The Arens product and multiplier operators

by

L. MÁTÉ (Budapest)

§ 0. Many problems of analysis are connected with operators of certain Banach algebras which commute with left multiplication (see [3], [4], [6], [8]). The operator T of a Banach algebra B with this property, i.e. with the property

$$T\varphi\psi = \varphi T\psi, \quad \varphi, \psi \in B,$$

is called a *multiplier*. It is obvious that the set A of multiplier operators forms an algebra and the operator

$$T_a\colon\ T_a\varphi = \varphi a, \quad \varphi \in B,$$

is a multiplier for every $a \in B$; hence A is an extension of B.

Civin and Yood [2] proposed and investigated a natural extension of a Banach algebra B namely the second conjugate space B^{**} with an *Arens product*. The Arens product in B^{**} is defined as follows: If $\Phi \in B^*$, $\varphi \in B$, then

$$\Phi * \varphi\colon\ (\Phi * \varphi)\psi = \Phi(\varphi\psi), \quad \psi \in B; \tag{1}$$

it is easy to show that $\Phi * \varphi \in B^*$ and $\|\Phi * \varphi\| \leqslant \|\Phi\|\|\varphi\|$. If $F \in B^{**}$, $\Phi \in B^*$ then

$$F * \Phi\colon\ (F * \Phi)\psi = F(\Phi * \psi), \quad \psi \in B, \tag{2}$$

and it is easy to show that $F * \Phi \in B^*$ and $\|F * \Phi\| \leqslant \|F\|\|\Phi\|$.

The definition of an Arens product based on (1) and (2) is the following:

If $F_1, F_2 \in B^{**}$, then

$$F_1 * F_2\colon\ (F_1 * F_2)\Phi = F_1(F_2 * \Phi), \quad \Phi \in B^*, \tag{3}$$

and B^{**} is a Banach algebra with the product (3). For more details and proofs, we refer the reader to [2].

In a previous paper [9] we dealt with the various connections between A and B^{**} with Arens product in case of B having a *weak right identity* (i.e. there being a sequence $\{e_\alpha\}$ in B such that $\|e_\alpha\| = 1$ and $\lim_\alpha \Phi(\varphi e_\alpha)$

$= \Phi(\varphi)$ for every $\varphi \in B$, $\Phi \in B^*$). The main point is that A is isomorphic to a certain subalgebra of $Y^* = B^{**}/Y^{\perp}$ in this case (for $Y^{\perp}$, see Definition 1), but this does not hold whenever there is no weak right identity in B.

In § 1 we shall deal with B without a weak right identity. The original norm in $Y \subset B^*$ will be replaced by a stronger one and the closure Y' will be considered. It will be shown that in the dual space Y'^* we can define a multiplication which is a counterpart of the Arens product and the connections between A and the algebra Y'^* are the same as those between A and Y^* in the case of B having a weak right identity.

The investigations proposed here cannot be carried out without any restriction considering the Banach algebra B. The following conditions are presupposed for B:

Condition 1. *B is without a left annihilator.*

Condition 2. *From*

$$\Phi_k \in B^*, \quad \varphi_k \in B, \quad \sum_{k=1}^{\infty} \|\Phi_k\| \|\varphi_k\| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \Phi_k * \varphi_k = 0$$

it follows that $\sum_{k=1}^{\infty} \Phi_k(\varphi_k) = 0$.

These two conditions together are weaker than the condition of having weak right identity. In fact, if there is weak right identity in B, then Conditions 1 and 2 are satisfied (see [9]). However, if G is a compact and non-discrete group, then $L^p(G)$ $(1 < p < \infty)$ is a Banach algebra satisfying Conditions 1 and 2 but without a weak right identity. Indeed, in the case of $L^p(G)$ $(1 < p < \infty)$

$$\Phi * \varphi = \int_G \Phi(t+\tau)\varphi(\tau)\,d\tau, \quad \Phi \in L^p, \ \varphi \in L^q$$

(see [10]), and if $\sum_{k=1}^{\infty} \|\Phi_k\| \|\varphi_k\| < \infty$, then $\sum_{k=1}^{\infty} \Phi_k * \varphi_k \in C(G)$ and $\sum_{k=1}^{\infty} \Phi_k(\varphi_k) = [\sum_{k=1}^{\infty} \Phi_k * \varphi_k](0)$. On the other hand, if G is not discrete, $L^p(G)$ $(1<p<\infty)$ is a reflexive space without a right identity; hence [2], Lemma 3.8, implies that it is also without a weak right identity.

In § 2 we shall show that if B has weak right identity, then Y'^* is isometric and isomorphic with Y^* and we give a sharper form of the main result of [9].

