Par suite, pour x assez grand

$$\operatorname{Re}\frac{\int_{\omega}^{\lambda} U_n(\mu, x)\,d\mu}{V_n(x)} = \operatorname{Re}\frac{B\Gamma(\gamma+1)}{A\Gamma(\delta+1)}\int_{\omega}^{\lambda}\varepsilon(\mu, x)\,d\mu\, x^{\delta-\gamma}$$

où $\varepsilon(\mu, x)\to 1$ uniformément quand $x\to\infty$, $\alpha\leqslant\mu\leqslant\beta$.

Supposons que $\operatorname{Re}(B/A)\geqslant 0$; alors les conditions du théorème 2 sont remplies.

Si, au contraire, $\operatorname{Re}(B/A)<0$, les conditions du théorème 1 sont satisfaites. La suite (z_m) doit rester sur la droite $\arg z=\theta$, $\cos(\delta-\gamma)\theta>0$.

Ces trois théorèmes ne peuvent pas épuiser toutes les possibilités que nous donne la théorie bien connue de l'intégrale de Laplace. Ils peuvent eux-mêmes aussi être modifiés.

Maintenant nous allons montrer que le théorème 3 a pour conséquence immédiate que la fonction exponentielle $\exp(-\lambda e^{\theta i}s^{\omega})$, $\omega>1$, $\lambda>0$, $0\leqslant\theta<2\pi$, n'existe pas.

Pour les fonctions $u(\lambda)$ et v de l'équation (1) on prend $u(\lambda)=l^2e^{\theta i}$, $v=l^{\omega+2}$, d'où $u(\lambda)/v=s^{\omega}e^{\theta i}$ et l'équation différentielle correspondant à (1) est $x'(\lambda)+s^{\omega}e^{\theta i}x(\lambda)=0$. Comme d'après le théorème 3 cette équation n'a pas de solutions, la fonction exponentielle $\exp(-\lambda e^{\theta i}s^{\omega})$ n'existe pas.

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On generalized topological divisors of zero in real m-convex algebras

by

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In a previous note [3] it was shown that a complex locally m-convex algebra either has generalized topological divisors of zero or it is homeomorphically isomorphic with the field of complex numbers. Here we extent this result onto real m-convex algebras: such an algebra either has generalized topological divisors of zero or it is isomorphically homeomorphic with one of the three finite-dimensional division algebras over real numbers (i.e. field of real numbers, field of complex numbers or division algebra of quaternions). The proof will be obtained by suitable modification of the proof given in [3], the "trick" lies here in considering complex-valued functionals in algebras over the field of real numbers. As before, we may limit ourselves to complete commutative algebras. In fact, it is sufficient to construct such divisors in any commutative m-convex algebra not being a field, since, if any commutative subalgebra of a real algebra A is a field, then A is a division algebra.

We assume here the same notation as in [3], moreover R will denote the field of real numbers and C — the field of complex numbers.

1. The complexification. Let A be an m-convex algebra over R. The *complexification* $\tilde{A}$ of A is defined as direct product $A\oplus A$ equipped with multiplication defined in the some way as multiplication of complex numbers defined as pairs of real numbers, i.e.

$$(x, y)(u, v) = (xu-yv, xv+yu),\qquad x, y, u, v\in A.$$

$\tilde{A}$ becomes an algebra over C with scalar multiplication defined as

$$(\alpha+\beta i)(x, y) = (\alpha x-\beta y, \alpha y+\beta x).$$

If A is an m-convex algebra with a system of submultiplicative pseudonorms $\boldsymbol{P}$, then $\tilde{A}$ is an m-convex algebra with submultiplicative pseudonorms given by

$$(1)\qquad \|(x, y)\| = \sup_{\theta}|e^{i\theta}(x, y)|,$$

where $|(x, y)| = \|x\| + \|y\|$, $\|\cdot\| \in \boldsymbol{P}$. Moreover $\tilde{A}$ is complete if A is complete (cf. [1], Theorem (1.3.1)). The system of pseudonorms in $\tilde{A}$, given by formula (1), will be also denoted by $\boldsymbol{P}$. The set $A' = \{(x, 0) \in \tilde{A} : x \in A\}$ is a real subalgebra of $\tilde{A}$ isomorphically homeomorphic with A since

$$\|x\| \leqslant \|(x, 0)\| \leqslant \sqrt{2}\|x\|, \qquad \|\cdot\| \in \boldsymbol{P}.$$

If A has a unit e, then $(e, 0)$ is the unit element of $\tilde{A}$.

