To obtain our theorem let us remark that in paper [3], when deriving the general result from the result on algebras with unit, we never used the fact that the field of scalars is the field of complex numbers. So these arguments work as well in the case of real scalars and we can formulate our main result:

THEOREM. *Let A be a real m-convex algebra. Then either A has generalized topological divisors of zero or A is isomorphically homeomorphic with one of the three finite-dimensional real division algebras (i.e. field of real numbers, field of complex numbers or division algebra of quaternions).*

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Continuity of operator functions

by

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1. Introduction. By $\mathscr{C}$ we shall denote the complex algebra, the elements of which are continuous complex-valued functions of a non-negative real variable. The operation of multiplication is defined by finite convolution; the operations of addition and scalar multiplication are defined in the usual way. $\mathscr{C}$ has no zero divisors, hence the quotient field may be constructed. This quotient field, which we denote by $\mathscr{M}$, is termed the *field of operators*.

It is the purpose of this paper to extend the definitions of operator function and continuous operator function as defined by Mikusiński in [3] (Part III, Chap. 1). A uniform convergence structure [2] is defined on $\mathscr{M}$ and is shown to be the direct limit of uniform structures of linear subspaces of $\mathscr{M}$. The Limitierung defined by M-convergence [4] is the Limitierung induced by the uniform convergence structure of $\mathscr{M}$. A uniform convergence structure is defined for a locally compact Hausdorff space as the direct limit of uniform structures of compact subspaces. It is shown that an operator function is continuous, in the generalized sense, if and only if it is uniformly continuous from a locally compact Hausdorff space provided with the "compact" uniform convergence structure to the uniform convergence space $\mathscr{M}$.

2. Preliminaries. Let X be an arbitrary set. $\boldsymbol{B}(X)$ shall denote the family of filter bases on the set X. If $\mathfrak{J}\in\boldsymbol{B}(X)$, then the filter generated by $\mathfrak{J}$ is denoted by $[\mathfrak{J}]$. $\boldsymbol{F}(X)$ shall denote the family of filters on the set X. The class $\boldsymbol{F}(X)$ is partially ordered by the relation $\leqslant$ defined by: $[\mathfrak{J}]\leqslant[\mathfrak{G}]$ iff for each $F\in\mathfrak{J}$ there exists a $G\in\mathfrak{G}$ such that $G\subset F$. This is equivalent to: $[\mathfrak{J}]\leqslant[\mathfrak{G}]$ iff $F\in[\mathfrak{J}]$ implies $F\in[\mathfrak{G}]$.

Let X and Y be sets and suppose $X\subset Y$. If $\mathfrak{J}\in\boldsymbol{B}(X)$, then $\mathfrak{J}\in\boldsymbol{B}(Y)$. In this case $[\mathfrak{J}]$ could refer to an element in $\boldsymbol{F}(X)$ or an element in $\boldsymbol{F}(Y)$. In those cases where confusion could arise a subscript will be used to indicate the precise meaning. Thus $[\mathfrak{J}]_Y$ refers to an element in $\boldsymbol{F}(Y)$; $[\mathfrak{J}]_X$ refers to an element in $\boldsymbol{F}(X)$.

Consider a non-empty subclass $\mathscr{W}$ of $\boldsymbol{F}(X)$ which satisfies: $[\mathfrak{F}] \in \mathscr{W}$ and $[\mathfrak{G}] \in \mathscr{W}$ implies there exists $[\mathfrak{B}] \in \mathscr{W}$ such that $[\mathfrak{B}] \leqslant [\mathfrak{F}] \cap [\mathfrak{G}]$.

One will recognize that $\mathscr{W}$ is a filter base, the elements of which are filters on X. $[\![\mathscr{W}]\!]$ shall denote the filter generated by $\mathscr{W}$, that is, $[\![\mathscr{W}]\!]$ is defined by: $[\mathfrak{F}] \in [\![\mathscr{W}]\!]$ iff there exists $[\mathfrak{B}] \in \mathscr{W}$ such that $[\mathfrak{B}] \leqslant [\mathfrak{F}]$.

