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Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions (I)

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The definition of hypoelliptic and entire elliptic convolution equations can be formulated in a general manner as follows. Let $\mathscr{H}'$ be a space of distributions in R^n, which may be the space $\mathscr{D}'$ of all distributions or one of its subspaces with a topology stronger than that induced in $\mathscr{H}'$ by $\mathscr{D}'$. We assume that:

(h_1) $\mathscr{H}'$ contains the space $\mathscr{E}'$ of distributions of compact support as a dense subset.

(h_2) $\mathscr{H}'$ is a module over the space $\mathscr{E}'$ under convolution, that is, for each $T \in \mathscr{H}'$ and $S \in \mathscr{E}'$, $S*T \in \mathscr{H}'$.

(h_3) The mapping $(S, T) \to S*T$ of $\mathscr{E}' \times \mathscr{H}'$ into $\mathscr{H}'$ is separately continuous.

Furthermore, let $\mathscr{O}_c'(\mathscr{H}' : \mathscr{H}')$ be the space of convolution operators in $\mathscr{H}'$, i.e. the space of continuous linear mappings of $\mathscr{H}'$ into $\mathscr{H}'$, which are convolution operators on $\mathscr{E}' \subset \mathscr{H}'$. $\mathscr{O}_c'(\mathscr{H}' : \mathscr{H}')$ can be identified with a subspace of $\mathscr{H}'$ (see section 1).

We introduce two classes of functions.

(I) $\mathscr{E}\mathscr{H}'$ is the set of all C^∞-functions $f \in \mathscr{H}'$ such that, for any $S \in \mathscr{O}_c'(\mathscr{H}' : \mathscr{H}')$, the convolution $S*f$ is a C^∞-function and $S \to S*f$ is a continuous mapping of $\mathscr{O}_c'(\mathscr{H}' : \mathscr{H}')$ into $\mathscr{E}$ — the space of all C^∞-functions on R^n. We show in section 1 that, in fact, $S*f$ is again in $\mathscr{E}\mathscr{H}'$.

(II) $\mathscr{A}\mathscr{H}'$ is a subset of $\mathscr{E}\mathscr{H}'$. A function $f \in \mathscr{E}\mathscr{H}'$ is in $\mathscr{A}\mathscr{H}'$, if, for every $S \in \mathscr{O}_c'(\mathscr{H}' : \mathscr{H}')$, the convolution $h = S*f$ can be continued analytically in the complex n-space C^n and the growth of the resulting entire function is restricted in the following way. In any horizontal strip V_b in C^n around R^n of width b, $|h(z)| \leqslant |g(\Re z)|$, where g is a function of $\mathscr{E}\mathscr{H}'$ depending on b and $\Re z$ is the real part of z.

Consider now the convolution equation

$$S*U = F, \tag{1}$$

where $S \in \mathscr{O}_c'(\mathscr{H}' : \mathscr{H}')$ and $U, F \in \mathscr{H}'$.

Definition. The convolution equation (1) and the distribution S are said to be *hypoelliptic* in $\mathcal{H}'$, if all solutions $U \in \mathcal{H}'$ are in $\mathcal{E}\mathcal{H}'$, when $F \in \mathcal{E}\mathcal{H}'$. Equation (1) and the distribution S are called *entire elliptic* in $\mathcal{H}'$, if all solutions $U \in \mathcal{H}'$ are in $\mathcal{A}\mathcal{H}'$, when $F \in \mathcal{A}\mathcal{H}'$.

Remark. One can also consider convolution operators of $\mathcal{H}'$ into another space of distributions and adapt suitably the definition of hypoelliptic and entire elliptic operators to that more general case.

Ehrenpreis [2] characterized completely hypoelliptic and entire elliptic convolution operators in the space $\mathcal{D}'$ of all distributions and in the space $\mathcal{D}'_F$ of distributions of finite order. In both cases the convolution operators are distributions of compact support. Recall that a distribution $S \in \mathcal{E}'$ is hypoelliptic in $\mathcal{D}'$, if and only if its Fourier transform $\hat{S}$ has the properties

(2) $$|\hat{S}(\xi)| \geqslant |\xi|^a \quad \text{for } \xi \in R^n,\ |\xi| \geqslant A,$$

and

(3) $$\frac{|\Im\zeta|}{\log|\zeta|} \to \infty, \quad \text{when } |\zeta| \to \infty,\ \zeta \in C^n,\ \hat{S}(\zeta) = 0,$$

where a and A are constants [1].

If $S = P(D)\delta$, where $P(D)$ is a polynomial of derivation and δ the Dirac measure, equation (1) turns into a linear partial differential equation with constant coefficients. Then, by a theorem of Hörmander [6], S is hypoelliptic in $\mathcal{D}'$, if and only if it satisfies the weaker condition

(4) $$|\Im\zeta| \to \infty, \quad \text{when } |\zeta| \to \infty,\ \zeta \in C^n,\ \hat{S}(\zeta) = 0.$$

Condition [4] is, however, not sufficient for an arbitrary $S \in \mathcal{E}'$ to be hypoelliptic in $\mathcal{D}'$.

On the other hand, condition (4) is implied by the requirement: "In the space of continuous functions of exponential growth in R^n, every solution of the homogeneus equation

$$S * U = 0$$

is a C^∞-function." This result is due to Grusin [4]. It shows that condition (4) is necessary and sufficient for $S = P(D)\delta$ to be hypoelliptic in every space $\mathcal{H}'$, which contains all continuous functions of exponential growth in R^n. Arbitrary convolution operators in $\mathcal{H}'$ behave, in general, differently. In fact, a convolution operator $S \in \mathcal{E}'$, hypoelliptic in a space of "distributions of exponential growth", need not to be hypoelliptic in $\mathcal{D}'$.