§ 1. If $\pi[B]$ is the natural embedding B into B^{**}, $F_\varphi = \pi[\varphi]$ $(\varphi \in B)$ and $F \in B^{**}$ such that

$$F_\varphi * F \in \pi[B] \quad \text{for every } \varphi \in B,$$

then the operator

$$T: \pi[T\varphi] = F_\varphi * F, \quad \varphi \in B, \tag{4}$$

is a multiplier. If $(F_\varphi * F)\Phi = 0$ for every $\Phi \in B^*$ and every $\varphi \in B$, then the operator T represented by F in the sense (4) is the null operator and

$$F(\Phi * \varphi) = (F * \Phi)\varphi = F_\varphi(F * \Phi) = (F_\varphi * F)\Phi = 0, \quad \varphi \in B, \ \Phi \in B^*. \tag{5}$$

Definition 1. Y is the *linear hull* of the set

$$\{\Phi * \varphi;\ \Phi \in B^*,\ \varphi \in B\}$$

and $Y^{\perp}$ is the *orthogonal complement* of Y in B^{**}.

On the basis of (5) and Definition 1 we can say that the operator T represented by F in the sense (4) is the null operator if and only if $F \in Y^{\perp}$.

Lemma 1. *The following two assertions are equivalent*:

(a) *B is without a left annihilator*;

(b) $\pi[B] \cap Y^{\perp} = 0$.

Proof. Let B be without a left annihilator and $\varphi_0 \in B \cap Y^{\perp}$; then $(\Phi * \varphi)\varphi_0 = \Phi(\varphi\varphi_0) = 0$, $\Phi \in B^*$, $\varphi \in B$; hence $\varphi_0 = 0$.

Conversely, let (b) be satisfied, $\varphi_0 \in B$ and $\varphi\varphi_0 = 0$ for every $\varphi \in B$; then

$$(\Phi * \varphi_0)\varphi = \Phi(\varphi\varphi_0) = 0, \quad \Phi \in B^*, \ \varphi \in B;$$

hence $\varphi_0 \in B \cap Y^{\perp}$ and consequently $\varphi_0 = 0$.

Definition 2. Y' is the *linear hull* of the set

$$\Big\{h = \sum_{k=1}^{\infty} \Phi_k * \varphi_k;\ \sum_{k=1}^{\infty} \|\Phi_k\| \|\varphi_k\| < \infty;\ \Phi_k \in B^*,\ \varphi_k \in B\Big\}. \tag{6}$$

Theorem 1. *Y' is a Banach space with the norm*

$$\|h\|' = \inf\Big\{\sum_{k=1}^{\infty} \|\Phi_k\| \|\varphi_k\|;\ \sum_{k=1}^{\infty} \Phi_k * \varphi_k = h\Big\};$$

moreover, Y' is a linear subset of B^* and

$$\|h\| \leqslant \|h\|', \quad h \in Y'.$$

Proof. It is obvious that $\|\ldots\|'$ is a norm. If $\{h_n\}$ $(h_n \in Y')$ is a Cauchy sequence, then there is a subsequence $\{h_k\}$ such that $\|h_{k+1} - h_k\|' < 1/2^k$. Hence, if

$$h = h_1 + \sum_{k=1}^{\infty} (h_{k+1} - h_k),$$

then $\|h\|' < \infty$, h is in form (6) and $\lim h_n = h$.

From (1) it follows that $\sum_{k=1}^{n}\Phi_k * \varphi_k \in B^*$ and it is obvious that $\|\sum_{k=1}^{n}\Phi_k * \varphi_k\| \leqslant \|\sum_{k=1}^{n}\Phi_k * \varphi_k\|'$; hence the second part of the theorem is also clear.

Remark. It is obvious that Y supplied with the norm $\|\ldots\|'$ is a dense subset of Y'.

Let us consider the dual space Y'^* of Y'. We are going to define a *modified Arens product* by which Y'^* becomes a Banach algebra.

For every $h \in Y'$ and $\varphi \in B$ the linear form $h * \varphi$ is defined by

$$(h*\varphi)\psi = h(\varphi\psi), \qquad \psi \in B;$$

for every $F \in Y'^*$ and $h \in Y'$ the linear form $F * h$ is defined by

$$(F*h)\varphi = F(h*\varphi), \qquad \varphi \in B.$$

PROPOSITION 1. *For every* $\varphi \in B$, $h \in Y'$, $F \in Y'^*$

$$h*\varphi \in Y' \quad \text{and} \quad F*h \in Y'.$$

Proof. Consider the linear form Ψ defined by

$$\Psi(\varphi) = F(\Phi*\varphi), \qquad \varphi \in B,$$

for $F \in Y'^*$, $\Phi \in B^*$. From

(7) $$|\Psi(\varphi)| \leqslant \|F\|\,\|\Phi*\varphi\|' \leqslant \|F\|\,\|\Phi\|\,\|\varphi\|$$

