$

$$= \sqrt{3}A \sup_{\|a\|=1} \Big(\sum_{m=1}^{\infty} |\langle x_m, a\rangle|^p\Big)^{1/p}.$$

This completes the proof.

References

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Supplement to my paper "On the convergence of superpositions of a sequence of operators"*

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The author is grateful to M. David for noting that Theorem 1 in the above paper remains true without the assumption $S_i S_j = S_j$. The proof (the idea of which is similar to that in the paper, but without using ($\bar{\bar{\text{a}}}$)) consists in showing by induction relative to k of the following formula:

$$T_n T_{n-1} T_{n-2} \ldots T_k - A T_{n-1} T_{n-2} \ldots T_k = (T_n - A)(T_{n-1} - S_{n-1}) \ldots (T_k - S_k)$$

starting with $k = n-1$ and going down to $k = 1$ (n fixed).

Since Theorem 1 is used in the next following theorems, these theorems also remain true if one omits in them the assumptions $S_i S_j = S_j$ in Theorem 2 and in the proof of sufficiency in Theorem 3 and $S = S^2$ in Theorems 4 and 5.

* See Studia Mathematica 25 (1965), p. 343-351.

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