[1] For a detailed study of the sufficiency of these conditions see Hörmander [8].

The purpose of this paper is to characterize hypoelliptic and entire elliptic operators in Schwartz's space $\mathcal{S}'$ of tempered distributions. In part II and part III we characterize hypoelliptic and entire elliptic convolution operators in the space $\mathcal{K}'_1$ $(= \Lambda_\infty)$ of "distributions of exponential growth" introduced by J. Sebastião e Silva and M. Hasumi. Other subspaces of $\mathcal{D}'$ and also partially hypoelliptic and entire elliptic operators will be discussed in future papers.

1. Convolution operators in $\mathcal{H}'$. We identify the space $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ of convolution operators in $\mathcal{H}'$ with the space of distributions, which consists of all $S \in \mathcal{H}'$ such that the mapping $T \to S * T$ of $\mathcal{E}'$ into $\mathcal{H}'$ can be extended to a continuous linear mapping of $\mathcal{H}'$ into $\mathcal{H}'$. The underlying isomorphism is clear. $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ is thus a subset of $\mathcal{H}'$ containing all distributions of compact support. We introduce in $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ the topology induced by the space $\mathcal{L}(\mathcal{H}', \mathcal{H}')$ of all continuous linear mappings of $\mathcal{H}'$ into itself, provided with the topology of uniform convergence on bounded sets in $\mathcal{H}'$. Then the injection $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}') \to \mathcal{H}'$ is continuous and the bilinear mapping $(S, T) \to S * T$ of $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}') \times \mathcal{H}'$ into $\mathcal{H}'$ is separately continuous [2].

For $S_1, S_2 \in \mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$, the convolution $S_1 * S_2$ is also in $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$. Moreover, the bilinear mapping $(S_1, S_2) \to S_1 * S_2$ of $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}') \times \mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ into $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ is separately continuous.

In fact, if $T \in \mathcal{E}'$, then

$$S_1 * (S_2 * T) = (S_1 * S_2) * T$$

and the mapping $T \to S_1 * (S_2 * T)$ can be extended to a continuous mapping of $\mathcal{H}'$ into $\mathcal{H}'$. This shows that $S_1 * S_2$ is in $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$. The second statement follows from general properties of the topology of the space $\mathcal{L}(\mathcal{H}', \mathcal{H}')$ ([1], chapt. III, § 4, Proposition 9).

From the above we also infer that the convolution of a finite number of distributions of $\mathcal{H}'$, all of which but one at most are in $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$, is associative.

Let now f be a function of $\mathcal{E}\mathcal{H}'$, i.e. a C^∞-function of $\mathcal{H}'$, such that $S \to S * f$ maps continuously $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ into $\mathcal{E}$. Then, for any $S_0 \in \mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$, the convolution $S_0 * f$ is a C^∞-function of $\mathcal{H}'$ and the mapping $S \to S * (S_0 * f) = (S * S_0) * f$ of $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ into $\mathcal{E}$ is continuous. Consequently $\mathcal{E}\mathcal{H}'$ is a module over $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ under the convolution operation.

In the main part of our investigations the space $\mathcal{H}'$ will have additional good properties. In particular, $\mathcal{H}'$ will be the dual of a space $\mathcal{H}$ of type $\mathcal{H}^\infty$ in the sense of L. Schwartz ([10], Exposé 10; see also [5], p. 95-96). We recall the definition.

[2] This continuity holds also, if the topology of $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$ is induced by the space $\mathcal{L}(\mathcal{H}', \mathcal{H}')$, provided with the weak topology.

Let Γ be a set of continuous functions on R^n such that, for every compact subset K of R^n, there is a function $\gamma \in \Gamma$, the values of which are on K different from zero. We say that a function $\varphi \in \mathcal{E}$ *satisfies the condition of growth defined by* Γ if, for any r ($r = (r_1, \ldots, r_n)$, r_j — integers $\geqslant 0$) and any $\gamma \in \Gamma$, the function $\gamma(x) D^r \varphi(x)$ is bounded on R^n.

A space $\mathcal{H}$ is said to be of *type* $\mathcal{H}^\infty$, if it satisfies the following conditions:

(H_1) $\mathcal{H}$ consists of functions $\varphi \in \mathcal{E}$ satisfying the condition of growth defined by Γ.

(H_2) $\mathcal{H}$ is a Hausdorff, complete, locally convex topological space and the injections $\mathcal{D} \to \mathcal{H} \to \mathcal{E}$ are continuous.

(H_3) A subset $B \in \mathcal{H}$ is bounded, if and only if, for any r and $\gamma \in \Gamma$, the set of numbers $\gamma(x) D^r \varphi(x)$, $\varphi \in B$, $x \in R^n$, is bounded.

(H_4) On any bounded set $B \subset \mathcal{H}$, the topology induced by $\mathcal{H}$ coincides with the topology induced by $\mathcal{E}$.