In lattice theoretic language one will observe:

(a) the class $\boldsymbol{F}(X)$ partially ordered by $\leqslant$ is a lattice;

(b) if intersection is identified with the "meet" operation, then $[\![\cdot]\!]$ is a "meet ideal" in the lattice $(\boldsymbol{F}(X), \leqslant)$.

Definition 1. Let X be a set and let $F \subset X \times X$ and $G \subset X \times X$. Then

(a) $F \circ G = \{(x, y) \in X \times X$: there exists a $z \in X$ such that $(x, z) \in G$ and $(z, y) \in F\}$;

(b) $F^{-1} = \{(y, x) \in X \times X: (x, y) \in F\}$;

(c) $\Delta_X = \{(x, x) \in X \times X\}$.

If $[\Phi] \in \boldsymbol{F}(X \times X)$ and $[\psi] \in \boldsymbol{F}(X \times X)$, then

(d) $[\psi] \circ [\Phi] = [\psi \circ \Phi]$. $\{F \circ G: F \in \Phi, G \in \psi\}$ is a base for the filter $[\psi \circ \Phi]$ provided $F \circ G \neq \emptyset$ for all $F \in \Phi$ and $G \in \psi$, otherwise not defined.

(e) $[\Phi]^{-1} = [\{F^{-1}: F \in \Phi\}]$.

If $[\mathfrak{F}] \in \boldsymbol{F}(X)$ and $[\mathfrak{G}] \in \boldsymbol{F}(X)$, then

(f) $[\mathfrak{F}] \times [\mathfrak{G}] = [\mathfrak{F} \times \mathfrak{G}] = [\{F \times G: F \in \mathfrak{F}$ and $G \in \mathfrak{G}\}]$.

Definition 2. Let X be a set. Then a meet ideal $[\![\mathscr{W}]\!]$ in $\boldsymbol{F}(X \times X)$ that satisfies:

(i) $[\Delta_X] \in [\![\mathscr{W}]\!]$;

(ii) $[\Phi] \in [\![\mathscr{W}]\!]$ implies $[\Phi]^{-1} \in [\![\mathscr{W}]\!]$;

(iii) if $[\Phi] \in [\![\mathscr{W}]\!]$ and $[\psi] \in [\![\mathscr{W}]\!]$, then $[\psi \circ \Phi] \in [\![\mathscr{W}]\!]$ whenever $[\psi \circ \Phi]$ is defined;

is termed a *uniform convergence structure* on X.

We provide $\mathscr{C}$ with the topology of compact convergence and by $\mathscr{C}_1$ we denote the topological complex algebra the elements of which are of the form $a+f$, a a complex number, $f \in \mathscr{C}$.

$\mathscr{C}^* = \mathscr{C} \cup \boldsymbol{C} - \{0\}$ where $\boldsymbol{C}$ denotes the complex field. $\mathscr{C}^*$ is preordered by the relation $|$ defined by: $p|q$ iff there exists $k \in \mathscr{C}^*$ such that $kp = q$.

$\mathscr{C}_p$ is the image of $\mathscr{C}_1$ under the mapping $\varphi_p: \mathscr{C}_1 \to \mathscr{M}$, defined by:

$$\varphi_p(a+f) = \frac{a}{p} + \frac{f}{p}, \quad p \in \mathscr{C}^*.$$

If the product $x\varphi_p(y)$ is defined by $x\varphi_p(y) = \varphi_p(xy)$, then since φ_p is a $\boldsymbol{C}$-module homomorphism, $\mathscr{C}_p$ is a $\mathscr{C}_1$-module. $\mathscr{C}_p$, provided with the topology T_p induced by the mapping φ_p, is a topological $\mathscr{C}_1$-module. With respect to the topology T_p on $\mathscr{C}_p$, φ_p is a homeomorphism.

