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Perron's integral for derivatives in L^r

by

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Introduction. The notion of the classical Perron integral is by now very familiar. It is based on the notions of major and minor functions and of upper and lower Dini derivates and serves the purpose of showing that an exact and finite classical derivative of a function is integrable and the function itself is the indefinite integral of the derivative.

Since the time the Perron integral was initially introduced the notion of derivative has developed and has undergone various generalizations. Every generalization of the derivative can serve as a basis of generalization of Perron's integral. The idea is not new. As far back as 1932 (see [1]) Burkill developed a theory of Perron integration based on approximate derivatives. There also have been other generalizations.

Here we return to this topic but base the theory of the Perron integral on the notion of derivative — and derivates — in the metric L^r. The notion of derivation in L^r has been introduced by Calderón and Zygmund [4] and unlike the idea of the approximate derivative has proved to be quite effective in applications (partial differential equations, area of surfaces, etc.). It seems likely that Perron's integral based on that notion deserves study. I would like to add that though the results of this paper have points in common with earlier results, the extension is not entirely routine.

The present paper consists of three parts. In the first part we define the notion of Dini derivates in the metric L^r (briefly, L^r-derivates) and prove a number of properties well known for the classical derivative (and due primarily to Denjoy and Lusin). In the second part, using previous results, we develop the theory of Perron's integral for derivates in L^r. In the third part we give applications to Fourier series.

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PART I

1. Definitions and elementary properties of L^r-derivates. Let $f(x)$ be finite and real-valued in an interval (a, b). (In what follows, unless stated otherwise, we consider only measurable real-valued functions and measurable sets.) The right-hand upper Dini derivate of f at x, $\bar{f}^+(x)$, may be described as the lower bound of all a such that

(1) $$\left[\frac{f(x+t)-f(x)}{t}-a\right]_+ = o(1) \quad \text{as } t\to+0 \text{ [1]}.$$

If no such a exists, then $\bar{f}^+(x)=+\infty$. Instead of (1) we may write

(2) $$[f(x+t)-f(x)-at]_+ = o(t) \quad \text{as } t\to+0.$$

Suppose now that $f\in L^r$, $1\leqslant r<\infty$, near the point x. We may then define the right-hand upper Dini derivate of f at x in the metric L^r, denoted by $\bar{f}_r^+(x)$, $1\leqslant r<\infty$. It is equal to the lower bound of all a such that

(3) $$\left\{\frac{1}{h}\int_0^h [f(x+t)-f(x)-at]_+^r\,dt\right\}^{1/r} = o(h) \quad \text{as} \quad h\to+0$$

or, what is the same,

(3′) $$\int_0^h [f(x+t)-f(x)-at]_+^r\,dt = o(h^{r+1}) \quad \text{as } h\to+0.$$

If no such a exists, then we set $\bar{f}_r^+(x)=+\infty$.

The classical case may be considered as the limiting case $r=\infty$ of our definition.

It is obvious what should be meant by $\bar{f}_r^-(x)$, the left-hand upper Dini derivate in the L^r-metric (more briefly, the left-hand upper L^r-derivate) and the two lower L^r-derivates, $\underline{f}_r^+(x)$, $\underline{f}_r^-(x)$. Specifically, we have

(4) $$\bar{f}_r^-(x) = \bar{g}^+(-x),$$

where the function g is defined as $g(-x)=-f(x)$ for all x in the domain of f. The (two-sided) upper L^r-derivate $\bar{f}_r(x)$ is equal to $\max[\bar{f}_r^+(x), \bar{f}_r^-(x)]$. The lower L^r-derivates of f at x are equal to the negatives of the corresponding upper derivates of $-f$ at x.

When $\bar{f}_r^+(x)=\underline{f}_r^+(x)$, the common value, denoted by $f_r'^+(x)$ is the right-hand L^r-derivative of f at x. If it is finite, say equal to a, then it is uniquely defined by the condition

(5) $$\int_0^h |f(x+t)-f(x)-at|^r\,dt = o(h^{r+1}) \quad \text{as} \quad h\to+0.$$

(1) By A_+ and A_- we denote $\max(A,0)$ and $\mathrm{Max}(-A,0)$.

A corresponding definition and formula holds for the left hand L^r-derivative, $f_r'^-(x)$. When all four L^r-derivates are equal, the common value, denoted by $f_r'(x)$ is the L^r-derivative of f at x. If it is finite, say equals a, it is uniquely defined by

(6) $$\int_{-h}^h |f(x+t)-f(x)-at|^r\,dt = o(h^{r+1}) \quad \text{as} \quad h\to 0.$$

In what follows we often consider L^r-derivates or derivatives at each point of a set E. Without loss of generality we may assume that the function is in L^r over some interval containing the set E (or even, if need be, that it is defined and in L^r over the whole real line).

The definition of $\bar{f}_r^+(x)$ could also have been based on (1) instead of (2) and we have the following

THEOREM 1. *Either $\bar{f}_r^+(x)=+\infty$ or it equals the lower bound of all a such that*

(7) $$\int_0^h \left[\frac{f(x+t)-f(x)-at}{t}\right]_+^r dt = o(h) \quad \textit{as} \quad h\to+0.$$

In fact, (7) obviously implies (3′). Using integration by parts, it can be easily shown that (3′) implies (7).

2. Relation between ordinary, approximate and L^r-derivates. We recall the definition of approximate derivates. The approximate right-hand upper derivate, $\bar{f}^+_{\mathrm{app}}(x)$, is the lower bound of all β such that the set

(8) $$\left\{t;\ t>0,\ \frac{f(x+t)-f(x)-\beta t}{t}>0\right\}$$

has $t=0$ as a point of dispersion. The relation of the other approximate derivates to the upper right-hand derivate is the same as in the case of the L^r-derivates. It is not difficult to see that if condition (7) is satisfied for a, then condition (8) is satisfied for any $\beta>a$. It follows that $\bar{f}^+_{\mathrm{app}}(x)\leqslant\bar{f}_r^+(x)$. Noting that the left-hand side of (3) increases with r and that (2) implies (3′), and utilizing the relations given in § 1 of the other Dini derivates to the right-hand upper derivate [2] we obtain

THEOREM 2. *For $1\leqslant r<s<\infty$, we have*

(9) $$\underline{f}^+(x)\leqslant\underline{f}_s^+(x)\leqslant\underline{f}_r^+(x)\leqslant\underline{f}^+_{\mathrm{app}}(x)\leqslant\bar{f}^+_{\mathrm{app}}(x)\leqslant\bar{f}_r^+(x)\leqslant\bar{f}_s^+(x)\leqslant\bar{f}^+(x).$$

Similar inequalities hold for left-hand and two-sided derivates. We obtain immediately the following

(2) These relations hold, of course, not only for the L^r-derivates but also for approximate and classical derivates.

Corollary. *Suppose* $1 \leqslant r < s < \infty$. *The existence of any one of the first three derivatives in the sequence*

$$f'^{+}(x_0),\ f_s'^{+}(x_0),\ f_r'^{+}(x_0),\ f_{\mathrm{app}}'^{+}(x_0)$$

implies the existence of the following derivative and the two are equal.

Similar results hold for left-hand and two-sided derivatives.