If $\mathcal{H}$ is a Montel space of type $\mathcal{H}^\infty$ and $\mathcal{H}'$ its strong dual, then the convolution operators in $\mathcal{H}'$ can be characterized by the following theorem ([10], Exposé 11, Théorème 1; see also [5], p. 103, Lemma 1):

A distribution $S \in \mathcal{H}'$ is in $\mathcal{O}'_c(\mathcal{H}' : \mathcal{H}')$, if and only if the function $\vec{S}: y - \tau_y S$ is infinitely differentiable with values in $\mathcal{H}'$ and such that $\langle \vec{S}, \varphi \rangle \in \mathcal{H}$ for every $\varphi \in \mathcal{H}$.

Here $\tau_y S$ denotes the translation of S by $y \in R^n$ and $\langle \, , \rangle$ are the duality brackets.

2. The case $\mathcal{H}' = \varphi'$. We recall briefly the main notions and characterize the basic spaces for the particular case, when $\mathcal{H}'$ is the space $\mathcal{S}'$ of tempered distributions on R^n.

A function $f(x)$ defined on R^n is *slowly increasing*, if there is a constant μ such that

$$f(x) = O(|x|^\mu), \tag{5}$$

as $|x| \to \infty$; $f(x)$ is *rapidly decreasing*, if condition (5) holds for every negative μ.

We denote by $\mathcal{S}$ the space of C^∞-functions on R^n, rapidly decreasing together with all their derivatives. The topology of $\mathcal{S}$ is defined by the system of semi-norms

$$v_{k,l}(\varphi) = \sup_{x \in R^n, |r| \leqslant l} (1 + |x|^k) D^r \varphi(x),$$

where k, l are integers. $\mathcal{S}$ is a Montel space of type $\mathcal{H}^\infty$; each function of $\mathcal{S}$ satisfies the condition of growth defined by $\Gamma = \{(1+|x|^k): k = 1, 2, \ldots\}$.

The dual $\mathcal{S}'$ of $\mathcal{S}$ is the space of tempered distributions. A distribution $T \in \mathcal{D}'$ is *tempered*, if and only if T is a derivative of a continuous, slowly increasing function. $\mathcal{S}'$ is provided with the strong dual topology.

We denote by $T \cdot \varphi$ or $T_x \cdot \varphi(x)$ the scalar product of a distribution $T \in \mathcal{S}'$ and a function $\varphi \in \mathcal{S}$.

The convolution operators in $\mathcal{S}'$ can be easily characterized by application of the theorem quoted at the end of section 1 and another theorem of Schwartz ([9], vol. II, p. 100, Théorème IX, 3°). They appear to be rapidly decreasing distributions. A distribution $S \in \mathcal{D}'$ is rapidly decreasing, if and only if, for every $\mu \geqslant 0$, S is a finite sum of derivatives of continuous functions, whose products with $|x|^\mu$ are bounded in R^n. $\mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$ is the space of rapidly decreasing distributions introduced as $\mathcal{O}'_c$ in [9] (vol. II, p. 100).

The Fourier transform $\hat{\varphi}$ of a function $\varphi \in \mathcal{S}$ is given by the usual formula

$$\hat{\varphi}(\xi) = \int \varphi(x) e^{-2\pi i \xi \cdot x} dx,$$

where $\xi \cdot x = \xi_1 x_1 + \ldots + \xi_n x_n$ and the integral is taken over R^n. $\hat{\varphi}$ is also a function of $\mathcal{S}$.

For a distribution $T \in \mathcal{S}'$ the Fourier transform $\hat{T} \in \mathcal{S}'$ is defined by the Parseval-Plancherel equation

$$\hat{T}_\xi \cdot \hat{\varphi}(\xi) = T_x \cdot \varphi(-x).$$

Fourier transforms of distributions of $\mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$ form the space $\mathcal{O}_M$ of C^∞-functions, slowly increasing together with all their derivatives (see [9], vol. II, p. 99). The topology of $\mathcal{O}_M$ is such that the Fourier transform is a topological isomorphism of $\mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$ onto $\mathcal{O}_M$. Moreover, the convolution $S * T$ of $S \in \mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$ and $T \in \mathcal{S}'$ is transformed into the product $\hat{S}\hat{T}$.

If $S \in \mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$ and T is a C^∞-function in $\mathcal{S}'$, then the convolution $S * T$ need not to be a C^∞-function; it is a tempered distribution. For illustration we give the following example.

Example. Assume that $n = 1$. We define the distribution $S \in \mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$ by the formula

$$S = \sum_{j=0}^{\infty} \frac{1}{2^j} \delta_{(-j)},$$

where $\delta_{(-j)}$ is the Dirac measure with support at $x = -j$. Take now an odd integer $k > 2 + 3\pi$. Then the function

$$f(x) = \cos(\pi k^x)$$

belongs to $\mathcal{S}'$ and is certainly infinitely differentiable. But the convolution

$$(S*f)(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \cos(\pi k^{j+x})$$

is a composition of the Weierstrass function corresponding to $1/2$ and k and the function k^x. Thus it is a continuous function without derivative at any point of R^1.

One can also show that the convolution $S*f$ of $S \in \mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ and $f \in \mathcal{O}_M$ need not to be a C^∞-function.

The set $\mathcal{E}\mathcal{S}'$ coincides with the space $\mathcal{O}_c(\mathcal{S}' : \mathcal{S}')$ of very slowly increasing C^∞-functions, which is the dual of $\mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ (see [3], p. 130). Recall that a C^∞-function f is in $\mathcal{O}_c(\mathcal{S}' : \mathcal{S}')$, if and only if its derivatives $D^r f$ have the same rate of increase as a power of $|x|$. In other words, there exists a constant μ, such that

$$D^r f(x) = O(|x|^\mu)$$

as $|x| \to \infty$, for all the derivatives.