LEMMA 3. *If $p|q$, then $\mathscr{C}_p \subset \mathscr{C}_q$.*

THEOREM 4. *Let the relation $\tau_M: \boldsymbol{F}(\mathscr{M}) \to \mathscr{M}$ be defined by: $[\mathfrak{F}]\, \tau_M x$ iff there exists $p \in \mathscr{C}^*$ such that $x \in \mathscr{C}_p$, $\mathscr{C}_p \in [\mathfrak{F}]$, and $[\mathfrak{N}_p(x)] \leqslant [\mathfrak{F}]$ (where $\mathfrak{N}_p(x)$ is a base for the neighborhood filter of x in $\mathscr{C}_p$). Then $(\mathscr{M}, \tau_M)$ is a limit space.*

The proofs to Theorem 4 and Lemma 3 and the verification of the statements preceding Lemma 3 may be found in [4].

3. The uniform convergence structure of Mikusiński operators. The linear topological space $\mathscr{C}_p$ has a natural uniform structure defined by $[\mathscr{W}_p] = [\{W_U: U \in \mathfrak{N}_p(0)\}]$ where $W_U = \{(x, y) \in \mathscr{C}_p \times \mathscr{C}_p: x-y \in U\}$ and $\mathfrak{N}_p(0)$ is a fundamental neighborhood system of 0 in $\mathscr{C}_p$ ([1], *Espaces Vectoriels Topologique*, Chapter 1).

LEMMA 5. *For each $p \in \mathscr{C}^*$, $(\mathscr{C}_p, T_p)$ is a uniform space in which addition is uniformly continuous.*

LEMMA 6. *If $p \in \mathscr{C}^*$ and $q \in \mathscr{C}^*$ and $p|q$, then the injection map $j_{p,q}$: $\mathscr{C}_p \to \mathscr{C}_q$ is uniformly continuous.*

Proof. By Lemma 3, $\mathscr{C}_p \subset \mathscr{C}_q$. Let $W'_{U'} = \{(x, y) \in \mathscr{C}_q \times \mathscr{C}_q: x-y \in U'\}$. Let $j(U) \subset U'$, $U \in \mathfrak{N}_p(0)$. If $x-y \in U$, then $j(x-y) = j(x)-j(y) \in U'$.

THEOREM 7. *If $\mathscr{W}_M = \{\{[\mathscr{W}_p]: p \in \mathscr{C}^*\}, [\Delta_{\mathscr{M}}]\}$, then $[\![\mathscr{W}_M]\!]$ is a uniform convergence structure on $\mathscr{M}$.*

Proof. (i) Let $[\mathscr{W}_p]$ and $[\mathscr{W}_q]$ be filters in $\mathscr{W}_M$. There exists $r \in \mathscr{C}^*$ such that $p|r$ and $q|r$. By Lemma 6, $[\mathscr{W}_r] \leqslant [\mathscr{W}_p]$ and $[\mathscr{W}_r] \leqslant [\mathscr{W}_q]$, thus

$$[\mathscr{W}_r] \leqslant [\mathscr{W}_p] \cap [\mathscr{W}_q]$$

and $\mathscr{W}_M$ generates a meet ideal in $\boldsymbol{F}(\mathscr{M} \times \mathscr{M})$.

(ii) $[\Delta_{\mathscr{M}}] \in [\![\mathscr{W}_M]\!]$ by construction.

(iii) Let $[\Phi] \in [\![\mathscr{W}_M]\!]$. Then either (a) there exists $p \in \mathscr{C}^*$ such that $[\mathscr{W}_p] \leqslant [\Phi]$ or (b) $[\Delta_{\mathscr{M}}] \leqslant [\Phi]$. If $[\mathscr{W}_p] \leqslant [\Phi]$, then since $[\mathscr{W}_p]$ is a uniform structure,

$$[\mathscr{W}_p] = [\mathscr{W}_p]^{-1} \leqslant [\Phi]^{-1},$$

hence $[\Phi]^{-1} \in [\![\mathscr{W}_M]\!]$. If $[\Delta_{\mathscr{M}}] \leqslant [\Phi]$, then $[\Delta_{\mathscr{M}}] = [\Delta_{\mathscr{M}}]^{-1} \leqslant [\Phi]^{-1}$ and $[\Phi]^{-1} \in [\![\mathscr{W}_M]\!]$.

