Par suite, pour \( \alpha \) assez grand
\[
\frac{\int_\gamma U_\alpha(\mu, \omega) \, d\mu}{V_\alpha(\omega)} = \frac{\Re}{\Re} \frac{B\Gamma(\gamma+1)}{\Gamma(\delta+1)} \int_{\omega} \omega(\mu, \omega) \, d\mu = \omega^{\delta-\gamma}
\]
où \( \omega(\mu, \omega) \to 1 \) uniformément quand \( \omega \to \infty \), \( \alpha \leq \mu \leq \beta \).
Supposons que \( \Re (B/A) > 0 \); alors les conditions du théorème 2 sont remplies.
Si, au contraire, \( \Re (B/A) < 0 \), les conditions du théorème 1 sont satisfaites. La suite \( (\omega_0) \) doit rester sur la droite arg \( \omega = \delta \), cos(\( \delta - \gamma \)) \( \theta > 0 \).
Ces trois théorèmes ne peuvent pas être tous les trois satisfaites que nous donnent la théorie bien connue de l'intégrale de Laplace. Ils peuvent eux-mêmes aussi être modifiés.
Maintenant nous allons montrer que le théorème 3 a pour conséquence immédiate que la fonction exponentielle \( \exp(-\lambda \omega^{\delta}) \), \( \omega > 1 \), \( \lambda > 0 \), \( 0 < \theta < 2\pi \), n'existe pas.
Pour les fonctions \( u(\lambda) \) et \( v \) de l'équation (1) on prend \( u(\lambda) = F\lambda^\alpha \),
\( v = \omega^{\delta}, \) d'où, \( u(\lambda)/v = \omega^{\delta-\lambda} \) et l'équation différentielle correspondant à (1) est \( \omega^{\lambda} + \omega^{\delta-\gamma} \omega^{\lambda} = 0 \). Comme d'après le théorème 3 cette équation ne possède pas de solutions, la fonction exponentielle \( \exp(-\lambda \omega^{\delta}) \) n'existe pas.

Traité utilisé


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where \((x, y) = \|x\| + \|y\|\), \(\|x||P\). Moreover \(\hat{A}\) is complete if \(A\) is complete (cf. [1], Theorem (3.3.1)). The system of pseudonorms in \(\hat{A}\), given by formula (1), will be also denoted by \(P\). The set \(A' = \{(x, 0) \in \hat{A} : x \in A\}\) is a real subalgebra of \(\hat{A}\) isomorphically homeomorphic with \(A\) since

\[
\|x\| < \|(x, 0)\| < \sqrt{2}\|x\|, \quad \|x||P.
\]

If \(A\) has a unit \(e\), then \((e, 0)\) is the unit element of \(\hat{A}\).

The following remarks will be useful in the sequel.

**Remark 1.** Let \(f\) be a continuous (complex-valued) multiplicative linear functional defined on \(A\); then its restriction \(f^*\) to \(A'\) may be considered, by the isomorphism between \(A'\) and \(A\), as a complex-valued multiplicative and linear functional in \(A\), i.e.

\[
\begin{align*}
(2) & \quad f(ax + by) = af(x) + bf(y), \quad a, y \in A; \\
(3) & \quad f(ax) = af(x), \quad a, x \in A, \quad a \in \mathbb{R}.
\end{align*}
\]

On the other hand, if \(f\) is a functional defined on \(A\) and satisfying (2) and (3), then the functional defined on \(A\) by the formula

\[
f^*([x, y]) = f(x) + if(y)
\]

is a multiplicative and linear functional defined on \(A\). Obviously, \(f^* = f^{**}\), so there is a one-one correspondence between multiplicative and linear functionals of \(\hat{A}\) and functionals satisfying (2) and (3) defined on \(A\). These functional will also be called multiplicative and linear. Note also that if \(f\) is a multiplicative and linear functional defined on \(A\), then its complex conjugate is also such a functional.

**Remark 2.** Let \(A\) be a complete commutative real \(m\)-convex algebra with unit \(e\); then an element \(x \in A\) is invertible in \(A\) if and only if \(f(x) \neq 0\) for any continuous multiplicative and linear functional \(f\) defined on \(A\). This follows from the fact that an element \((x, 0) \in \hat{A}\) is invertible in \(\hat{A}\) if and only if \(x\) is invertible in \(A\), from the Remark 1, and from corresponding fact on complex complete \(m\)-convex algebras.

**Remark 3.** If \(A\) is a real \(m\)-convex algebra with unit \(e\), and \(W_0\) is a sequence of polynomials with real coefficients of one complex variable, which tends uniformly to \(0\) on each compact subset of \(C\), then \(\|W_0(x)\| \to 0\) for each \(x \in A\) and \(\|x||P\). In fact, \(\|W_0([x, y])\| \to 0\) for each \((x, y) \in \hat{A}\) and \(\|x||P\), so the conclusion holds for \(\hat{A}\) and therefore for \(A\).

**Remark 4.** \(\hat{A}\) is semisimple if and only if \(\lim \|\sqrt{x}\| = 0\) for each \(x \in A\). When \(\hat{A}\) is a commutative complete real \(m\)-convex algebra. This condition implies therefore that in \(\hat{A}\) there is a total family of continuous multiplicative and linear functionals.

2. Proof of the theorem

**Lemma 1.** Suppose that \(A\) is a commutative complete real \(m\)-convex algebra with unit \(e\) and that its complexification \(\hat{A}\) is a semisimple algebra. Then either \(A\) has generalized topological divisors of zero or \(A\) is isomorphically homeomorphic with one of the three finite-dimensional division algebras over real numbers.