In order to prove the above coincidence we observe that each function $f \in \mathcal{E}\mathcal{S}'$ can be identified with the continuous linear functional on $\mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$, whose value at $S \in \mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ is the value of the convolution $S*f(-x)$ at the origin.

Conversely, for every $S \in \mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ and $f \in \mathcal{O}_c(\mathcal{S}' : \mathcal{S}')$, the convolution $S*f$ is a C^∞-function and the mapping $S \to S*f$ of $\mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ into $\mathcal{E}$ is continuous. This is a consequence of the following facts:

(a) f can be represented in the form $f = Pg$, where P is a polynomial and g belongs to $\mathcal{D}_{L^1}$ — the space of C^∞-functions, which are in L^1 together with all their derivatives.

(b) The mapping $S \to S*g$ maps continuously $\mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ into D_{L^1} ([9], vol. II, p. 104).

The set $\mathcal{A}\mathcal{S}'$ consists of functions $f \in \mathcal{E}\mathcal{S}'$ extendable over C^n as entire functions, slowly increasing in any horizontal strip $V_b = \{z: |\Im z_j| \leqslant b, j = 1, 2, \ldots, n\}$. More precisely, an entire function f is in $\mathcal{A}\mathcal{S}'$, if and only if, in every strip V_b,

$$|f(z)| \leqslant M(1+|z|^\mu),$$

where M and μ are constants depending on b.

We have to prove that the set defined above as $\mathcal{A}\mathcal{S}'$ is a module over $\mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ under convolution. For that purpose we use

PROPOSITION 1. *A distribution $T \in \mathcal{S}'$ has as its Fourier transform $\hat{T}$ an entire function f, slowly increasing in any strip V_b, if and only if the product $e^{u \cdot x} T_x$ is in $\mathcal{S}'$, for every $u \in R^n$* (³).

If now T satisfies the "growth condition" of proposition 1, then so does its product φT with any $\varphi \in \mathcal{O}_M$. Hence, passing to the Fourier transform and applying proposition 1, we obtain the desired result.

Remark. Distributions satisfying the "growth condition" of proposition 1 form the space $\mathcal{O}_c'(\mathcal{K}_1' : \mathcal{K}_1')$ of convolution operators in $\mathcal{K}_1'$ (see [5]). $\mathcal{O}_c'(\mathcal{K}_1' : \mathcal{K}_1')$ is shown in [11] to be the dual of $\mathcal{O}_c(\mathcal{K}_1' : \mathcal{K}_1')$, the space of C^∞-functions φ with the following property: There is a constant μ (depending on φ) such that

$$D^r \varphi(x) = O(e^{\mu|x|})$$

as $|x| \to \infty$, for all the derivatives.

A function $f(z)$ analytic in the strip V_b is said to be *rapidly decreasing in V_b*, if

$$\sup_{z \in V_b} |z|^\nu |f(z)| < \infty,$$

for every $\nu > 0$.

We have

PROPOSITION 2. *A function $\varphi \in \mathcal{S}$ has as its Fourier transform $\hat{\varphi}$ an entire function f, rapidly decreasing in any strip V_b, if and only if the product $e^{u \cdot x}\varphi(x)$ is in $\mathcal{S}$ for every $u \in R^n$.*

Proposition 2 was proved by Hasumi ([5], Proposition 4).

If f is an entire function, rapidly decreasing in any strip V_b, then obviously $f \in \mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ and, for every $T \in \mathcal{S}'$, the convolution $f*T$ is in $\mathcal{A}\mathcal{S}'$, by proposition 2 and proposition 1.

3. Differential operators of infinite order. In section 5 we use functions of the form

$$\sigma_\lambda(z) = \prod_{j=1}^{n} (e^{\lambda z_j} + e^{-\lambda z_j})$$

and the corresponding differential operators of infinite order

$$\sigma_\lambda\left(\frac{1}{2\pi i} D\right) = 2^n \sum_r \left(-\frac{\lambda^2}{4\pi^2}\right)^{|r|} \frac{D^{2r}}{(2r)!};$$

(³) This result is contained in [5] (Proposition 8 and Proposition 9); it is also easy to prove it directly.

here λ is a non-negative number, $z = (z_1, \ldots, z_n) \in C^n$ and r runs through all n-tuples of non-negative integers. For any $\lambda \geqslant 0$, the operator $\sigma_\lambda\left(\frac{1}{2\pi i}D\right)$ is defined by equality

$$\sigma_\lambda\left(\frac{1}{2\pi i}D\right)f = 2^n \sum_r \left(-\frac{\lambda^2}{4\pi^2}\right)^{|r|} \frac{D^{2r}f}{(2r)!}$$

on a class of C^∞-functions $f \in \mathscr{S}'$, such that the series on the right-hand side converges in $\mathscr{S}'$.