3. Further properties of L^r-derivates. We have

Theorem 3. *The L^r-derivates and L^r-derivatives are measurable.*

Proof. As mentioned above, we may assume that $f \in L^r(-\infty, \infty)$. By Lusin's theorem we need only prove that $\bar{f}_r^{+}(x)$ is measurable on every closed set T such that f restricted to T is continuous. Let a be any real number and k, m, n positive integers. Let

$$g(x, h, a) = h^{-r-1}\int_0^h [f(x+t)-f(x)-at]_+^r\, dt \tag{10}$$

and let

$$E(a, k, m, n) = \{x \in T;\ g(x, h, a) < k^{-1} \text{ when } (m+n)^{-1} \leqslant h \leqslant m^{-1}\}. \tag{11}$$

Clearly, for each x_0 in $E(a, k, m, n)$ there exists a positive $\varepsilon_0 < 1$ such that $g(x_0, h, a) < \varepsilon_0 k^{-1}$ for all $h \in [(m+n)^{-1}, m^{-1}]$; and it is consequently easy to show that for any $x \in T$ sufficiently close to x_0, $g(x, h, a) < k^{-1}$ for $(m+n)^{-1} \leqslant h \leqslant m^{-1}$. Thus $E(a, k, m, n)$ is open in T. Let

$$E(a) = \{x \in T;\ \bar{f}_r^{+}(x) \leqslant a\} = \{x \in T;\ g(x, h, a) = o(1), \text{ as } h \to +0\}.$$

Then $E(a) = \bigcap_k \bigcup_m \bigcap_n E(a, k, m, n)$ is measurable and the theorem is established.

Theorem 4. *Suppose that $\bar{f}_r^{+}(x) < \infty$ at each point of a (measurable) set E. Then a.e. in E we have $-\infty < \underline{f}_r^{-}(x) = \bar{f}_r^{+}(x) < \infty$.*

We shall use the following

Lemma 1. *Suppose that $\bar{f}_r^{+}(x) < \infty$ on a set E, $|E| > 0$. Then there is a subset $T \subset E$, $|T| > 0$ such that $f_T'(x)$ (the derivative of f relative to T) exists and is finite at each point of T.*

In view of the fact that the hypothesis implies $\bar{f}_{\mathrm{app}}^{+}(x) < \infty$ on E (see Theorem 2), this lemma is contained implicitly in the proof of Theorem 10.1, in [8], Chap. 9.

Proof of Theorem 4. It is sufficient to show that the conclusion holds a.e. on a subset A of E, $|A| > 0$. Let $E(a, k, m, n)$ be as in (11) with T the subset of E given in the lemma, and let $E(a, k, m) = \bigcap_n E(a, k, m, n)$. Since $\bar{f}_r^{+}(x) < \infty$ in T, we have $T = \bigcup_a \bigcap_k \bigcup_m E(a, k, m)$, where a takes on all integral values. Hence there exist a_0, k_0, m_0 such that $|E(a_0, k_0, m_0)| > 0$.

Let A, $|A| > 0$, be a closed subset of $E(a_0, k_0, m_0)$. Let x_0 be a point of density of A. Since $f_A'(x_0) = f_T'(x_0)$ is finite, our theorem will be established when we prove that $\bar{f}_r^{+}(x_0) = f_A'(x_0) = \underline{f}_r^{-}(x_0)$.

We may assume $x_0 = 0$ and $f(0) = f_A'(0) = 0$. Consequently $f_{\mathrm{app}}'(0) = 0$ and (by Theorem 2) $\bar{f}_r^{+}(0) \geqslant 0$, $\underline{f}_r^{-}(0) \leqslant 0$. It remains to be shown that $\bar{f}_r^{+}(0) \leqslant 0$, and $\underline{f}_r^{-}(0) \geqslant 0$, or which is the same that

$$\int_0^h [f(t)]_+^r\, dt = o(h^{r+1}), \qquad \int_{-h}^0 [f(t)]_+^r\, dt = o(h^{r+1}). \tag{12}$$

Now

$$\int_{-h}^h [f(t)]_+^r\, dt = I + J,$$

where I is the integral of $[f(t)]_+^r$ taken over the set $A \cap [-h, h]$ and

$$J = \sum_i \int_{a_i}^{b_i} [f(t)]_+^r\, dt,$$

the intervals $[a_i, b_i]$ being the intersections of $[-h, h]$ with the intervals contiguous to A (³). Now

$$J = \sum_i \int_0^{b_i - a_i} [f(a_i + t)]_+^r\, dt$$

and since $[f(a_i+t)]_+ \leqslant [f(a_i+t) - f(a_i) - a_0 t]_+ + |f(a_i)| + |a_0 t|$, it follows by applying Minkowski's inequality twice that $J^{1/r} \leqslant P^{1/r} + Q^{1/r} + R^{1/r}$ where

$$P = \sum_i \int_0^{b_i - a_i} [f(a_i+t) - f(a_i) - a_0 t]_+^r\, dt,$$

$$Q = \sum_i \int_0^{b_i - a_i} |f(a_i)|^r\, dt, \qquad R = \sum_i \int_0^{b_i - a_i} |a_0 t|^r\, dt.$$

Since $a_i \in A \subset E(a_0, k_0, m_0)$, we have

$$\int_0^{b_i - a_i} [f(a_i+t) - f(a_i) - a_0 t]_+^r\, dt \leqslant (b_i - a_i)^{r+1} k_0^{-1} \leqslant h^r (b_i - a_i) k_0^{-1}$$

for $0 < h < m_0^{-1}$, and furthermore $f(a_i) = o(a_i) = o(h)$. Also, since x_0 is a point of density of T, $\sum_i (b_i - a_i) = o(h)$. It follows that P, Q, R are $o(h^{r+1})$ so that $J = o(h^{r+1})$. Since the integrand of I is $o(t^r)$, we also have $I = o(h^{r+1})$. Thus (12) holds and our theorem is established.

(³) Since x_0 is a point of density of A, we may, if necessary, increase h slightly so that $-h \in A$.

We have the following immediate

COROLLARY. a) *If the two upper [lower] L^r-derivates are less than $+\infty$ [greater than $-\infty$] on a set E, then a.e. on E the L^r-derivative exists and is finite.*

b) *If the two L^r-derivates on the right [left] are finite on a set E, then a.e. on E the L^r-derivative exists and is finite.*

For later application we need the following

THEOREM 5. *Suppose that $F\epsilon L^r[a,b]$ and that $\underline{F}_r(x)\geqslant 0$ except possibly on a countable set E' where, however F is L^r continuous* ([4]). *Then $F(x)$ is non-decreasing on $[a,b]$.*

The proof will be based on the following three lemmas.

LEMMA 2. *Suppose that the interval $[a,b]$ is the union of two disjoint measurable sets L and R such that each point in R is a point of right-hand density* ([5]) *of R and each point of L is a point of left-hand density of L. Then every point of R lies to the right of every point of L.*

Proof. Suppose, to the contrary, that $x_1\epsilon R$, $x_2\epsilon L$ and $a\leqslant x_1<x_2\leqslant b$. Let

$$g(x)=(x-d)^{-1}\int_a^x C_R(t)\,dt,$$

where $d<x_1$ and $C_R(t)$ is the characteristic function of R. Since $g(x_0)<1$ for $x_1\leqslant x_0\leqslant x_2$, it follows that when $x_0\epsilon R$ $[x_0\epsilon L]$, $g(x)$ increases as we move away from x_0 slightly to the right [left]. On the other hand, since $g(x)$ is continuous on $[x_1,x_2]$ it must attain a maximum at some point x_0, $x_1\leqslant x_0\leqslant x_2$. We thus obtain a contradiction.

