Proof of Theorem 3.3. In the case of S associated with a symmetric basis of an F-space it is necessary to prove that $S^{\sigma\sigma} = S^{\gamma\gamma}$, and this follows from 1.3 and 2.6.

If S is a σ-perfect FK-space, then either $S = s$ in which case we are finished or $S \subseteq m$ [4]. In the second case for each u in S^{a} define

$$P_u^{\sigma}(s) = \sup_{\pi}\left\{\sum_{i=1}^{\infty}|u_{\pi(i)}s_i|: \pi \text{ is a permutation on the natural numbers}\right\}$$

and proceed as in the γ-perfect case. The seminorms P_i defined in the course of the argument will all be symmetric by 2.10 so that $\mathscr{E}$ will be a symmetric basis for its closed linear span by Definition 1.2.

The following is a generalization of $(\mathrm{SB}_1) \Leftrightarrow (\mathrm{SB}_3)$ of Singer [8], 6, Theorem 5.3.

3.4. Corollary. *A basis $\{x_n\}$ of a locally convex F-space X is a symmetric basis if and only if every permutation $\{x_{\pi(n)}\}$ of $\{x_n\}$ is a basis of X equivalent to $\{x_n\}$.*

Proof. Without loss of generality we may restrict our attention to $\mathscr{E}$ a basis for a locally convex space S.

The necessity of the condition was given in Lemma 1.3.

If every permutation of $\mathscr{E}$ is a basis for S equivalent to $\mathscr{E}$, then $s \in S$ implies $(s_{\pi(i)}) \in S$ so that S is symmetric. Therefore, $S^{aa} = S^{\sigma\sigma}$ so that, by 3.2 and 3.3, $\mathscr{E}$ is a symmetric basis for S.

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An example concerning reflexivity

by

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The spaces c_0 and l not only fail to be reflexive but contain no infinite-dimensional reflexive subspace [7, 12]. It is natural to conjecture that each non-reflexive space contains an infinite-dimensional closed subspace with this property; this conjecture is false. Here we give an example of a Banach space which is not reflexive (or even quasi-reflexive [4]) yet has the property that each of its closed infinite-dimensional subspaces contains a subspace isomorphic to the Hilbert space l^2. We also discuss this type of non-reflexive space and show that it has some properties in common with reflexive and quasi-reflexive spaces.

Lemma 1. *Let X be the quasi-reflexive space constructed by R. C. James* ([8], *also see* [9], *p.* 198). *Every infinite-dimensional closed subspace of X contains a subspace isomorphic to l^2.*

Proof. We recall that the space X consists of vectors $x = (a_1, a_2, \ldots)$, x a sequence of scalars, where x is in X if and only if $\lim a_n = 0$ and

$$\|x\| = \sup\Big[\sum_{i=1}^{n}|a_{p_{2i-1}}-a_{p_{2i}}|^2+|a_{p_{2n+1}}|^2\Big]^{1/2}$$

is finite, where the supremum is taken over all finite increasing (or one term) sequences. The vectors x_i, with a one in the i-th place and zeros elsewhere, constitute a Schauder basis for X.

Using a theorem due to Bessaga and Pełczyński ([2], C.2, p. 157), each infinite-dimensional closed subspace M of X contains a sequence $\{y_n\}$ which is basic ($\{y_n\}$ is a Schauder basis for its closed linear span $[y_n]$) and is equivalent to a block basis $[z_n]$, with respect to x_n, i.e. each z_n is given by

$$z_n = \sum_{i=q_n+1}^{q_{n+1}} a_i^n x_i$$

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with $q_n \lneq q_{n+1}$ for all n, and we may choose $\|z_n\| = 1$. By the equivalence of the basis $\{y_n\}$ and $\{z_n\}$ we mean that the linear operator induced by sending y_n to z_n is an isomorphism of $[y_n]$ with $[z_n]$. Because of this equivalence it is enough to show that $[z_n]$ contains a subspace isomorphic to l^2. We will show, in fact, that $[z_{2n}]$ is isomorphic to l^2. To simplify our indexing we will suppose that we have already discarded every other block in the basis $\{z_n\}$ so that there is a space between each block in the sense that $a^n_{q_n+1} = 0$ for all n.