(iv) We note $[\Delta] \circ [\Delta] = [\Delta]$ and $[\Delta] \circ [\mathscr{W}_p] = [\mathscr{W}_p] \circ [\Delta] = [\mathscr{W}_p] = [\mathscr{W}_p] \circ [\mathscr{W}_p]$ for any $p \in \mathscr{C}^*$. Further, if $p \in \mathscr{C}^*$ and $q \in \mathscr{C}^*$, there exists $r \in \mathscr{C}^*$ such that $[\mathscr{W}_r] \leqslant [\mathscr{W}_p]$ and $[\mathscr{W}_r] \leqslant [\mathscr{W}_q]$. Thus for $[\Phi] \in [\![\mathscr{W}_M]\!]$, $[\psi] \in [\![\mathscr{W}_M]\!]$ if $[\Phi \circ \psi]$ is defined, it follows from the above remarks that $[\Phi \circ \psi] \in [\![\mathscr{W}_M]\!]$.

THEOREM 8. *Let X be an arbitrary set and let $[\![\mathscr{W}]\!]$ be a uniform convergence structure on X. Define $\tau_{\mathscr{W}}\colon \boldsymbol{F}(X)\to X$ by: $[\mathfrak{F}]\tau_{\mathscr{W}}x$ iff $[\{x\}]\times[\mathfrak{F}]\in[\![\mathscr{W}]\!]$. Then $\tau_{\mathscr{W}}$ is a Limitierung on X and is referred to as the Limitierung induced by $[\![\mathscr{W}]\!]$.*

A proof of this theorem may be found in [2].

LEMMA 9. *Let X be an arbitrary set, let $[\Phi]\in\boldsymbol{F}(X\times X)$, and let $[\mathfrak{F}]\in\boldsymbol{F}(X)$. Then $[\Phi]\leqslant[\{x\}]\times[\mathfrak{F}]$ iff $[\Phi(x)]\in\boldsymbol{F}(X)$ and $[\Phi(x)]\leqslant[\mathfrak{F}]$ where $\Phi(x)=\{V(x)\colon V\in\Phi\}$ and $V(x)=\{y\in X\colon (x,y)\in V\}$.*

Proof. First note $[\Phi(x)]\in\boldsymbol{F}(X)$ iff $V(x)\neq\emptyset$ for all $V\in\Phi$. If $[\Phi(x)]\in\boldsymbol{F}(X)$ and $[\Phi(x)]\leqslant[\mathfrak{F}]$, then for each $V(x)\in\Phi(x)$ there exists $F\in\mathfrak{F}$ such that $F\subset V(x)$. Thus $\{x\}\times F\subset\{x\}\times V(x)\subset V$. Hence $[\Phi]\leqslant[\{x\}]\times[\mathfrak{F}]$. Conversely, if $[\Phi]\leqslant[\{x\}]\times[\mathfrak{F}]$ and $V\in\Phi$, then for some $F\in\mathfrak{F}$, $\{x\}\times F\subset V$. But then $F\subset V(x)$, hence $[\Phi(x)]\leqslant[\mathfrak{F}]$.

Lemma 9 provides an equivalent definition of the Limitierung $\tau_{\mathscr{W}}$ induced by the uniform convergence structure $\mathscr{W}$: $[\mathfrak{F}]\ \tau_{\mathscr{W}}x$ iff for some $[\Phi]\in\mathscr{W}$, $[\Phi(x)]\in\boldsymbol{F}(X)$ and $[\Phi(x)]\leqslant[\mathfrak{F}]$.

Definition 10. A limit space (X,τ) is *uniformizable* iff there exists a uniform convergence structure $[\![\mathscr{W}]\!]$ on X such that $\tau=\tau_{\mathscr{W}}$.