LEMMA 3. *Let E' be a countable subset of $[a,b]$ and E the complement of E'. Suppose* (i) *$F(x)$ is approximately continuous at each point of E', and* (ii) *each point x_0 of E is a point of right-hand density of the set $\{x;F(x)\geqslant F(x_0)\}$ and a point of left-hand density of the set $\{x;F(x)\leqslant F(x_0)\}$. Then $F(x)$ is non-decreasing on $[a,b]$.*

Proof. Suppose that $x_1,x_2\epsilon[a,b]$ and $F(x_1)<F(x_2)$. We shall show that $x_1<x_2$. Choose $\varepsilon>0$ so that $F(x_1)<F(x_2)-\varepsilon$ and such that $F(x)\neq F(x_2)-\varepsilon$ for any $x\epsilon E'$. Let $R=\{x;\,a\leqslant x\leqslant b,\,F(x)\geqslant F(x_2)-\varepsilon\}$ and $L=\{x;\,a\leqslant x\leqslant b,\,F(x)<F(x_2)-\varepsilon\}$. Then clearly, R and L satisfy the hypothesis of Lemma 2, and since $x_1\epsilon L$ and $x_2\epsilon R$ it follows that $x_1<x_2$.

([4]) A function $f\epsilon L^r[a,b]$ is said to be *L^r-continuous* at a point x_0 in $[a,b]$ if $\int_H|f(x)-f(x_0)|^r dx=o(h)$, where $H=[a,b]\frown[x_0-h,x_0+h]$.

([5]) For convenience, when the point $x=b$ $[x=a]$ belongs to a given subset S of $[a,b]$, we consider $x=b$ $[x=a]$ a point of right-hand [left-hand] density of S.

LEMMA 4. *Let E' be a countable subset of $[a,b]$ and E the complement of E'. Suppose that $\underline{F}_{\rm app}(x)\geqslant 0$ for all $x\epsilon E$ and F is approximately continuous at each point of E'. Then $F(x)$ is non-decreasing on $[a,b]$.*

Proof. Replacing $F(x)$ by $F(x)+\varepsilon x$, $\varepsilon>0$, we may assume that $\underline{F}_{\rm app}(x)>0$ for all $x\epsilon E$. But then E and E' satisfy the hypothesis of Lemma 3 and consequently $F(x)$ is non-decreasing on $[a,b]$.

Proof of Theorem 5. By Theorem 2, $\underline{F}_{\rm app}(x)\geqslant 0$ for all $x\epsilon E$. At each point of E', F is L^r-continuous and a fortiori approximately continuous. The conclusion now follows from Lemma 4.

Remarks. 1) In the proof of Lemma 3 it is clear that the hypothesis on E' could be replaced by the weaker hypothesis that $F(E')$ does not contain an interval. Consequently, this can be done in Lemma 4 and Theorem 5.

2) The proof of Lemma 3 goes through if each point $x_0\epsilon E$ is merely a point of right-hand density of the set $\{x;F(x)\geqslant F(x_0)\}$, provided $F(x)$ is approximately continuous at each point of $[a,b]$. Consequently, the hypothesis in Lemma 4 could be changed to require that $\underline{F}^+_{\rm app}(x)\geqslant 0$ for $x\epsilon E$ and $F(x)$ is approximately continuous on $[a,b]$. Theorem 5 will then hold under the hypothesis that $\underline{F}^+_r(x)\geqslant 0$ for each $x\epsilon E$ and $F(x)$ is L^r-continuous on $[a,b]$.

3) Furthermore, the hypothesis of approximate continuity in Lemmas 3 and 4 may be replaced by the weaker requirement that F be approximately lower semi-continuous on the right and approximately upper semi-continuous on the left. The requirement of L^r-continuity in Theorem 5 may be weakened in a similar manner.

PART II

4. The Perron integral in the L^r-sense.

Definition 1. Given a function $f(x)$ on $[a,b]$. A finite-valued function $\psi(x)\epsilon L^r[a,b]$, $1\leqslant r<\infty$, is said to be an *L^r-major function* of $f(x)$ if (i) $\psi(a)=0$, (ii) $\psi(x)$ is L^r-continuous on $[a,b]$ and (iii) except for an at most denumerable subset of $[a,b]$ we have $-\infty\neq\underline{\psi}_r(x)\geqslant f(x)$. A function $\varphi(x)$ is an *L^r-minor function of $f(x)$* if $-\varphi(x)$ is an L^r-major function of $-f(x)$.

THEOREM 6. *Suppose that $\psi(x)$ and $\varphi(x)$ are, respectively, L^r-major and L^r-minor functions of $f(x)$ on $[a,b]$. Then $u(x)=\psi(x)-\varphi(x)$ is non-decreasing on $[a,b]$.*

Proof. Except for an at most denumerable set we have $-\infty\neq\underline{\psi}_r(x)\geqslant f(x)\geqslant\overline{\varphi}_r(x)\neq\infty$. We shall show that $\underline{u}_r(x)\geqslant 0$. For $\varepsilon>0$

there exist $\alpha, \beta, \alpha \leqslant \beta+\varepsilon$ such that

$$\int_0^h [S(x,t)]_-^r dt = o(h^{r+1}), \qquad \int_0^h [T(x,t)]_+^r dt = o(h^{r+1})$$

where

$$S(x,t) = \psi(x+t)-\psi(x)-\beta t, \qquad T(x,t) = \varphi(x+t)-\varphi(x)-\alpha t.$$

Let $U(x,t) = u(x+t)-u(x)-(\beta-\alpha)t = S(x,t)-T(x,t)$. Then $[U(x,t)]_- \leqslant [S(x,t)]_- + [T(x,t)]_+$ and so, by Minkowski's inequality,

$$\int_0^h [U(x,t)]_-^r dt = o(h^{r+1}) \quad \text{and} \quad \underline{u}_r^+(x) \geqslant \beta-\alpha \geqslant -\varepsilon.$$

Since ε is arbitrary, $\underline{u}_r^+(x) \geqslant 0$. Similarly $\underline{u}_r^-(x) \geqslant 0$. Since $u(x)$ is L^r-continuous, our conclusion now follows from Theorem 5.

Definition 2. Let $f(x)$ be defined on $[a, b]$. If $\inf \psi(b)$ for all L^r-major functions ψ of $f(x)$ equals $\sup \varphi(b)$ for all minor functions φ of $f(x)$, then the common value, denoted by

$$(P_r)\int_a^b f(x)\,dx$$

is called the *L^r-Perron integral* of f on $[a, b]$, and f is said to be *P_r-integrable* on $[a, b]$.

Remark 1. The P_r-integrability of a function f and the value of the integral are not affected when we extend the class $\{\psi\}$ of L^r-major functions and the class $\{\varphi\}$ of L^r-minor functions of $f(x)$ by allowing the inequality $\underline{\psi}_r(x) \geqslant f(x) \geqslant \overline{\varphi}_r(x)$ to fail on a set of measure zero. (However, the condition $\underline{\psi}_r(x) > -\infty$ and $\overline{\varphi}_r(x) < \infty$ is required to hold nearly everywhere [6]. A function $f(x)$ may, therefore, be P_r-integrable even if it is not defined on a set of measure zero. The proof is the same as for the classical Perron integral [5]. It is not difficult to see that Theorem 6 will hold also for these extended families of L^r-major and L^r-minor functions.