**Proof.** If \(A\) were finite-dimensional, then it would be a Banach algebra and the conclusion of Lemma 1 holds for example by (3). Suppose that \(A\) is an infinite-dimensional algebra. By Remark 4 it follows that in \(A\) there is defined an infinite family of (non-zero) multiplicative and linear continuous functionals. Take two of them, say \(f_1\) and \(f_2\), one being not the complex conjugate of the other. There exists an element \(x \in A\) such that \(f_1(x) \neq f_2(x) \neq f_1^*(x)\). The polynomial \(W_1(\xi) = (\lambda - f_1^*(x))(\lambda - f_2(x))\) has real coefficients, and \(\zeta = W_1(f_1(x)) 
eq 0\). Taking \(V(\xi) = (W_1(\xi) - \zeta) \times (W_2(\xi) - \zeta)\), we obtain again a polynomial with real coefficients. Setting \(y = V(\xi)\), we get \(y \in A\), \(f_1(y) = V(f_1(x)) = \|x\| \neq 0\) and \(f_2(y) = V(f_2(x)) = 0\). So, setting \(z = (2y - |\xi|\xi)/|\xi|^2\), we get \(z \in A\) and \(f_1(x) = f_1(y) = 1\), so \((x, y)\) form a pair of generalized topological divisors of zero in \(A\).

**Lemma 2.** Suppose that \(A\) is a commutative complete real \(m\)-convex algebra with unit \(e\). If there exists in \(A\) a non-zero element \(x\) such that \(\lim \|\sqrt{x}\| = 0\) for each \(\|\cdot||P\), then \(A\) has generalized topological divisors of zero.

**Proof.** Take the subalgebra \(A_e\) of \(A\) defined as the smallest complete subalgebra of \(A\) containing \(e\). By Remark 2 the argument of Lemma 3 of (3) works as well in this case, so there are in \(A_e\) generalized topological divisors of zero.

Combining these two lemmas with the fact that any \(m\)-convex algebra is a dense subalgebra of a complete algebra and that any dense subalgebra of an algebra with generalized topological divisors of zero also has such divisors, and also with the fact that any \(m\)-convex division algebra over reals is isomorphically homeomorphic with one of the three standard finite-dimensional division algebras over reals and that any \(m\)-convex algebra which is not a division algebra contains a commutative subalgebra which is not a field, we get the following

**Proposition.** In any real \(m\)-convex algebra \(A\) with unit \(e\) there are generalized topological divisors of zero or \(A\) is isomorphically homeomorphic with one of the three finite-dimensional real division algebras.
To obtain our theorem let us remark that in paper [3], when deriving the general result from the result on algebras with unit, we never used the fact that the field of scalars is the field of complex numbers. So these arguments work as well in the case of real scalars and we can formulate our main result:

**Theorem.** Let $A$ be a real $m$-convex algebra. Then either $A$ has generalized topological divisors of zero or $A$ is isomorphically homeomorphic with one of the three finite-dimensional real division algebras (i.e., field of real numbers, field of complex numbers or division algebra of quaternions).

**References**


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**Continuity of operator functions**

by

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(Reino)

**1. Introduction.** By $\mathcal{A}$ we shall denote the complex algebra, the elements of which are continuous complex-valued functions of a non-negative real variable. The operation of multiplication is defined by finite convolution; the operations of addition and scalar multiplication are defined in the usual way. $\mathcal{A}$ has no zero divisors, hence the quotient field may be constructed. This quotient field, which we denote by $\mathcal{M}$, is termed the field of operators.

It is the purpose of this paper to extend the definitions of operator function and continuous operator function as defined by Mikusiński in [3] (Part II, Chap. 1). A uniform convergence structure [2] is defined on $\mathcal{M}$ and is shown to be the direct limit of uniform structures of linear subspaces of $\mathcal{M}$. The Limitierung defined by $M$-convergence [4] is the Limitierung induced by the uniform convergence structure of $\mathcal{M}$. A uniform convergence structure is defined for a locally compact Hausdorff space as the direct limit of uniform structures of compact subspaces. It is shown that an operator function is continuous, in the generalized sense, if and only if it is uniformly continuous from a locally compact Hausdorff space provided with the “compact” uniform convergence structure to the uniform convergence space $\mathcal{M}$.

**2. Preliminaries.** Let $X$ be an arbitrary set. $\mathcal{B}(X)$ shall denote the family of filter bases on the set $X$. If $3 \in \mathcal{B}(X)$, then the filter generated by $3$ is denoted by $[3]$. $\mathcal{F}(X)$ shall denote the family of filters on the set $X$. The class $\mathcal{F}(X)$ is partially ordered by the relation $\preceq$ defined by:

$[3] \preceq [3']$ iff for each $F \in [3]$ there exists a $G \in [3']$ such that $G \subseteq F$. This is equivalent to: $[3] \preceq [3']$ iff $F \in [3]$ implies $F \subseteq [3']$.

Let $X$ and $Y$ be sets and suppose $X \subseteq Y$. If $3 \in \mathcal{B}(X)$, then $3 \in \mathcal{B}(Y)$.

In this case $[3]$ could refer to an element in $\mathcal{F}(X)$ or an element in $\mathcal{F}(Y)$. In those cases where confusion could arise a subscript will be used to indicate the precise meaning. Thus $[3]_X$ refers to an element in $\mathcal{F}(X)$; $[3]_Y$ refers to an element in $\mathcal{F}(Y)$. [3]