THEOREM 11. *τ_M is the Limitierung induced by the uniform convergence structure $[\![\mathscr{W}_M]\!]$.*

Proof. $[\mathfrak{F}]\tau_M x$ iff there exists $p\in\mathscr{C}^*$ such that $x\in\mathscr{C}_p$, $\mathscr{C}_p\in[\mathfrak{F}]$, and $[\mathfrak{N}_p(x)]\leqslant[\mathfrak{F}]$. Let $W_U\in\mathscr{W}_p$; then there exists $U_p(x)\in\mathfrak{N}_p(x)$ such that $U_p(x)\subset W_U(x)$. Since $[\mathfrak{N}_p(x)]\leqslant[\mathfrak{F}]$, there exists $F\in\mathfrak{F}$, $F\subset\mathscr{C}_p$ such that $F\subset U_p(x)\subset W_U(x)$. Thus $[\mathscr{W}_p]\leqslant[\{x\}]\times[\mathfrak{F}]$ and $[\mathfrak{F}]\tau_{\mathscr{W}}x$.

Conversely, suppose $[\mathfrak{F}]\tau_{\mathscr{W}}x$. Then either (a) there exists $p\in\mathscr{C}^*$ such that $[\mathscr{W}_p(x)]\leqslant[\mathfrak{F}]$ or (b) $[\{x\}]\leqslant[\mathfrak{F}]$. If (a) and if $U_p(x)\in\mathfrak{N}_p(x)$, there exists $W_U(x)\in\mathfrak{N}_p(x)$ such that $W_U(x)\subset U_p(x)$. Since $[\mathscr{W}_p(x)]\leqslant[\mathfrak{F}]$, there exists $F\in\mathfrak{F}$, $F\subset\mathscr{C}_p$, such that $F\subset W_U(x)$. Hence $[\mathfrak{N}_p(x)]\leqslant[\mathfrak{F}]$ and $[\mathfrak{F}]\tau_M x$. If $[\{x\}]\leqslant[\mathfrak{F}]$, then $[\{x\}]=[\mathfrak{F}]$ and $[\mathfrak{F}]\tau_M x$.

Definition 12. Let $(X,[\![\mathscr{W}_1]\!])$ and $(Y,[\![\mathscr{W}_2]\!])$ be uniform convergence spaces and let f be a mapping: $f\colon (X,[\![\mathscr{W}_1]\!])\to(Y,[\![\mathscr{W}_2]\!])$. f is *uniformly continuous* on X if and only if $[(f\times f)(\Phi)]\in[\![\mathscr{W}_2]\!]$ for each $[\Phi]\in\mathscr{W}_1$.

Definition 13. Given two uniform convergence structures $[\![\mathscr{W}_1]\!]$ and $[\![\mathscr{W}_2]\!]$ on a set X, $[\![\mathscr{W}_2]\!]$ is *finer* than $[\![\mathscr{W}_1]\!]$ if and only if $[\Phi]\in[\![\mathscr{W}_2]\!]$ implies $[\Phi]\in[\![\mathscr{W}_1]\!]$.

$[\![\mathscr{W}_2]\!]$ is finer than $[\![\mathscr{W}_1]\!]$ if and only if the identity map $j\colon (X,[\![\mathscr{W}_2]\!])\to(X,[\![\mathscr{W}_1]\!])$ is uniformly continuous.

THEOREM 14. *Let $\mathscr{A}=\{(E_\lambda,[\![\mathscr{W}_\lambda]\!])\colon \lambda\in\Lambda,\ \Lambda$ an arbitrary index set$\}$ be a collection of uniform convergence spaces. Further suppose that Λ is directed by the pre-order $\leqslant$ which satisfies: If $\lambda\leqslant\mu$, then $E_\lambda\subset E_\mu$ and the injection map $j_{\lambda,\mu}\colon E_\lambda\to E_\mu$ is uniformly continuous. If $E=\bigcup_{\lambda\in\Lambda}E_\lambda$ and if $[\![\mathscr{W}]\!]$ is the meet ideal generated by $[\Delta_E]$ and the filters $[\Phi]_{(E\times E)}$, $[\Phi]\in\mathscr{W}_\lambda$ for some $\lambda\in\Lambda$, then*

(1) $[\![\mathscr{W}]\!]$ is the (unique) finest uniform convergence structure on E such that for each $\lambda\in\Lambda$, $j_\lambda\colon E_\lambda\to E$ is uniformly continuous;

(2) If $(A,[\![\mathscr{I}]\!])$ is a uniform convergence space and f is a mapping from E to A, then $f\colon E\to A$ is uniformly continuous iff $fj_\lambda\colon E_\lambda\to A$ is uniformly continuous for each λ.