Remark 2. As in the case of the classical Perron integral, it follows from Theorem 6 that $\psi(b) \geqslant \varphi(b)$ and consequently, f is P_r-integrable on $[a, b]$ if and only if there exists a sequence $\{\psi_n\}$ of L^r-major functions and a sequence $\{\varphi_n\}$ of L^r-minor functions such that $\lim_n [\psi_n(b)-\varphi_n(b)] = 0$, and then

$$\lim_n \psi_n(b) = \lim_n \varphi_n(b) = (P_r)\int_a^b f\,dx.$$

[6] By "nearly everywhere" we mean everywhere except for at most a countable set of points.

In particular, if F is both an L^r-major and L^r-minor function of f, then $F(b) = (P_r)\int_a^b f\,dx$.

5. Elementary properties of the P_r-integral. As an immediate consequence of Remark 1 following definition 2 we have

THEOREM 7. *Suppose $f(x) = g(x)$ a.e. and f is P_r-integrable on $[a, b]$. Then g is also P_r-integrable and*

$$(P_r)\int_a^b f\,dx = (P_r)\int_a^b g\,dx.$$

From Theorem 6, we obtain immediately the following two theorems.

THEOREM 8. *If f is P_r-integrable on $[a, b]$, then f is P_r-integrable on any subinterval of $[a, b]$.*

THEOREM 9. *Let*

$$F(x) = (P_r)\int_a^x f\,dt, \qquad a \leqslant x \leqslant b.$$

Then for any L^r-major function ψ and any L^r-minor function φ of f, $\psi - F$ and $F - \varphi$ are non-decreasing on $[a, b]$.

We also have

THEOREM 10. *If f is P_r-integrable on $[a, b]$ and $[b, c]$, then it is also P_r-integrable on $[a, c]$ and we have*

$$(P_r)\int_a^c f\,dx = (P_r)\int_a^b f\,dx + (P_r)\int_b^c f\,dx.$$

THEOREM 11. *If $F(x)$ is L^r-continuous on $[a, b]$ and $-\infty < \underline{F}_r \leqslant \overline{F}_r < \infty$ nearly everywhere, then $F'_r(x)$ exists a.e. on $[a, b]$ and*

$$F(x)-F(a) = (P_r)\int_a^x F'_r(t)\,dt.$$

In fact, by the corollary to Theorem 4, F'_r exists and is finite a.e., and since $F(x)-F(a)$ is both an L^r-major and L^r-minor function of $F'_r(x)$, it is the indefinite P_r-integral of $F'_r(x)$. (See Remarks 1 and 2 following definition 2.)

THEOREM 12. *Suppose that f is P_r-integrable on $[a, b]$ and let*

$$F(x) = (P_r)\int_a^x f\,dt, \qquad a \leqslant x \leqslant b.$$

Then (i) *$F(x)$ is L^r-continuous on $[a, b]$,* (ii) *$F'_r(x)$ exists and is finite a.e. on $[a, b]$ and* (iii) *$F'_r(x) = f(x)$ a.e. on $[a, b]$.*

Proof of (i). Given $\varepsilon > 0$. Choose an L^r-minor function $\varphi(x)$ of f so that for the non-decreasing function $u(x) = F(x) - \varphi(x)$ we have $0 \leqslant u(x) < \varepsilon$ and consequently $|F(x+t) - F(x)| < |\varphi(x+t) - \varphi(x)| + +\varepsilon(x, x+t \epsilon [a, b])$. Since φ is L^r-continuous we see that, for h sufficiently small,

$$\frac{1}{2h} \int_{-h}^{h} |F(x+t) - F(x)|^r dt < \varepsilon^r.$$

The proof of (ii) and (iii) is carried out exactly as in the classical case. From Theorems 12 and 3 we obtain immediately the following

COROLLARY. *Every P_r-integrable function f on an interval $[a, b]$ is measurable and finite a.e. on $[a, b]$.*

THEOREM 13. *Suppose that f is P_r-integrable on every interval $[a, x]$, $a \leqslant x < b$, and let*

$$F(x) = (P_r) \int_a^x f\, dt.$$

Suppose also that it is possible to define $F(b)$ so that $F(x)$ is L^r-continuous at $x = b$. Then f is P_r-integrable on $[a, b]$ and

$$F(b) = (P_r) \int_a^b f\, dt.$$

Proof. Since F is L^r-continuous at b, given $\delta > 0$ we have

$$\int_{b-h}^{b} |F(b) - F(x)|^r < \delta^r h$$

for h sufficiently small and, consequently, $|F(b) - F(x_0)| < \delta$ for some x_0, $b-h < x_0 < b$. It follows that we can construct an increasing sequence of points $\{b_n\}$, $n = 0, 1, 2, \ldots$, on $[a, b]$ such that $a = b_0$, $b_n \to b$ and $F(b_n) \to F(b)$.

Given $\varepsilon > 0$, we shoose for each interval $[b_{i-1}, b_i]$ an L^r-major function $\psi_i(x)$ of f so that $\psi_i(b_{i-1}) = 0$ and

$$\int_{b_{i-1}}^{b_i} f\, dt \leqslant \psi_i(b_i) \leqslant \int_{b_{i-1}}^{b_i} f\, dt + \varepsilon/2^i.$$

Let

$$\psi(x) = \begin{cases} \sum\limits_{i=1}^{\infty} \psi_i(b_i), & x = b, \\ \sum\limits_{i=1}^{k-1} \psi_i(b_i) + \psi_k(x), & b_{k-1} \leqslant x \leqslant b_k. \end{cases}$$

It follows that

$$|\psi(b) - \psi(x)| \leqslant |F(b) - F(x)| + \varepsilon/2^{k-1}, \quad b_{k-1} \leqslant x \leqslant b_k.$$

Suppose now that $b_{n-1} \leqslant b - h < b_n$. Then

$$|\psi(b) - \psi(x)| \leqslant |F(b) - F(x)| + \varepsilon/2^{n-1}, \quad b - h \leqslant x < b,$$

and by Minkowski's inequality and the L^r-continuity of F at b, we obtain

$$\left\{\frac{1}{h} \int_{b-h}^{b} |\psi(b) - \psi(x)|^r dx\right\}^{1/r} \leqslant o(1) + \varepsilon/2^{n-1} = o(1), \quad \text{as } h \to +0,$$

so that ψ is L^r-continuous at $x = b$. It is now easy to see that ψ is an L^r-major function of f on $[a, b]$ and that $F(b) \leqslant \psi(b) \leqslant F(b) + \varepsilon$.

Similarly, we can construct a minor function φ of f on $[a, b]$ so that $F(b) - \varepsilon \leqslant \varphi(b) \leqslant F(b)$. Since ε is arbitrary, f is P_r-integrable on $[a, b]$ and

$$F(b) = (P_r) \int_a^b f\, dt.$$

Remark. We do not attempt to construct ψ so that $\underline{\psi}_r(b) > -\infty$. As a matter of fact, there are cases (see Example 1, § 7) where it is impossible to fulfill this requirement, and it is exactly for this reason that exceptional points in the definition are unavoidable.