If, for example, the Fourier transform $\hat{f}$ of f is defined as

$$\hat{f}(\xi) = \frac{1}{\sigma_\lambda(\xi)},$$

where λ is fixed, then the series

$$2^n \sum_r \frac{(\lambda\xi)^{2r}}{(2r)!}\hat{f}(\xi)$$

converges uniformly in R^n and its sum is the constant function $\equiv 1$. Therefore f belongs to the domain of the operator $\sigma_\lambda\left(\frac{1}{2\pi i}D\right)$ and

$$\sigma_\lambda\left(\frac{1}{2\pi i}D\right)f = \delta.$$

The operators $\sigma_\lambda\left(\frac{1}{2\pi i}D\right)$ have the following useful properties:

(a) Given any $\lambda \geqslant 0$, a function $f \in \mathscr{A}\mathscr{S}'$ is in the domain of $\sigma_\lambda\left(\frac{1}{2\pi i}D\right)$ and the image $\sigma_\lambda\left(\frac{1}{2\pi i}D\right)f$ is also in $\mathscr{A}\mathscr{S}'$. Moreover, if $\hat{f}$ is the Fourier transform of f, then the product $\sigma_\lambda(\xi)\hat{f}_\xi$ is in $\mathscr{S}'$ and

$$\widehat{\sigma_\lambda\left(\frac{1}{2\pi i}D\right)f(x)} = \sigma_\lambda(\xi)\hat{f}_\xi.$$

(b) Suppose that, for some $\lambda \geqslant 0$, $S \in \mathscr{O}_c'(\mathscr{S}' : \mathscr{S}')$ is in the domain of $\sigma_\lambda\left(\frac{1}{2\pi i}D\right)$ and $\sigma_\lambda\left(\frac{1}{2\pi i}D\right)S \in \mathscr{O}_c'(\mathscr{S}' : \mathscr{S}')$. Then, for any $T \in \mathscr{S}'$, the convolution $S * T$ is in the domain of $\sigma_\lambda\left(\frac{1}{2\pi i}D\right)$ and

$$\sigma_\lambda\left(\frac{1}{2\pi i}D\right)(S * T) = \sigma_\lambda\left(\frac{1}{2\pi i}D\right)S * T.$$

The proof in both cases is simple and we leave it out.

4. Hypoelliptic operators in $\mathscr{S}'$. It appears that in $\mathscr{S}'$ a hypoelliptic convolution operator is also entire elliptic. Moreover, the latter implication cannot be inverted: an entire elliptic operator in $\mathscr{S}'$ need not to be hypoelliptic in $\mathscr{S}'$. Recall that, according to the results of Ehrenpreis [2], the relation between hypoelliptic and entire elliptic operators in $\mathscr{D}'$ and $\mathscr{D}_F'$ is converse.

We now prove a necessary and sufficient condition for a convolution operator in $\mathscr{S}'$ to be hypoelliptic in $\mathscr{S}'$. We make use of the following lemma:

LEMMA 1. *Let T be a distribution, whose Fourier transform $\hat{T}$ is of the form*

$$\hat{T} = \sum_{j=1}^{\infty} a_j \delta_{({}_j\xi)} \tag{6}$$

where the ${}_j\xi$ satisfy condition

$$|{}_j\xi| > 2|{}_{j-1}\xi| > 2^j \tag{7}$$

and a_j are complex numbers, such that

$$a_j = O(|{}_j\xi|^\mu) \tag{8}$$

for some μ; then the series in (6) *converges in $\mathscr{S}'$. We assert that $T \in \mathscr{E}\mathscr{S}'$. if and only if*

$$a_j = o(|{}_j\xi|^{-\nu}) \tag{9}$$

for every $\nu \geqslant 0$.

Proof. Because of (6) and (8),

$$T = \sum_{j=1}^{\infty} a_j e^{2\pi i x \cdot {}_j\xi},$$

where the series converges in $\mathscr{S}'$. If the coefficients a_j satisfy condition (9), then the last series and all its term-by-term derivatives converge uniformly in R^n. Consequently T is a C^∞-function bounded together with all its derivatives, and therefore belongs to $\mathscr{E}\mathscr{S}'$.

Conversely, let T be a function of $\mathscr{E}\mathscr{S}'$. Then, for every integer $\nu \geqslant 0$ and every $\varphi \in \mathscr{S}$,

$$e^{2\pi i u \cdot x} \Delta^\nu T_x \cdot \varphi(-x) \to 0$$

as $|u| \to \infty$, $u \in R^n$; here Δ^ν is the iterated Laplace operator. Hence, passing to the Fourier transform, we get

$$\tau_u(|\xi|^{2\nu}\hat{T}_\xi)\cdot\hat{\varphi}(\xi) = \sum_{j=1}^{\infty} a_j |{}_j\xi|^{2\nu}\hat{\varphi}({}_j\xi - u) \to 0, \tag{10}$$

as $|u| \to \infty$. We fix a function φ such that

(11) $$|\hat{\varphi}(0)| \geqslant 1$$

and

(12) $$\hat{\varphi}(\xi) = 0 \quad \text{for} \quad |\xi| \geqslant 1.$$

Suppose now that condition (9) is not satisfied. Then there is a $\varrho > 0$ and a positive integer ν_0, such that

(13) $$|{}_j\xi|^{2\nu_0}|a_j| \geqslant \varrho$$

for a subsequence of $\{a_j\}$, which we may take as the whole sequence without loss of generality.

We set now ${}_ju = {}_j\xi$. Making use of (7) and (12) we obtain

$$\sum_{\substack{j=1\\ j\neq k}}^{\infty} a_j |{}_j\xi|^{2\nu_0}\hat{\varphi}({}_j\xi - {}_ku) = 0.$$

On the other hand, conditions (11) and (13) imply that

$$|a_k|\,|{}_k\xi|^{2\nu_0}|\hat{\varphi}(0)| \geqslant \varrho.$$

This contradicts the convergence (10). Our assertion is thus established.