Proof. 1° (a) $[\Delta_E]\in\mathscr{W}$ by construction.

(b) If $[\Phi]_{E\times E}\in\mathscr{W}$, then there exists $\lambda\in\Lambda$ such that $[\Phi]\in[\![\mathscr{W}_\lambda]\!]$. $[\Phi]^{-1}\in[\![\mathscr{W}_\lambda]\!]$, hence $[\Phi]^{-1}_{E\times E}\in\mathscr{W}$.

(c) If $[\Phi]_{E\times E}\in\mathscr{W}$ and $[\Psi]_{E\times E}\in\mathscr{W}$, then there exist $\lambda\in\Lambda$, $\mu\in\Lambda$ such that $[\Phi]\in[\![\mathscr{W}_\lambda]\!]$ and $[\Psi]\in[\![\mathscr{W}_\mu]\!]$. But there exists $\omega\in\Lambda$ such that $E_\lambda\cup E_\mu\subset E_\omega$ and the injection maps $j_{\lambda,\omega}\colon (E_\lambda,[\![\mathscr{W}_\lambda]\!])\to(E_\omega,[\![\mathscr{W}_\omega]\!])$ and $j_{\mu,\omega}\colon (E_\mu,[\![\mathscr{W}_\mu]\!])\to(E_\omega,[\![\mathscr{W}_\omega]\!])$ are uniformly continuous. Hence

$$[(j_{\lambda,\omega}\times j_{\lambda,\omega})([\Phi])]_{E_\omega\times E_\omega}=[\Phi]_{E_\omega\times E_\omega}\in[\![\mathscr{W}_\omega]\!];$$

similarly

$$[(j_{\mu,\omega}\times j_{\mu,\omega})([\Psi])]_{E_\omega\times E_\omega}=[\Psi]_{E_\omega\times E_\omega}\in[\![\mathscr{W}_\omega]\!].$$

Thus $\big[[\Phi]_{E_\omega\times E_\omega}\circ[\Psi]_{E_\omega\times E_\omega}\big]_{E_\omega\times E_\omega}\in[\![\mathscr{W}_\omega]\!]$ whenever $[\Phi]_{E_\omega\times E_\omega}\circ[\Psi]_{E_\omega\times E_\omega}$ is defined. If $[\Phi]_{E\times E}\circ[\Psi]_{E\times E}$ is defined, then $[\Phi]_{E_\omega\times E_\omega}\circ[\Psi]_{E_\omega\times E_\omega}$ is defined and $\big[[\Phi]_{E\times E}\circ[\Psi]_{E\times E}\big]_{E\times E}\in[\![\mathscr{W}]\!]$.

For each $\lambda\in\Lambda$ if $[\Phi]\in[\![\mathscr{W}_\lambda]\!]$, then $[(j_\lambda\times j_\lambda)([\Phi])_{E\times E}=[\Phi]_{E\times E}\in\mathscr{W}$; hence the injection map $j_\lambda\colon (E_\lambda,[\![\mathscr{W}_\lambda]\!])\to(E,[\![\mathscr{W}]\!])$ is uniformly continuous.

Finally, suppose $\mathscr{W}'$ is another uniform convergence structure on X such that the injection map $j_\lambda\colon (E_\lambda,[\![\mathscr{W}_\lambda]\!])\to(E,[\![\mathscr{W}]\!])$ is uniformly continuous for each $\lambda\in\Lambda$. Then if $[\Phi]\in\mathscr{W}$ and $[\Phi]\neq[\Delta]$, there exists $\lambda\in\Lambda$, such that $[\Phi]\in[\![\mathscr{W}_\lambda]\!]$. $\big[(j_\lambda\times j_\lambda)[\Phi]\big]_{E\times E}=[\Phi]_{E\times E}\in[\![\mathscr{W}']\!]$, thus $[\![\mathscr{W}']\!]\leqslant[\![\mathscr{W}]\!]$.