THEOREM 14. *Suppose $1 \leqslant r < q < \infty$. Then any function which is either P_q-integrable or Perron integrable is also P_r-integrable and the values of the integrals are equal.*

Proof. By Theorem 2, every major function in the classical sense and every L^q-major function is also an L^r-major function. Similarly for minor functions.

Remark 1. There are cases (see Example 2, § 7) of a continuous function $F(x)$, $a \leqslant x \leqslant b$, which has a finite L^r-derivative at each point of $[a, b]$ for all r, $1 \leqslant r < \infty$, and at the same time there is a set C, $|C| > 0$, such that at each $x \epsilon C$, $F'(x)$ fails to exist. By the corollary to Theorem 2, for $a \leqslant x \leqslant b$ we have $F'_r(x) = f(x)$ where $f(x)$ is the same for all r. By Theorem 11,

$$F(x) = (P_r) \int_a^x f\, dt, \quad 1 \leqslant r < \infty.$$

On the other hand, if $f(x)$ were Perron integrable in the classical sense we would have, by Theorem 14

$$(P) \int_a^x f\, dt = (P_r) \int_a^x f\, dt = F(x)$$

and consequently $F'(x) = f(x)$ a.e. contradicting the fact that $F'(x)$ does not exist for x in C.

We see thus that a function $f(x)$ may be P_r-integrable for all $r, 1 \leqslant r < \infty$, yet fail to be Perron integrable.

Similarly (see Example 3, § 7) for any $r, 1 \leqslant r < \infty$, it is possible to construct a continuous function $F(x)$ on $[a, b]$ such that F'_r exists and is finite everywhere, and at the same time there is a set C, $|C| > 0$, such that $F'_{r+\varepsilon}(x)$ does not exist for any x in C and any $\varepsilon > 0$. It follows that $F'_r(x) = f(x)$ is P_r-integrable on $[a, b]$ but is not $P_{r+\varepsilon}$-integrable for any $\varepsilon > 0$.

Remark 2. It can be easily verified that any L^r-major function is a C_1-major function (see [2]), and similarly for minor functions. It follows that any function which is L^r-integrable is also C_1-integrable and the values of the integrals are equal. Thus our scale of integration is intermediate between the classical Perron integral and the scale of integration of Burkill [3].

THEOREM 15. *If $f(x)$ and $g(x)$ are P_r-integrable on $[a, b]$ and a and β are real numbers then the function $h = af + \beta g$ is also P_r-integrable and we have*

$$(P_r)\int_a^b h\,dx = a\cdot(P_r)\int_a^b f\,dx + \beta\cdot(P_r)\int_a^b g\,dx.$$

The proof is the same as in the case of the classical Perron integral.

THEOREM 16. 1) *If f is P_r-integrable on $[a, b]$ and $f \geqslant 0$ a.e., then f is also Lebesgue integrable on $[a, b]$.*

2) *If $f_n(x)$ is a non-decreasing sequence of P_r-integrable functions and $(P_r)\int_a^b f_n dx$ is bounded, then*

$$f(x) = \lim_n f(x)$$

is also P_r-integrable and we have

$$(P_r)\int_a^b f\,dx = \lim_n \int_a^b f_n\,dx.$$

3) *If f and g are P_r-integrable on $[a, b]$ and $f \geqslant g$ a.e., then*

$$(P_r)\int_a^b f\,dx \geqslant (P_r)\int_a^b g\,dx.$$

The proof, making use of Theorems 5 and 15, is the same as for the classical case.

6. Integration by parts.

THEOREM 17. *Suppose that, on the interval $[a, b]$, f is P_r-integrable, G is absolutely continuous and $G'(x) \equiv g(x)$ is in $L^{r'}$, where $1 \leqslant r < \infty$, $r' = r/(r-1)$. Then fG is P_r-integrable on $[a, b]$ and if $F(x) = C + \int_a^x f\,dt$, then*

$$(1)\qquad (P_r)\int_a^b fG\,dx = FG]_a^b - \int_a^b Fg\,dx,$$

where the integral on the right exists as a Lebesgue integral.

(If $r = 1$, the condition on G becomes $G \in \mathrm{Lip}\ 1$).

It is well known that in the classical case $r = \infty$ $(r' = 1)$ it is enough to assume that G is of bounded variation. In that case the integral on the right is the Riemann-Stieltjes integral $\int_a^b F\,dG$.

Proof. Since (1) is obvious when either F or G is constant, and since $g = g^+ - g^-$, we may assume without loss of generality that $F(a) = 0$ and G is non-negative and non-decreasing.

Let $\psi(x)$ and $\varphi(x)$ be a major and a minor function for f. We consider the function

$$(2)\qquad M(x) = \psi(x)G(x) - \int_a^x \varphi(u)g(u)\,du$$

and we will show that it is a major function for the product fG.

Clearly, $M(a) = 0$ and $M(x)$ is L^r-continuous on $[a, b]$. We shall show that $\underline{M}_r(x) \geqslant \underline{\psi}_r(x)G(x)$ nearly everywhere and it will follow that $\underline{M}_r(x) > -\infty$ nearly everywhere and $\underline{M}_r(x) \geqslant f(x)G(x)$ a.e. on $[a, b]$.

Let x be any point such that $\underline{\psi}_r(x) > -\infty$, $\overline{\varphi}_r(x) < \infty$, and let a, β be finite numbers such that $a < \underline{\psi}_r(x)$, $\beta > \overline{\varphi}_r(x)$. Then, for $t > 0$,

$$\begin{aligned}
&M(x+t) - M(x) - aG(x)t \\
&= G(x+t)\psi(x+t) - G(x)\psi(x) - aG(x)t - \int_0^t \varphi(x+s)g(x+s)\,ds \\
&= G(x+t)[\psi(x+t) - \psi(x) - at] + at[G(x+t) - G(x)] + [\psi(x) - \varphi(x)][G(x+t) - \\
&\quad - G(x)] - \int_0^t [\varphi(x+s) - \varphi(x) - \beta s]g(x+s)\,ds - \int_0^t \beta s g(x+s)\,ds \\
&= A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned}$$

Clearly, $A_2 = o(t)$ as $t \to 0$, and by Theorem 6, $A_3 \geqslant 0$. Also, using Hölder's inequality in case $r > 1$ and the essential boundedness of g

in case $r = 1$, we see that $A_5 = o(t)$. We thus obtain

$$[M(x+t)-M(x)-aG(x)t]_-$$
$$\leqslant G(x+t)[\psi(x+t)-\psi(x)-at]_- + \int_0^t [\varphi(x+s)-\varphi(x)-\beta s]_+ g(x+s)\,ds + o(t)$$
$$= P+Q+o(t).$$

Again, using Hölder's inequality for $r>1$ and the essential boundedness of g in case $r=1$ we see that $Q = o(t)$. We now have

$$[M(x+t)-M(x)-aG(x)t]_- \leqslant G(b)[\psi(x+t)-\psi(x)-at]_- + o(t).$$

It follows, by Minkowski's inequality, that

(3) $$\int_0^h [M(x+t)-M(x)-aG(x)t]_-^r\,dt = o(h^{r+1}).$$

Similarly, using the equality

$$M(x)-M(x-t)-aG(x)t = G(x-t)[\psi(x)-\psi(x-t)-at] + at[G(x-t)-G(x)] + \int_0^t [\psi(x)-\psi(x-s)-as]g(x-s)\,ds + \int_0^t [\psi(x-s)-\varphi(x-s)]g(x-s)\,ds + \int_0^t asg(x-s)\,ds$$

we can show that

(4) $$\int_0^h [M(x)-M(x-t)-aG(x)t]_-^r\,dt = o(h^{r+1}).$$

It follows from (3) and (4) that $\underline{M}_r^+(x)$, $\underline{M}_r^-(x)$, and consequently $\underline{M}_r(x)$, are nearly everywhere, $\geqslant aG(x)$.