We are now in a position to prove

THEOREM 1. *If a distribution $S \in \mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ is hypoelliptic in $\mathcal{S}'$, then its Fourier transform $\hat{S}$ satisfies condition* (2), *i.e. there are constants a and A, such that*

$$|\hat{S}(\xi)| \geqslant |\xi|^a \quad \textit{for} \quad \xi \in R^n,\ |\xi| \geqslant A.$$

Proof. Suppose that the condition is not satisfied. Then there are ${}_j\xi \in R^n$ defined as in lemma 1 and such that

(14) $$|\hat{S}({}_j\xi)| \leqslant |{}_j\xi|^{-j}.$$

The series

$$\sum_{j=1}^{\infty} \delta_{({}_j\xi)}$$

converges in $\mathcal{S}'$ to $\hat{U}$, say. Hence $U \in \mathcal{S}'$ and, by lemma 1, U is not in $\mathcal{E}\mathcal{S}'$. But, on the other hand,

$$\widehat{S * U} = \hat{S}\hat{U} = \sum_{j=1}^{\infty} \hat{S}({}_j\xi)\,\delta_{({}_j\xi)}.$$

Applying now inequality (14) and once more lemma 1, we conclude that $S * U$ is in $\mathcal{E}\mathcal{S}'$. Thus S is not hypoelliptic in $\mathcal{S}'$, q.e.d.

For the proof of the convers to theorem 1 we need a *parametrix* for S, i.e. a distribution P such that

(15) $$S * P = \delta - W,$$

where W is a function with suitable ragularity properties. For our purpose W has to be an analytic function, rapidly decreasing in a horizontal strip V_b, $b > 0$. However, we impose on W a more restrictive assumption, namely, that W is an entire function, rapidly decreasing in every strip V_b. In that case a distribution $P \in \mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ satisfying equation (15) will be called *rapidly decreasing parametrix* for S.

THEOREM 2. *If $S \in \mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$ satisfies condition* (2), *then there exists a rapidly decreasing parametrix for S.*

Proof. Let us take a C^∞-function ψ, such that

(16) $$\psi(\xi) = \begin{cases} 1 & \text{for} \quad |\xi| \leqslant A, \\ 0 & \text{for} \quad |\xi| \geqslant A+1, \end{cases}$$

where A is the constant in (2). Then we define the Fourier transform $\hat{P}$ of P by the formula

$$\hat{P}(\xi) = \begin{cases} 0 & \text{for} \quad |\xi| \leqslant A, \\ \dfrac{1-\psi(\xi)}{\hat{S}(\xi)} & \text{for} \quad |\xi| > A. \end{cases}$$

Obviously $\hat{P}$ is a C^∞-function, slowly increasing together with all its derivatives, and so P is in $\mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$. Furthermore,

$$\hat{S}(\xi)\hat{P}(\xi) = 1 - \psi(\xi),$$

whence, passing to the inverse Fourier transform, we see that P is a rapidly decreasing parametrix for S with $\hat{W} = \psi$, q.e.d.

From the existence of a rapidly decreasing parametrix for S it follows that S is hypoelliptic and entire elliptic in $\mathcal{S}'$. For, assume that

$$S * U = F,$$

where $S \in \mathcal{O}_c'(\mathcal{S}' : \mathcal{S}')$, $U \in \mathcal{S}'$ and $F \in \mathcal{E}\mathcal{S}'$ (or $F \in \mathcal{A}\mathcal{S}'$ respectively). Then, making use of (15), we can write

$$\begin{aligned} U = U * \delta &= U * (S * P) + U * W \\ &= (U * S) * P + U * W = F * P + U * W. \end{aligned}$$

But $F * P$ is in $\mathcal{E}\mathcal{S}'$ (in $\mathcal{A}\mathcal{S}'$) and $U * W$ is in $\mathcal{A}\mathcal{S}'$, so that U is in $\mathcal{E}\mathcal{S}'$ (in $\mathcal{A}\mathcal{S}'$).

Combining the above with theorem 2, we obtain

THEOREM 3. *If $S \in \mathcal{O}_c'(\mathcal{S}': \mathcal{S}')$ satisfies condition (2), then S is hypoelliptic and entire elliptic in $\mathcal{S}'$.*

From theorem 1 and theorem 3 we can draw the following corollaries.

COROLLARY 1. *Condition (2) is necessary and sufficient for S to be hypoelliptic in $\mathcal{S}'$.*

COROLLARY 2. *Every hypoelliptic in $\mathcal{S}'$ distribution $S \in \mathcal{O}_c'(\mathcal{S}': \mathcal{S}')$ is entire elliptic in $\mathcal{S}'$.*

Finally we remark that condition (2) does not guarantee the existence of a solution of equation (1), even if F is in $\mathcal{S}$. In fact, one can easily choose the S and F so that the equation has no solution.

5. Entire elliptic operators in $\mathcal{S}'$. As said before, an entire elliptic convolution operator in $\mathcal{S}'$ need not to be hypoelliptic. We illustrate this by an example.