The following remarks will be useful in the sequel.

Remark 1. Let f be a continuous (complex-valued) multiplicative linear functional defined on A; then its restriction f^0 to A' may be considered, by the isomorphism between A' and A, as a complex-valued multiplicative and linear functional in A, i.e.

$$f^0(xy) = f^0(x)f^0(y), \qquad x, y \in A, \tag{2}$$

$$f^0(ax+\beta y) = af^0(x)+\beta f^0(y), \qquad x, y \in A;\ a, \beta \in R. \tag{3}$$

On the other hand, if f is a functional defined on A and satisfying (2) and (3), then the functional defined on $\tilde{A}$ by the formula

$$f^{\sim}[(x, y)] = f(x)+if(y) \tag{4}$$

is a multiplicative and linear functional defined on $\tilde{A}$. Obviously, $f^{\sim 0} = f^{0\sim}$, so there is a one-one correspondence between multiplicative and linear functionals of $\tilde{A}$ and functionals satisfying (2) and (3) defined on A. These functional will also be called multiplicative and linear. Note also that if f is a multiplicative and linear functional defined on A, then its complex conjugate is also such a functional.

Remark 2. Let A be a complete commutative real m-convex algebra with unit e; then an element $x \in A$ is invertible in A if and only if $f(x) \neq 0$ for each continuous multiplicative and linear functional f defined on A. This follows from the fact that an element $(x, 0) \in A'$ is invertible in $\tilde{A}$ if and only if x is invertible in A, from the Remark 1, and from corresponding fact on complex complete m-convex algebras.

Remark 3. If A is a real m-convex algebra with unit e, and W_n is a sequence of polynomials with real coefficients of one complex variable, which tends uniformly to zero on each compact subset of C, then $\|W_n(x)\| \to 0$ for each $x \in A$ and $\|\cdot\| \in \boldsymbol{P}$. In fact, $\|W_n[(x, y)]\| \to 0$ for each $(x, y) \in \tilde{A}$ and $\|\cdot\| \in \boldsymbol{P}$, so the conclusion holds for A' and therefore for A.

Remark 4. $\tilde{A}$ is semisimple if and only if $\lim \sqrt[n]{\|x^n\|} = 0$ for each $\|\cdot\| \in \boldsymbol{P}$ implies $x = 0$, $x \in A$, where A is a commutative complete real m-convex algebra. This condition implies therefore that in A there is a total family of continuous multiplicative and linear functionals.

2. Proof of the theorem

Lemma 1. *Suppose that A is a commutative complete real m-convex algebra with unit e and that its complexification $\tilde{A}$ is a semisimple algebra. Then either A has generalized topological divisors of zero or A is isomorphically homeomorphic with one of the three finite-dimensional division algebras over real numbers.*

Proof. If A were finite-dimensional, then it would be a Banach algebra and the conclusion of Lemma 1 holds for example by [2]. Suppose that A is an infinite-dimensional algebra. By Remark 4 it follows that in A there is defined an infinite family of (non-zero) multiplicative and linear continuous functionals. Take two of them, say f_1 and f_2, one being not the complex conjugate of the other. There exists an element $x \in A$ such that $f_1(x) \neq f_2(x) \neq \overline{f_1(x)}$. The polynomial $W_1(\lambda) = (\lambda - f_1(x))(\lambda - \overline{f_1(x)})$ has real coefficients, and $\zeta = W_1(f_2(x)) \neq 0$. Taking $V(\lambda) = (W_1(\lambda)-\zeta) \times \times (W_1(\lambda)-\bar{\zeta})$, we obtain again a polynomial with real coefficients. Setting $y = V(x)$, we get $y \in A$, $f_1(y) = V(f_1(x)) = |\zeta|^2 \neq 0$ and $f_2(y) = V(f_2(x)) = 0$. So, setting $z = (2y - |\zeta|^2 e)/|\zeta|^2$ we get $z \in A$ and $f_1(z) = 1$, $f_2(z) = -1$. Taking now polynomials φ_n and ψ_n of Lemma 1 of [3] and observing that φ_n and ψ_n have real coefficients we see that $x_n = \varphi_n(z)$ and $y_n = \psi_n(z)$ belong to A, $\lim \|x_n y_n\| = 0$ for each $\|\cdot\| \in \boldsymbol{P}$, and $f_1(x_n) = f_2(y_n) = 1$, so (x_n) and (y_n) form a pair of generalized topological divisors of zero in A.