Let $p^j = \{p^j_1, \ldots, p^j_{2n_j+1}\}$ be a sequence which yields the norm of z_j. Let an integer m be given. Combine the first m sequences $p^1, \ldots, p^m$ into an increasing sequence I as follows:

$$I = \{p^1_1, \ldots, p^1_{2n_1+1}, r_1, p^2_1, \ldots, p^2_{2n_2+1}, r_2, \ldots, r_m, p^m_1, \ldots, p^m_{2n_m+1}, r_{m+1}, p^{m+1}_1\},$$

where $r_1, \ldots, r_{m+1}$ are chosen so that each z_j is zero in the r_i-th coordinate for each i; note that we need the space between the z_j, mentioned in the paragraph above, to do this.

Set $I = \{q_1, \ldots, q_{2k+1}\}$ and let $\sum t_i z_i = \{b_i\}$ be an element in $[z_i]$. Then

$$\begin{aligned}\Big\|\sum_1^m t_i z_i\Big\|^2 &\geqslant \sum_{i=1}^{k} |b_{q_{2i-1}} - b_{q_{2i}}|^2 \\ &= \sum_{i=1}^{n_1} |t_1|^2 |a^1_{p^1_{2i-1}} - a^1_{p^1_{2i}}|^2 + |t_1|^2 |a^1_{p^1_{2n_1+1}} - 0|^2 + \\ &\quad + \sum_{i=1}^{n_2} |t_2|^2 |a^2_{p^2_{2i-1}} - a^2_{p^2_{2i}}|^2 + |t_2|^2 |a^2_{p^2_{2n_2+1}} - 0|^2 + \ldots \\ &= |t_1|^2 \|z_1\|^2 + \ldots + |t_m|^2 \|z_m\|^2 = \sum_{i=1}^{m} |t_i|^2.\end{aligned}$$

Since this is true for each m, $\|\sum t_i z_i\| \geqslant [\sum |t_i|^2]^{1/2}$.

For any element $\sum t_1 z_i = \{b_i\}$ in X and any increasing sequence $p_1, \ldots, p_{2n+1}$ we consider the sum

$$(*) \qquad \sum_{i=1}^{n} |b_{p_{2i-1}} - b_{p_{2i}}|^2 + |b_{p_{2n+1}}|^2.$$

A typical term $|b_{p_{2i-1}} - b_{p_{2i}}|^2$ has two possible forms. The first form is $|t_j a^j_{n_i} - t_j a^j_{m_i}|^2$. Let A be the set of all the indices i for which a term of this form occurs in $(*)$. We have

$$\sum_{i \in A} |b_{p_{2i-1}} - b_{p_{2i}}|^2 = \sum_{i \in A} |t_j|^2 |a^j_{n_i} - a^j_{m_i}|^2 \leqslant \sum_{j=1}^{\infty} |t_j|^2.$$

For indices i not in A the term is of the form $|t_j a^j_{n_i} - t_{j+k} a^{j+k}_{m_i}|^2$. Using the fact that $|a^m_i| \leqslant \|z_m\| = 1$ for all m, we see that this term is bounded above by $2[|t_j|^2 + |t_{j+k}|^2]$. Since p_i is an increasing sequence, for any j, the scalar t_j can occur in at most two terms of this second type. Hence

$$\sum_{i \notin A} |b_{p_{2i-1}} - b_{p_{2i}}|^2 \leqslant 4 \sum_{i=1}^{\infty} |t_i|^2.$$

Putting these two estimates together we see that

$$\Big\|\sum t_i z_i\Big\| \leqslant \sqrt{5}\Big[\sum_{i=1}^{\infty} |t_i|^2\Big]^{1/2}.$$

From the upper and lower bounds we have obtained for $\|\sum t_i z_i\|$ we can conclude that the map T from l^2 to $[z_i]$ defined by $T(t_i) = \sum t_i z_i$ is an isomorphism of l^2 onto $[z_i]$ and that completes the proof.

Lemma 1 suggests the next result.