Example. Assume again that $n = 1$. We fix a $\lambda > 0$ and set

$$\hat{S}(\xi) = \frac{1}{\sigma_\lambda(\xi)}; \tag{17}$$

$\sigma_\lambda(\xi)$ is the function defined in section 3. Then S is a function of $\mathcal{S}$ and therefore a convolution operator in $\mathcal{S}'$. Being a function of $\mathcal{S}$, S is not hypoelliptic in $\mathcal{S}'$. But

$$\sigma_\lambda\left(\frac{1}{2\pi i}D\right) S = \delta,$$

because of (17) (see section 3). If now

$$S * U = F$$

and $F \in \mathcal{A}\mathcal{S}'$, then

$$U = \delta * U = \sigma_\lambda\left(\frac{1}{2\pi i}D\right) S * U = \sigma_\lambda\left(\frac{1}{2\pi i}D\right)(S * U) = \sigma_\lambda\left(\frac{1}{2\pi i}D\right) F \in \mathcal{A}\mathcal{S}'.$$

This shows that S is entire elliptic in $\mathcal{S}'$.

The aim of this section is to prove a necessary and sufficient condition for a distribution $S \in \mathcal{O}_c'(\mathcal{S}': \mathcal{S}')$ to be entire elliptic in $\mathcal{S}'$.

LEMMA 2. *Under the conditions of lemma 1, the distribution T is in $\mathcal{A}\mathcal{S}'$, if and only if*

$$a_j = o(e^{-\nu|_j\xi|}), \tag{18}$$

for every $\nu \geqslant 0$.

Proof. If condition (18) is satisfied, then the series

$$\sum_{j=1}^{\infty} a_j e^{2\pi i z \cdot {}_j\xi}$$

converges uniformly in every strip V_b. Therefore T is in $\mathcal{A}\mathcal{S}'$; moreover, T is bounded in every strip V_b.

The proof of the "only if" part runs parallel to the corresponding part of the proof of lemma 1. Merely the iterated Laplace operator has now to be replaced by the differential operator of infinite order $\sigma_\nu\left(\frac{1}{2\pi i}D\right)$. The passage to the Fourier transform can be achieved by means of the formula given in section 3.

Lemma 2 enables us to prove

THEOREM 4. *If a distribution $S \in \mathcal{O}_c'(\mathcal{S}': \mathcal{S}')$ is entire elliptic in $\mathcal{S}'$, then there are constants a and A, such that*

$$|\hat{S}(\xi)| \geqslant e^{-a|\xi|} \quad \textit{for} \quad \xi \in R^n,\ |\xi| \geqslant A. \tag{19}$$

Proof. We proceed similarly as in the proof of theorem 1. Suppose that condition (19) is not satisfied. Then there is a sequence $\{_j\xi\}$ defined as in lemma 1 and such that

$$|\hat{S}(_j\xi)| \leqslant e^{-j|_j\xi|}. \tag{20}$$

We take the distribution

$$\hat{U} = \sum_{j=1}^{\infty} \delta_{(_j\xi)},$$

which is in $\mathcal{S}'$. By lemma 1, U is not in $\mathcal{E}\mathcal{S}'$. But from (20) and lemma 2 it follows that the convolution $S * U$ is in $\mathcal{A}\mathcal{S}'$, its Fourier transform being of the form

$$\widehat{S * U} = \hat{S}\hat{U} = \sum_{j=1}^{\infty} \hat{S}(_j\xi)\,\delta_{(_j\xi)}.$$

Therefore S is not entire elliptic in $\mathcal{S}'$, q.e.d.

THEOREM 5. *If a distribution $S \in \mathcal{O}_c'(\mathcal{S}': \mathcal{S}')$ is entire elliptic in $\mathcal{S}'$, then there are constants a and A such that*

$$\hat{S}(\xi) \neq 0 \quad \textit{for} \quad \xi \in R^n,\ |\xi| \geqslant A \tag{21}$$

and

$$D^r\left[\frac{1}{\hat{S}(\xi)}\right] = O(e^{a|\xi|}) \quad \textit{as} \quad |\xi| \to \infty, \tag{22}$$

for all the derivatives.

Proof. By theorem 4, there are constants a and A such that $\hat{S}(\xi)$ satisfies condition (21) and

$$\frac{1}{\hat{S}(\xi)} = O(e^{a|\xi|}) \quad \text{as} \quad |\xi| \to \infty.$$

We have to prove the existence of an a, which is common for all the derivatives in (22).

Instead of the function $1/\hat{S}(\xi)$, which is defined for $|\xi| > A$, we consider the function

$$\chi(\xi) = \begin{cases} 0 & \text{for} \quad |\xi| \leqslant A, \\ \dfrac{1-\psi(\xi)}{\hat{S}(\xi)} & \text{for} \quad |\xi| > A, \end{cases}$$

where $\psi(\xi)$ is given by equality (16). $\chi(\xi)$ is a C^∞-function on R^n and $\chi(\xi) = 1/\hat{S}(\xi)$ for $|\xi| \geqslant A+1$.

First we observe that, for any $T \in \mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$, the product χT is also in $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$. In fact, suppose that $T_1 \in \mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ and $\chi T_1 \notin \mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$. Then, for a sufficiently large λ,

$$T_2 = \frac{\chi T}{\sigma_\lambda}$$

is a tempered distribution, which does not belong to $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$. But the product

$$\hat{S} T_2 = \frac{\hat{S}\chi T_1}{\sigma_\lambda} = \frac{(1-\psi)T_1}{\sigma_\lambda}$$

is in $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$. In view of proposition 1, this is a contradiction to the assumption that S is entire elliptic in $\mathcal{S}'$.