2° Since the composition of uniformly continuous functions is a uniformly continuous function [2], it follows that $fj_\lambda\colon E_\lambda\to A$ is uniformly continuous for each $\lambda\in\Lambda$ if $f\colon E\to A$ is uniformly continuous.

Conversely, if $fj_\lambda\colon E_\lambda\to A$ is uniformly continuous for each $\lambda\in\Lambda$, then if $[\Phi]_{E\times E}\in\mathscr{W}$, there exists $\lambda\in\Lambda$ such that we have $[\Phi]\in[\![\mathscr{W}_\lambda]\!]$. But $\big[(f\times f)([\Phi])\big]_{A\times A}=\big[(fj_\lambda\times fj_\lambda)([\Phi]_{E_\lambda\times E_\lambda})\big]_{A\times A}\in[\![\mathscr{I}]\!]$.

The uniform convergence space $(E,[\![\mathscr{W}]\!])$ is the *direct limit* of the family $\mathscr{A}$ of uniform convergence spaces.

LEMMA 15. *A filter $[\mathscr{W}]$ on $X\times X$ is a uniform structure on X if and only if the meet ideal $[\![[\mathscr{W}]]\!]$ generated by $[\mathscr{W}]$ is a uniform convergence structure on X [2].*

THEOREM 16. *The uniform convergence structure $[\![\mathscr{W}_M]\!]$ on $\mathscr{M}$ is the direct limit of the uniform convergence structures $[\![[\mathscr{W}_p]]\!]$ on the spaces $\mathscr{C}_p$.*

Proof. By definition of $\mathscr{W}_M$, $[\Phi]\in\mathscr{W}_M$ if either $\Phi = [\Delta_{\mathscr{M}}]$ or $[\Phi] = [\mathscr{W}_p]_{\mathscr{M}\times\mathscr{M}} = [(j_p\times j_p)([\![\mathscr{W}_p]\!])]_{\mathscr{M}\times\mathscr{M}}$ for some $p\in\mathscr{C}^*$. Thus $[\![\mathscr{W}_M]\!]$ is the direct limit structure of Theorem 14.

4. Continuity. Mikusiński in [3], Part III, Chapter I, defines an operator function as a mapping from the real numbers to the field of operators. In this same section he defines a notion of continuity for operator functions as follows: An operator function g defined on an interval I of the real line is M-continuous if and only if for each finite closed subinterval $J\subset I$ there exists a $p_J\in\mathscr{C}^*$ such that fp_J is a continuous function of x and t in $J\times R^+$, where $\boldsymbol{R}^+$ denotes the non-negative real numbers.

Definition 17. A mapping $f: X\to\mathscr{M}$, X an arbitrary space, $\mathscr{M}$ the operator field, will be called an *operator function*.

Definition 18. Let X be a locally compact Hausdorff (separated) topological space. $f: X\to\mathscr{M}$ is *M-continuous* if for any compact set $K\subset X$ there exists $p\in\mathscr{C}^*$, depending in general on K, such that

(i) $f(K)\subset\mathscr{C}_p$ and

(ii) the induced map $\mathscr{C}_p|f: K\to\mathscr{C}_p$ is continuous.

For the special case $X = I$, I an interval of the real line, Definition 18 is equivalent to the definition given by Mikusiński.

Let X be a locally compact Hausdorff space and let $\mathscr{K} = \{K\subset X: K$ is compact$\}$.

LEMMA 19. *If K is a compact Hausdorff space, then:*

(a) *K has a unique uniform structure $[\mathfrak{A}_K]$ compatible with the topology on K where $[\mathfrak{A}_K]$ is the filter of neighborhoods of Δ_K.*

(b) *If f is a continuous function from the compact space K to the uniform space Y, then f is uniformly continuous.*

For a proof of (a) and (b), the reader is referred to [1] (*Topologie Générale*, Chapter II, Section 4).

LEMMA 20. *If $K\in\mathscr{K}$, $K'\in\mathscr{K}$ and $K\subset K'$, then the injection map $j_{K,K'}: K\to K'$ is uniformly continuous.*

Proof. $j_{K,K'}$ is continuous and thus, by Lemma 19, $j_{K,K'}$ is uniformly continuous.