Since a is any finite number less than $\underline{\psi}_r(x)$, it follows that

$$\underline{M}_r(x) \geqslant \underline{\psi}_r(x)G(x) \quad \text{nearly everywhere.}$$

Thus $M(x)$ is a major function of $f(x)G(x)$.

Similarly,

$$m(x) = \varphi(x)G(x) - \int_a^x \psi(s)g(s)\,ds$$

is a minor function of fG.

Since

$$M(x)-m(x) = [\psi(x)-\varphi(x)]G(x) + \int_a^x [\psi(s)-\varphi(s)]g(s)\,ds$$

is uniformly small together with $\psi-\varphi$, it follows that fG is P_r-integrable on $[a,b]$. Making $\psi\to F$ and $\varphi\to F$ we obtain from (2) the formula

$$\int_a^b fG\,dx = F(b)G(b) - \int_a^b Fg\,dx$$

and the theorem is established.

As a corollary, we obtain the following second mean value theorem for the P_r-integral.

THEOREM 18. *Suppose that f is P_r-integrable and G a non-decreasing absolutely continuous function on $[a,b]$ such that $G'(x) \equiv g(x)$ is in $L^{r'}$ $[a,b]$, $r' = r/(r-1)$. Then there exists ξ, $a<\xi<b$, such that*

$$(P_r)\int_a^b fG\,dx = G(a)\int_a^\xi f\,dx + G(b)\int_\xi^b f\,dx.$$

The proof proceeds as in the classical case and is based upon the following (well known)

LEMMA. *Suppose that F is L^r-continuous (or even merely approximately continuous) on an interval $[a,b]$; then F has the Darboux property on this interval.*

Proof of Lemma. Suppose that F fails to have the Darboux property on $[a,b]$. Then there exists a constant k and a subinterval I of $[a,b]$ such that I is the union of the two disjoint non-empty sets $A = \{x\in I;\ F(x)<k\}$, $B = \{x\in I;\ F(x)>k\}$. By the approximate continuity of F, every point of A is both a point of right-hand density of A and a point of left-hand density of A, and similarly for B. It now follows from Lemma 2 that every point of A is to the right and to the left of every point of B which is absurd.

To complete the proof of the theorem, we observe that since the theorem is obvious when G is constant, we may assume that $\int_a^b g\,dx \neq 0$. By Theorem 17,

$$(P_r)\int_a^b fG\,dx = FG\Big]_a^b - \int_a^b Fg\,dx.$$

Now,

$$\int_a^b Fg\,dx = k\int_a^b g\,dx$$

for some constant k, and clearly we cannot have $k>F(x)$ a.e. or $k<F(x)$ a.e. Hence there exist x', x'' in the interior of $[a,b]$ such that $F(x') \leqslant k \leqslant F(x'')$. By the Darboux property for F, $k = F(\xi)$ for some ξ in $[x', x'']$ and our theorem is established.

7. Examples.

Example 1 (See Remark following Theorem 13).

For $n = 2, 3, \ldots$, let I_n and J_n denote intervals with center at $1-1/n$ and of length equal to n^{-3} and $2^{-1}n^{-3}$ respectively. Let $F_n(x)$ be a differentiable non-negative function such that (i) $F_n(x) \leqslant 2n^{s(n)}$, $s(n) = (\log n)^{-1/2}$, (ii) the support of $F_n(x)$ is contained in I_n and (iii) for x in J_n, $F_n(x) \geqslant n^{s(n)}$.

Let

$$F(x) = \sum_{n=2}^{\infty} F_n(x).$$

Then, for $0 \leqslant x < 1$, $F'(x)$ exists and is finite; and for $1 \leqslant r < \infty$, $F(x)$ is L^r-continuous at $x = 1$. Defining $f(x) = F'(x)$, $0 \leqslant x < 1$, $f(x) = 0$, $x = 1$, it follows from Theorems 2, 11 and 13 that $f(x)$ is P_r-integrable on $[0, 1]$ and

$$F(x) = (P_r)\int_0^x f\,dt, \quad 0 \leqslant x \leqslant 1,\ 1 \leqslant r < \infty.$$

Let now ψ be any major function of f on $[0, 1]$. Since $\psi \geqslant F$, we have for $1-t \in J_n$ and n sufficiently large, and whatever the value of $\psi(1)$ and of β may be,

$$\psi(1-t) - \psi(1) + \beta t \geqslant \tfrac{1}{2}n^{s(n)}.$$

It follows that

$$\int_0^h [\psi(1) - \psi(1-t) - \beta t]_-^r\,dt \neq o(h^{r+1}).$$

Hence $\underline{\psi}_r(1) = -\infty$.

Example 2 (See Remark following Theorem 14).

On the interval $[0, 1]$ we construct a Cantor set C of measure $\mu > 0$ by the standard process of removing open intervals in successive stages. At the n^{th} stage, $n = 1, 2, 3, \ldots$, we start with 2^{n-1} closed intervals of equal length and remove from the center of each one an open interval $B_{n,k}$ $(k = 1, 2, 3, \ldots, 2^{n-1})$ of length $|B_{n,k}| = \lambda n^{-2} 2^{-n+1}$, where λ satisfies the equality

$$\lambda \sum_{n=1}^{\infty} n^{-2} = 1 - \mu.$$

This leaves 2^n closed intervals $W_{n,l}$ $(l = 1, 2, 3, \ldots, 2^n)$ of equal length. The desired set C is the complement of the union of all the $B_{n,k}$.

For each $B_{n,k}$ let $I_{n,k}$ and $J_{n,k}$ be subintervals in the center of $B_{n,k}$ of length $2^{-n-1}|B_{n,k}|$ and $2^{-n}|B_{n,k}|$ respectively. For each $B_{n,k}$ we construct a non-negative differentiable function $F_{n,k}$ such that (i) the support of $F_{n,k}$ is contained in $J_{n,k}$, (ii) $F_{n,k}(x) \geqslant 2^{-n}$ when x is in $I_{n,k}$ and (iii) $F_{n,k}(x) \leqslant 2^{-n+1}$. Let

$$F(x) = \begin{cases} 0, & x \in C, \\ F_{n,k}(x), & x \in B_{n,k}. \end{cases}$$

Clearly, $F(x)$ is continuous. Moreover, $F'(x)$ exists and is finite for $x \notin C$. It can also be shown that for any $x \in C$ and any r, $1 \leqslant r < \infty$, we have $F'_r(x) = 0$, and $\overline{F}'(x) \neq 0$. It follows from the Corollary to Theorem 2 that $F'_r(x)$ exists for all $r, 1 \leqslant r < \infty$, and all x, $0 \leqslant x \leqslant 1$, and that $F'(x)$ does not exist for any x in C.

Example 3 (See Remark following Theorem 14).