Lemma 2. *Suppose that A is a commutative complete real m-convex algebra with unit e. If there exists in A a non-zero element x such that $\lim \sqrt[n]{\|x^n\|} = 0$ for each $\|\cdot\| \in \boldsymbol{P}$, then A has generalized topological divisors of zero.*

Proof. Take the subalgebra A_0 of A defined as the smallest complete subalgebra with unit of A containing x. By Remark 2 the argument of Lemma 3 of [3] works as well in this case, so there are in A_0 generalized topological divisors of zero.

Combining these two lemmas with the fact that any m-convex algebra is a dense subalgebra of a complete algebra and that any dense subalgebra of an algebra with generalized topological divisors of zero also has such divisors, and also with the fact that any m-convex division algebra over reals is isomorphically homeomorphic with one of the three standard finite-dimensional division algebras over reals and that any m-convex algebra which is not a division algebra contains a commutative subalgebra which is not a field, we get the following

Proposition. *In any real m-convex algebra A with unit either there are generalized topological divisors of zero or A is isomorphically homeomorphic with one of the three standard finite-dimensional real division algebras.*

To obtain our theorem let us remark that in paper [3], when deriving the general result from the result on algebras with unit, we never used the fact that the field of scalars is the field of complex numbers. So these arguments work as well in the case of real scalars and we can formulate our main result:

THEOREM. *Let A be a real m-convex algebra. Then either A has generalized topological divisors of zero or A is isomorphically homeomorphic with one of the three finite-dimensional real division algebras (i.e. field of real numbers, field of complex numbers or division algebra of quaternions).*

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Continuity of operator functions

by

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1. Introduction. By $\mathscr{C}$ we shall denote the complex algebra, the elements of which are continuous complex-valued functions of a non-negative real variable. The operation of multiplication is defined by finite convolution; the operations of addition and scalar multiplication are defined in the usual way. $\mathscr{C}$ has no zero divisors, hence the quotient field may be constructed. This quotient field, which we denote by $\mathscr{M}$, is termed the *field of operators.*

It is the purpose of this paper to extend the definitions of operator function and continuous operator function as defined by Mikusiński in [3] (Part III, Chap. 1). A uniform convergence structure [2] is defined on $\mathscr{M}$ and is shown to be the direct limit of uniform structures of linear subspaces of $\mathscr{M}$. The Limitierung defined by M-convergence [4] is the Limitierung induced by the uniform convergence structure of $\mathscr{M}$. A uniform convergence structure is defined for a locally compact Hausdorff space as the direct limit of uniform structures of compact subspaces. It is shown that an operator function is continuous, in the generalized sense, if and only if it is uniformly continuous from a locally compact Hausdorff space provided with the "compact" uniform convergence structure to the uniform convergence space $\mathscr{M}$.

2. Preliminaries. Let X be an arbitrary set. $\boldsymbol{B}(X)$ shall denote the family of filter bases on the set X. If $\mathfrak{J}\in\boldsymbol{B}(X)$, then the filter generated by $\mathfrak{J}$ is denoted by $[\mathfrak{J}]$. $\boldsymbol{F}(X)$ shall denote the family of filters on the set X. The class $\boldsymbol{F}(X)$ is partially ordered by the relation $\leqslant$ defined by: $[\mathfrak{J}]\leqslant[\mathfrak{G}]$ iff for each $F\in\mathfrak{J}$ there exists a $G\in\mathfrak{G}$ such that $G\subset F$. This is equivalent to: $[\mathfrak{J}]\leqslant[\mathfrak{G}]$ iff $F\in[\mathfrak{J}]$ implies $F\in[\mathfrak{G}]$.

Let X and Y be sets and suppose $X\subset Y$. If $\mathfrak{J}\in\boldsymbol{B}(X)$, then $\mathfrak{J}\in\boldsymbol{B}(Y)$. In this case $[\mathfrak{J}]$ could refer to an element in $\boldsymbol{F}(X)$ or an element in $\boldsymbol{F}(Y)$. In those cases where confusion could arise a subscript will be used to indicate the precise meaning. Thus $[\mathfrak{J}]_Y$ refers to an element in $\boldsymbol{F}(Y)$; $[\mathfrak{J}]_X$ refers to an element in $\boldsymbol{F}(X)$.