THEOREM 21. *$X = \bigcup_{K\in\mathscr{K}} K$ and if $[\![\mathscr{V}]\!]$ is the meet ideal generated by $[\Delta_K]$ and the filters $\{[\mathfrak{A}_K]: K\in\mathscr{K}\}$, then $(X, [\![\mathscr{V}]\!])$ is the direct limit of $\{(K, [\mathfrak{A}_K]: K\in\mathscr{K}\}$.*

The proof parallels that of Theorem 14.

THEOREM 22. *$f: X\to\mathscr{M}$ is M-continuous iff f is uniformly continuous from $(X, [\![\mathscr{V}]\!])$ to $(\mathscr{M}, [\![\mathscr{W}_M]\!])$.*

Proof. Let $[\Phi]\in[\![\mathscr{V}]\!]$; then if $[\Delta_x]\leqslant[\Phi]$, then $[(f\times f)(\Phi)]\geqslant[(f\times f)(\Delta_x)]\geqslant[\Delta_{\mathscr{M}}]$. Suppose there exists $K\in\mathscr{K}$ such that $[\mathfrak{A}_K]\leqslant[\Phi]$. Since f is M-continuous, there exists $p\in\mathscr{C}^*$ such that $f(K)\subset\mathscr{C}_p$ and $\mathscr{C}_p|f: K\to\mathscr{C}_p$ is continuous. Thus $(f\times f)(\mathfrak{A}_K)\subset\mathscr{C}_p\times\mathscr{C}_p$ and the continuity of $\mathscr{C}_p|f: K\to\mathscr{C}_p$ implies uniform continuity by Lemma 19, hence $[(f\times f)(\mathfrak{A}_K)]\geqslant[\mathscr{W}_p]$. This shows that $f: X\to\mathscr{M}$ is uniformly continuous.

If $f: (X, [\![\mathscr{V}]\!])\to(\mathscr{M}, [\![\mathscr{W}_M]\!])$ is uniformly continuous, then for each $K\in\mathscr{K}$, $[(f\times f)(\mathfrak{A}_K)]\in[\![\mathscr{W}_M]\!]$. Hence either there exists $p\in\mathscr{C}^*$ such that $[\mathscr{W}_p]_{\mathscr{C}_p\times\mathscr{C}_p}\leqslant[(f\times f)(\mathfrak{A}_K)]_{\mathscr{C}_p\times\mathscr{C}_p}$ or $[\Delta_{\mathscr{M}}]\leqslant[(f\times f)(\mathfrak{A}_K)]$. In the first case $f(K)\subset\mathscr{C}_p$ and $\mathscr{C}_p|f: K\to\mathscr{C}_p$ is (uniformly) continuous. If $[\Delta_{\mathscr{M}}]\leqslant[(f\times f)(\mathfrak{A}_K)]$, then there exists $U_K\in\mathfrak{A}_K$ such that $(f\times f)(\mathfrak{A}_K)\subset\Delta_{\mathscr{M}}$. For $(x, y)\in U_K$, $\big(f(x), f(y)\big)\in\Delta_{\mathscr{M}}$ hence $f(x) = f(y)$. By definition of $\mathfrak{A}_K$, $\mathfrak{A}_K(x)$ is an open neighborhood of x in K. Since K is compact, if $\mathscr{B} = \bigcup_{x\in K} U_K(x)$ is an open cover of K, there exists a finite subcover $\mathscr{B}' = \bigcup_{x\in F} U_K(x)$, F a finite set. f constant on U_K implies $f\big(U_K(x)\big) = z/p_x$, $z\in\mathscr{C}_1$, $p_x\in\mathscr{C}^*$, $x\in F$. Thus $\prod_{x\in F} p_x = p\in\mathscr{C}^*$ and $f(K)\subset\mathscr{C}_p$. If $A\in T_p$, then $f^{-1}(A) = \bigcup_{x\in F'} U_K(x)$, $F'\subset F$, which is an open set in K. Thus f is M-continuous.

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