Let now $\{\varphi_j\}$ be a sequence of C^∞-functions with compact support, which converges in $\mathcal{O}_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ to the constant function $\equiv 1$. Then the products $\chi\varphi_j$ converge to χ in $\mathcal{E}$. Moreover, for any $T \in \mathcal{O}'_1(\mathcal{K}'_1 : \mathcal{K}'_1)$, the scalar products

$$T \cdot \chi\varphi_j = \chi T \cdot \varphi_j$$

converge, since $\chi T \in \mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$. Thus the $\chi\varphi_j$ converge weakly in $\mathcal{O}_c(\mathcal{K}'_1 : \mathcal{K}'_1)$. But $\mathcal{O}_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ is a quasi-complete Montel space (see [11]). Consequently the $\chi\varphi_j$ converge to χ in the topology of $\mathcal{O}_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ and $\chi \in \mathcal{O}_c(\mathcal{K}'_1 : \mathcal{K}'_1)$. This shows, by the remark in section 2, that χ satisfies the required growth condition together with all its derivatives, q.e.d.

The converse to theorem 5 also holds. We have

THEOREM 6. *If $S \in \mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$ satisfies conditions* (21) *and* (22), *then it is entire elliptic in $\mathcal{S}'$.*

Proof. Instead of a parametrix we take now the distribution $Q \in \mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$, the Fourier transform of which is defined as

$$\hat{Q}(\xi) = \begin{cases} 0 & \text{for} \quad |\xi| \leqslant A, \\ \dfrac{1-\psi(\xi)}{\sigma_a(\xi)\hat{S}(\xi)} & \text{for} \quad |\xi| > A, \end{cases}$$

where $\psi(\xi)$ is the C^∞-function satisfying equality (16). As an immediate consequence of this definition we obtain equality

$$\sigma_a\left(\frac{1}{2\pi i}D\right)(S*Q) = \delta - W, \tag{23}$$

with $\hat{W} = \psi$.

Assume now that

$$S*U = F,$$

$U \in \mathcal{S}'$ and $F \in \mathcal{A}\mathcal{S}'$. Then, in view of (23), we can write

$$U = U*\delta = U*\sigma_a\left(\frac{1}{2\pi i}D\right)(S*Q) + U*W$$
$$= \sigma_a\left(\frac{1}{2\pi i}D\right)(U*S*Q) + U*W = \sigma_a\left(\frac{1}{2\pi i}D\right)(F*Q) + U*W.$$

Since the last two terms are in $\mathcal{A}\mathcal{S}'$, we conclude that $U \in \mathcal{A}\mathcal{S}'$, and thus S is entire elliptic in $\mathcal{S}'$.

COROLLARY. *Conditions* (21) *and* (22) *are necessary and sufficient for $S \in \mathcal{O}'_c(\mathcal{S}' : \mathcal{S}')$ to be entire elliptic in $\mathcal{S}'$.*

Partial differential operators with constant coefficients, entire elliptic in $\mathcal{S}'$, are also hypoelliptic in $\mathcal{S}'$. This follows from theorem 3, theorem 4 and a lemma of Hörmander (see [7], p. 557, lemma 2). Thus for partial differential operators with constant coefficients both notions "hypoellipticity in $\mathcal{S}'$" and "entire ellipticity in $\mathcal{S}'$" coincide.

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Absolut p-summierende Abbildungen in normierten Räumen

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Eine lineare Abbildung T von einem normierten Raum E in einen normierten Raum F wird *absolut p-summierend* genannt, wenn es eine nicht negative Zahl ϱ gibt, so daß für jedes endliche System $x_1, \ldots, x_k$ von Elementen aus E die Ungleichung

$$\Big\{\sum_{i=1}^{k} \|Tx_i\|^p\Big\}^{1/p} \leqslant \varrho \sup_{\|a\|\leqslant 1}\Big\{\sum_{i=1}^{k} |\langle x_i, a\rangle|^p\Big\}^{1/p}$$

besteht.

Im folgenden entwickeln wir eine Theorie [1] dieser Abbildungen, die auch dadurch charakterisiert sind, daß sie alle p-summierbaren Folgen in absolut p-summierbare Folgen überführen. Es zeigt sich, daß die absolut 2-summierenden Abbildungen eine sehr natürliche Verallgemeinerung der Hilbert-Schmidt-Abbildungen sind. Außerdem erweisen sich die absolut p-summierenden Abbildungen in der Theorie der nuklearen lokalkonvexen Räume als überaus nützlich. Unter Verwendung ihrer Multiplikationseigenschaften erhält man insbesondere einen sehr einfachen Beweis des verallgemeinerten Dvoretzky-Rogers-Theorems.

Von fundamentaler Bedeutung ist die Tatsache, daß sich die absolut p-summierenden Abbildungen durch das Bestehen einer Ungleichung der Form [1]

$$\|Tx\| \leqslant \varrho\Big\{\int_{U^0} |\langle x, a\rangle|^p d\mu\Big\}^{1/p}$$

charakterisieren lassen. Dabei ist μ ein normiertes positives Radonsches Maß auf der schwach kompakten Einheitskugel U^0 des dualen Banachraumes E'. Die wesentliche Idee zum Beweis dieses Kriteriums stammt von Herrn S. Kwapień, der mich auf einen Satz von Mazur und Orlicz aufmerksam machte (vgl. [12]).

1. Einfache Eigenschaften der absolut p-summierenden Abbildungen. Eine lineare Abbildung T von einem normierten Raum E in einen normierten Raum F heißt *absolut p-summierend* $(1 \leqslant p < +\infty)$, wenn

(1) Für $p = 1$ vgl. [7] und [8].