LEMMA 2. *Let X be a quasi-reflexive space* [4]. *Then every infinite-dimensional closed subspace of X contains an infinite-dimensional reflexive subspace.*

Proof. Since each subspace of a quasi-reflexive space is quasi-reflexive ([4], corollary 4.2, p. 909), it suffices to show that each infinite-dimensional quasi-reflexive space contains an infinite-dimensional reflexive subspace. From theorem 3.5 of [4], p. 908, for each quasi-reflexive space X there is a quasi-reflexive space Y with $Y^* = X$. Since for M reflexive $({}_{\perp}M)^{\perp} = M$, an easy duality argument shows that X contains an infinite-dimensional reflexive subspace if and only if Y has an infinite-dimensional reflexive quotient ([12], theorem 5.2, p. 258). The space Y is not reflexive and so by [3], problem 9, p. 119, there is a closed non-reflexive subspace M of Y with Y/M infinite-dimensional. (We remark that a proof of this result can be obtained from the theorem of [10]). Suppose that Y is quasi-reflexive of order n. As in [4] for a Banach space X we let $\mathrm{Ord}(X)$ be the dimension of $X^{**}/\pi(X)$, where π is the natural map of X into X. Then we have $\mathrm{Ord}(Y/M) + \mathrm{Ord}(M) = \mathrm{Ord}\, Y = n$ ([4], corollary 4.2, p. 909). Thus Y/M is an infinite-dimensional quotient of Y with $\mathrm{Ord}(Y/M) \leqslant n-1$. An inductive argument completes the proof.

For a sequence $\{X_i\}$ of Banach spaces, the l^2-sum $X = \sum^2 X_i$ is the Banach space X of sequences $\{x_i\}$ with x_i in X_i and with finite norm

$$\|\{x_i\}\| = \Big[\sum \|x_i\|^2\Big]^{1/2} \quad \text{([5], p. 31).}$$

LEMMA 3. *Let X_i be a sequence of Banach spaces, $X = \sum^2 X_i$ the l^2-sum of these spaces. Suppose that M is an infinite-dimensional closed* *subspace of X. Then either M contains an infinite-dimensional closed subspace which is isomorphic to a subspace of some X_i or M contains a subspace isomorphic to l^2.*

Proof. Let Q_i be the map of X into X_i given by $Q_i(\{x_j\}) = (0, 0, \ldots, 0, x_i, 0, \ldots)$, where x_i occurs in the i-th position. Let M be an infinite-dimensional closed subspace of X. If any Q_i has a bounded inverse on an infinite-dimensional closed subspace of M, then M contains an infinite-dimensional closed subspace isomorphic to a subspace of X_i and we are through. Thus we may assume that each Q_i is strictly singular ([6], p. 76) and in this case $P_n = \sum_{i=1}^{n} Q_i$ is also strictly singular ([6], theorem III.2.4, p. 86). In particular, no P_n has a bounded inverse on M. We now use a standard gliding hump argument. Let y^0 be any element of M of norm 1. Choose N_1 so that

$$\Big[\sum_{i=N_1}^{\infty} \|y_i^0\|^2\Big]^{1/2} < 1/2^3.$$

Since P_{N_1} has no bounded inverse on M, we can find an element y^1 in M, $\|y^1\| = 1$, with $\|P_{N_1} y^1\| < 1/2^5$. Choose N_2 so that

$$\Big[\sum_{i=N_2}^{\infty} \|y_i^1\|^2\Big]^{1/2} < 1/2^5.$$

Continuing we obtain a sequence $y^1, y^2, \ldots$ of elements of M and integers $N_1 \lneqq N_2 \lneqq N_3 \lneqq \ldots$ such that

(1) $$\|y^n\| = 1 \quad \text{for all } n,$$

and

(2) $$\Big[\sum_{i=1}^{N_n} \|y_i^n\|^2\Big]^{1/2} + \Big[\sum_{i=N_{n+1}}^{\infty} \|y_i^n\|^2\Big]^{1/2} < 1/2^{n+2}.$$

Define z^n by $z_i^n = y_i^n$ for $N_n < i < N_{n+1}$ and $z_i^n = 0$ otherwise. We have

$$\Big\|\sum a_i y^i\Big\| \leqslant \Big\|\sum a_i z^i\Big\| + \sup|a_i| \sum \|y^i - z^i\| \leqslant 2\Big[\sum |a_i|^2\Big]^{1/2}.$$

We also have

$$\Big\|\sum a_i y^i\Big\| \geqslant \Big\|\sum a_i z^i\Big\| - \sup|a_i| \sum \|y^i - z^i\| \geqslant (1/2)\Big[\sum |a_i|^2\Big]^{1/2}.$$

Hence the map $T(a_i) = \sum a_i y^i$ is an isomorphism of l^2 onto the subspace $[y^i]$ of M.