Let the intervals $B_{n,k}$ and the set C be as in Example 2. Given r, $1 \leqslant r < \infty$, we construct on each $B_{n,k}$ a differentiable function $F_{n,k}$ such that (i) The support of $F_{n,k}$ is contained in a subinterval in the center of $B_{n,k}$ of length $2^{-n}n^{-1}|B_{n,k}|$, (ii) $F_{n,k}(x) \geqslant 2^{(n/r)-1}|B_{n,k}|$ on a subinterval in the center of $B_{n,k}$ of length $n^{-1}2^{-n-1}|B_{n,k}|$ and (iii) $F_{n,k}(x) \leqslant 2^{n/r}|B_{n,k}|$. Let

$$F(x) = \begin{cases} 0, & x \in C, \\ F_{n,k}(x), & x \in B_{n,k}. \end{cases}$$

Again, $F(x)$ is continuous and $F'_r(x) = F'(x)$ exists and is finite for $x \notin C$. It can also be shown that for any $x \in C$, $F'_r(x) = 0$ and $F'_{r+\varepsilon}(x) \neq 0$, for any $\varepsilon > 0$. It follows as in example 2 that $F'_{r+\varepsilon}(x)$ does not exist for $x \in C$.

PART III

8. Application to Fourier series. In the final section of this paper we apply previously obtained results to the theory of Fourier series.

Suppose $f(x)$ is real valued, periodic of period 2π and P_r-integrable, $1 \leqslant r < \infty$, over any interval $[a, a+2\pi]$. Clearly,

$$(P_r)\int_a^{a+2\pi} f\,dx$$

is independent of a. In view of Theorem 17, the Fourier coefficients

$$a_n = \frac{1}{2\pi}(P_r)\int_0^{2\pi} f(t)\cos nt\,dt, \quad b_n = \frac{1}{2\pi}(P_r)\int_0^{2\pi} f(t)\sin nt\,dt$$

exist $(n = 0, 1, 2, \ldots)$. We are interested in the behavior of the Fourier series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

of f. If we suppose for simplicity that $a_0 = 0$, then the indefinite integral

$$F(x) = (P_r)\int_0^x f(t)\,dt$$

is also a periodic function, and as integration by parts shows, its Fourier series is obtained by a termwise integration of the Fourier series of f, i.e.

$$F(x) \sim C + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}.$$

The function F belongs to L^r, and in particular to L, so that its Fourier coefficients tend to zero. It follows that if f is periodic and P_r-integrable, $1 \leqslant r < \infty$, over a period, then the Fourier coefficients a_n, b_n of f are $o(n)$ (as in the case of the classical Perron integral; [9], Vol. II, p. 85), and so the method $(C, 1)$ seems to be appropriate to apply here.

Before stating the next theorem, let us also observe that at each point where F'_r exists, the function F satisfies the Dini condition (this is easily seen by integration by parts) so that the Fourier series of F converges a.e. to F.

THEOREM 19. *If f is of period 2π and P_r-integrable, $1 \leqslant r < \infty$, over a period, then the Fourier series of f*

$$\tfrac{1}{2}a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is almost everywhere $(C, 1)$-summable to $f(x)$. Likewise, the conjugate series

$$\sum_{1}^{\infty} (a_n \sin nx - b_n \cos nx)$$

is almost everywhere $(C, 1)$-summable to the function

$$(1) \qquad \tilde{f}(x) = \lim_{\varepsilon \to 0}\left[-\frac{1}{\pi}(P_r)\int_{\varepsilon}^{\pi} \frac{f(x+t) - f(x-t)}{2\tan\frac{1}{2}t}\,dt\right],$$

where the limit is taken in the L^r-sense (7).

In the classical Perron case this result has been obtained by Marcinkiewicz (see [9], Vol. II, p. 85) and his argument is easily adaptable to our case. For this reason we can be brief.

(7) i.e. denoting the expression in brackets on the right-hand side of (1) by $g(x, \varepsilon)$, we have

$$\int_0^h |\tilde{f}(x) - g(x, s)|^r\,ds = o(h) \quad \text{as} \quad h \to 0.$$

We begin with the first part of the Theorem. In view of the fact that under our hypothesis $f = F'_r$ almost everywhere, the result follows from

THEOREM 20. *Suppose F is periodic of period 2π and is in $L^r(0, 2\pi)$, $1 \leqslant r < \infty$. Then if F'_r exists and is finite on a set E, the differentiated Fourier series of F is $(C, 1)$-summable to F'_r almost everywhere in E.*

For $r = \infty$, this is the result of Marcinkiewicz. In the proof which follows we assume, as we may, that $r = 1$.

We denote the Fourier series of F by $S[F]$ and the termwise differentiated series by $S'[F]$. For the conjugate series we correspondingly use the notation $\tilde{S}[F]$ and $\tilde{S}'[F]$.

The hypothesis that F'_1 exists and is finite in E implies that given $\varepsilon > 0$ we can find a closed subset E_1 of E, $|E_1| > |E| - \varepsilon$, such that F'_1 exists uniformly in E_1 and $F(x)$, $F'_1(x)$ are uniformly bounded in E_1. By a decomposition theorem of Calderón and Zygmund ([4], Theorem 9) we may write

$$(2) \qquad F = G + H$$

where G is in C' (and of period 2π) and $G = F$ on E_1. It follows that $H = 0$ on E_1. In particular, since $H'_{\text{app}} = 0$ at each point of density of E_1, $H'_1 = 0$ almost everywhere on E_1. Let E_2 be a closed subset of E_1, $|E_2| > |E| - \varepsilon$ and such that on E_2, H'_1 exists uniformly, and $H'_1 = 0$. If we set $f = F'_1$, $h = H'_1$, $g = G'$, then by (2) we have in E_2, $f = g + h$ and $h = 0$. Also by (2)

$$S'[F] = S'[G] + S'[H] = S[g] + S'[H].$$

Since $S[g]$ is (uniformly) $(C, 1)$-summable to g, and $g = f$ in E_2, the proof will be completed if we show that $S'[H]$ is $(C, 1)$-summable to 0 at almost every point of E_2. Denote by $\sigma_n(x)$ the $(C, 1)$-means of $S'[H]$ and consider a point $x_0 \in E_2$. Since H'_1 exists and equals zero at x_0, we have

$$(3) \qquad \int_0^t |H(x_0 + s)|\,ds = o(t^2).$$

If $K_n(s)$ is the Fejér kernel

$$\frac{1}{2(n+1)}\left(\frac{\sin\frac{1}{2}(n+1)s}{\sin\frac{1}{2}s}\right)^2,$$

then

$$\sigma_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} H(x_0 + s)K'_n(s)\,ds = -\frac{1}{\pi}\int_{|s| \leqslant 1/n} - \frac{1}{\pi}\int_{1/n \leqslant |s| \leqslant \pi} = P_n + Q_n,$$

say. Since $K'_n = O(n^2)$,

$$|P_n| \leqslant O(n^2) \int_{-1/n}^{1/n} |H(x_0+s)|\, ds = O(n^2)\cdot o(n^{-2}) = o(1),$$

by (3), and it remains to estimate Q_n.

It is clear that for $1/n \leqslant s \leqslant \pi$,

$$|K'_n(s)| \leqslant A(n^{-1}|s|^{-3}+s^{-2}) \leqslant A/s^2$$

so that

$$|Q_n| \leqslant A \int_{1/n\leqslant|s|\leqslant\pi} |H(x_0+s)| \frac{ds}{s^2}. \tag{4}$$

We shall now use the fact that if $\delta(x)$ is the function equal to 0 on E_2 and equal to (b_j-a_j) on any interval (a_j, b_j) contiguous to E_2, then the integral

$$\int_{t\leqslant\pi} \frac{\delta(x+t)}{t^2}\, dt \tag{5}$$

is finite for a.e. x in E_2 ([9], Vol. I, p. 130). In view of the hypothesis that $H'_1(x)$ exists uniformly in E_2 and equals zero there, we have

$$\int_{a_j}^{b_j} |H(t)|\, dt \leqslant A(b_j-a_j)^2$$

for all intervals (a_j, b_j) contiguous to E_2. It is not difficult to prove (see e.g. the argument in [9], Vol. I, p. 131) that at every point of density of E_2 the integral in (4) is majorized by some multiple of the integral (5) and therefore $Q_n = O(1)$ almost everywhere in E_2. We refine here the O to o in a routine way, observing that the behavior of $\sigma_n(x_0)$ depends only on the behavior of H in the immediate neighbourhood of x_0. Hence $\sigma_n(x) \leqslant |P_n|+|Q_n| = o(1)$ almost everywhere in E_2, and so almost everywhere in E. This completes the proof of Theorem 20 and so also of the first part of Theorem 19.

We now use a well known result (see [6] or [7]) stating that if any trigonometric series is $(C,1)$-summable in a set E, then the conjugate series is $(C,1)$-summable almost everywhere in E. Hence under the hypothesis of Theorem 20, $\tilde{S}'[F]$ is $(C,1)$-summable almost everywhere in E. In particular, under the hypothesis of Theorem 19, $\tilde{S}[f]$ is $(C,1)$-summable almost everywhere and it remains only to show that the $(C,1)$-sum of $\tilde{S}[f]$ is given by formula (1).

This can be proved e.g. as follows. $(C,1)$-summability of $\tilde{S}[f]$ $(=\tilde{S}'[F])$ implies A (Abel) summability so that

$$\lim_{\varrho\to1} \frac{1}{\pi} \int_{-\pi}^{\pi} F(x_0+t) \frac{\partial}{\partial t} Q(\varrho, t)\, dt \tag{6}$$

exists almost everywhere, where

$$Q(\varrho, t) = \frac{\varrho \sin t}{1-2\cos t+\varrho^2}$$

is the conjugate Poisson kernel.

In the case where $F(x_0+t)+F(x_0-t)-2F(x_0) = o(t)$, as $t\to0$, it is known (see [9], Vol. I, p. 103) that the limit in (6) equals

$$\lim_{\varepsilon\to0} \left[-\frac{1}{\pi} \int_{\varepsilon}^{\pi} \frac{F(x_0+t)+F(x_0-t)-2F(x_0)}{(2\sin\frac{1}{2}t)^2} \right] \tag{7}$$

and with some slight modifications, the proof goes through also for the case where

$$\int_0^h |F(x_0+t)+F(x_0-t)-2F(x_0)|\, dt = o(h^2) \quad \text{as} \quad h\to0. \tag{8}$$

Since F'_1 exists a.e., F satisfies (8) a.e. and so $S[F]$ is almost everywhere $(C,1)$-summable to (7). Integrating by parts, the expression inside the brackets in (7) is seen to equal

$$\frac{1}{\pi} \frac{F(x_0+\varepsilon)+F(x_0-\varepsilon)-2F(x_0)}{2\tan\frac{1}{2}\varepsilon} - \frac{1}{\pi} \int_{\varepsilon}^{\pi} \frac{f(x_0+t)-f(x-t)}{2\tan\frac{1}{2}t}\, dt. \tag{9}$$

At each point x_0 where F'_r exists we have (see Theorem 1)

$$\int_0^h \left| \frac{F(x_0+\varepsilon)-F(x_0-\varepsilon)-2F(x_0)}{2\tan\frac{1}{2}\varepsilon} \right|^r d\varepsilon = o(h).$$

Thus the integrated term in (9) approaches zero in the L^r-sense as $\varepsilon\to0$ and so for almost every x,

$$\tilde{S}[f] = \lim_{\varepsilon\to0}(L^r) \left[-\frac{1}{\pi}(P_r) \int_{\varepsilon}^{\pi} \frac{f(x+t)-f(x-t)}{2\tan\frac{1}{2}t}\, dt \right].$$

Theorem 19 is thus established.

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Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions (I)

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The definition of hypoelliptic and entire elliptic convolution equations can be formulated in a general manner as follows. Let $\mathscr{H}'$ be a space of distributions in R^n, which may be the space $\mathscr{D}'$ of all distributions or one of its subspaces with a topology stronger than that induced in $\mathscr{H}'$ by $\mathscr{D}'$. We assume that:

(h_1) $\mathscr{H}'$ contains the space $\mathscr{E}'$ of distributions of compact support as a dense subset.

(h_2) $\mathscr{H}'$ is a module over the space $\mathscr{E}'$ under convolution, that is, for each $T \in \mathscr{H}'$ and $S \in \mathscr{E}'$, $S*T \in \mathscr{H}'$.

(h_3) The mapping $(S, T) \to S*T$ of $\mathscr{E}' \times \mathscr{H}'$ into $\mathscr{H}'$ is separately continuous.

Furthermore, let $\mathscr{O}'_c(\mathscr{H}' : \mathscr{H}')$ be the space of convolution operators in $\mathscr{H}'$, i.e. the space of continuous linear mappings of $\mathscr{H}'$ into $\mathscr{H}'$, which are convolution operators on $\mathscr{E}' \subset \mathscr{H}'$. $\mathscr{O}'_c(\mathscr{H}' : \mathscr{H}')$ can be identified with a subspace of $\mathscr{H}'$ (see section 1).

We introduce two classes of functions.

(I) $\mathscr{E}\mathscr{H}'$ is the set of all C^∞-functions $f \in \mathscr{H}'$ such that, for any $S \in \mathscr{O}'_c(\mathscr{H}' : \mathscr{H}')$, the convolution $S*f$ is a C^∞-function and $S \to S*f$ is a continuous mapping of $\mathscr{O}'_c(\mathscr{H}' : \mathscr{H}')$ into $\mathscr{E}$ — the space of all C^∞-functions on R^n. We show in section 1 that, in fact, $S*f$ is again in $\mathscr{E}\mathscr{H}'$.

(II) $\mathscr{A}\mathscr{H}'$ is a subset of $\mathscr{E}\mathscr{H}'$. A function $f \in \mathscr{E}\mathscr{H}'$ is in $\mathscr{A}\mathscr{H}'$, if, for every $S \in \mathscr{O}'_c(\mathscr{H}' : \mathscr{H}')$, the convolution $h = S*f$ can be continued analytically in the complex n-space C^n and the growth of the resulting entire function is restricted in the following way. In any horizontal strip V_b in C^n around R^n of width b, $|h(z)| \leqslant |g(\Re z)|$, where g is a function of $\mathscr{E}\mathscr{H}'$ depending on b and $\Re z$ is the real part of z.

Consider now the convolution equation

$$S*U = F, \tag{1}$$

where $S \in \mathscr{O}'_c(\mathscr{H}' : \mathscr{H}')$ and $U, F \in \mathscr{H}'$.