We use Lemma 1 and Lemma 3 to give our example.

EXAMPLE. For each i let X_i be the quasi-reflexive space described by James [8]. Let $X = \sum^2 X_i$ be the l^2-sum of these spaces. Note that X is not quasi-reflexive since $X^{**} = \pi(X) \oplus l^2$. Then the space X has the property that any infinite-dimensional closed subspace of X contains a subspace isomorphic to the Hilbert space l^2.

Say that a Banach space X is *somewhat reflexive* if each closed infinite-dimensional subspace of X contains an infinite-dimensional reflexive subspace. From Lemma 2 we see that a quasi-reflexive space is somewhat reflexive and from Lemma 3 we see that the l^2-sum of somewhat reflexive spaces is again somewhat reflexive. We do not know much about somewhat reflexive spaces. We ask, for example, whether the quotient or the conjugate of a somewhat reflexive space must be somewhat reflexive. It is interesting, however, that some of the results of [4], proved there for quasi-reflexive spaces, are true for the somewhat larger class of somewhat reflexive spaces.

LEMMA 4. *Let X be a somewhat reflexive space as defined in the above paragraph.*

(1) *If X has an unconditional basis, then X is reflexive* ([4], corollary 4.5, p. 910).

(2) *Any bounded linear map of X into l is compact* ([4], theorem 5.1, p. 9.1).

Proof. A space with an unconditional basis which is not reflexive contains a copy of either c_0 or l ([5], theorem 4, p. 76), neither of which is somewhat reflexive and (1) follows.

We claim that there is a non-compact continuous linear operator from X to l if and only if there is a continuous projection of X onto a subspace isomorphic to l. Result (2) follows from this. To see that the claim is true, suppose that X contains no complemented subspace isomorphic to l. Then by [2], theorem 4, p. 155, X^* contains no subspace isomorphic to c_0. Since $m = l^*$ is a space of type $C(S)$, each map from m to X^* is weakly compact by [11], theorem 1, p. 35. Thus the conjugate of each map from X to l is weakly compact and so each map of X to l is itself weakly compact. Weak and norm convergence is the same for sequences in l ([1], p. 137) and so each weakly compact map with range in l is compact. The converse is obvious; hence our claim is verified.

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Perron's integral for derivatives in L^r

by

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Introduction. The notion of the classical Perron integral is by now very familiar. It is based on the notions of major and minor functions and of upper and lower Dini derivates and serves the purpose of showing that an exact and finite classical derivative of a function is integrable and the function itself is the indefinite integral of the derivative.

Since the time the Perron integral was initially introduced the notion of derivative has developed and has undergone various generalizations. Every generalization of the derivative can serve as a basis of generalization of Perron's integral. The idea is not new. As far back as 1932 (see [1]) Burkill developed a theory of Perron integration based on approximate derivatives. There also have been other generalizations.

Here we return to this topic but base the theory of the Perron integral on the notion of derivative — and derivates — in the metric L^r. The notion of derivation in L^r has been introduced by Calderón and Zygmund [4] and unlike the idea of the approximate derivative has proved to be quite effective in applications (partial differential equations, area of surfaces, etc.). It seems likely that Perron's integral based on that notion deserves study. I would like to add that though the results of this paper have points in common with earlier results, the extension is not entirely routine.

The present paper consists of three parts. In the first part we define the notion of Dini derivates in the metric L^r (briefly, L^r-derivates) and prove a number of properties well known for the classical derivative (and due primarily to Denjoy and Lusin). In the second part, using previous results, we develop the theory of Perron's integral for derivates in L^r. In the third part we give applications to Fourier series.

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PART I

1. Definitions and elementary properties of L^r-derivates. Let $f(x)$ be finite and real-valued in an interval (a, b). (In what follows, unless