it follows that $\Psi \in B^*$. For every $\varphi_0 \in B$

$$(\Psi*\varphi)\varphi_0 = \Psi(\varphi\varphi_0) = F(\Phi*\varphi\varphi_0) = F([\Phi*\varphi]*\varphi_0)$$
$$= (F*[\Phi*\varphi])\varphi_0;$$

hence

(8) $$\Psi*\varphi = F*(\Phi*\varphi).$$

From (7) and (8) we obtain

$$F*h = \sum_{k=1}^{\infty}\Psi_k*\varphi_k \in Y',$$

where $h = \sum_{k=1}^{\infty}\Phi_k*\varphi_k$ and Ψ_k: $\Psi_k(\varphi) = F(\Phi_k*\varphi)$.

The first part of the assertion, namely that $h*\varphi \in Y$, can be proved in the same way by taking into account that for every $\varphi \in B$

(9) $$F\varphi:\ F_\varphi(h) = h(\varphi), \qquad h \in Y',$$

defines a unique element of Y'^* (see Lemma 1).

Definition 3. For every $F_1, F_2 \in Y'^*$

$$F_1*F_2:\ (F_1*F_2)h = F_1(F_2*h), \qquad h \in Y'.$$

As in the case of an Arens product, it is easy to show that $*$ is a multiplication and $\|F_1*F_2\| \leqslant \|F_1\|\,\|F_2\|$.

PROPOSITION 2. *The one-to-one mapping from* B *into* Y'^* *defined by* (9) *is an algebraic isomorphism.*

The proof is obvious.

THEOREM 2. *If* B *is a Banach algebra satisfying Condition* 1, *then for every* $T \in A$ *there is a unique multiplier extension onto* Y'^*.

Proof. I. If T^* is a conjugate operator of T, then (see [9])

$$T^*(\Phi*\varphi) = (T^*\Phi)*\varphi$$

and if $h = \sum_{k=1}^{\infty}\Phi_k*\varphi_k$, then $\sum_{k=1}^{\infty}(T^*\Phi_k)*\varphi_k \in Y'$ and $T^*h = \sum_{k=1}^{\infty}(T^*\Phi_k)*\varphi_k$; hence Y' is invariant for T^*. It is obvious that T^* is bounded concerning also the norm $\|\ldots\|'$.

II. Let us consider T^* as the bounded operator of Y' and T' as the conjugate operator in Y'^*. Then T' is an extension of T onto Y'^*. Indeed, if $\varphi \to F_\varphi$ by the isomorphism in Proposition 2, then

$$(T'F_\varphi)h = F_\varphi(T^*h) = (T^*h)\varphi = h(T\varphi), \qquad h \in Y',$$

and from Lemma 1 it follows that $T\varphi \to T'F_\varphi$.

III. $$([T'F]*h)\varphi = [T'F](h*\varphi) = F(T^*(h*\varphi)]$$
$$= F([T^*h]*\varphi = (F*T^*h)\varphi, \qquad \varphi \in B;$$

hence

(10) $$[T'F]*h = F*T^*h, \qquad F \in Y'^*,\ h \in Y'.$$

From the basis of (10) for every $F_1, F_2 \in Y'^*$

$$[T'(F_1*F_2)]h = (F_1*F_2)T^*h = F_1(F_2*T^*h)$$
$$= F_1([T'F_2]*h) = (F_1*T'F_2)h, \qquad h \in Y';$$

hence $T'(F_1*F_2) = F_1*T'F_2$.

THEOREM 3. *If* B *is a Banach algebra satisfying Conditions* 1 *and* 2, *then for every* $T \in A$ *there is a unique* $F_T \in Y'^*$ *such that, by the isomorphism* (9),

(11) $$T\varphi = F_\varphi*F_T, \qquad \varphi \in B.$$

Proof. If

$$I:\ Ih = \sum_{k=1}^{\infty}\Phi_k(\varphi_k), \qquad h = \sum_{k=1}^{\infty}\Phi_k*\varphi_k \in Y',$$

then it is easy to show that $I \in Y'^*$ and

$$F*I = F, \quad F \in Y'^*;$$

hence for every multiplier T'

$$T'F_T = T'(F*I) = F*T'I.$$

If $F_T = T'I$, then Theorem 2 implies (11).

If $F_\varphi * F_T = 0$ for every $\varphi \in B$, then

$$(F_T*h)\varphi = F_\varphi(F_T*h) = (F_\varphi*F_T)h = 0, \quad \varphi \in B;$$

hence

(12) $$F_T*h = 0, \quad h \in Y'.$$

The identity operator E is represented by I in (11) and, for every F_T, $I*F_T = F_T*I$; hence

(13) $$I(F_T*h) = (I*F_T)h = F_T(h);$$

comparing (12) and (13) we obtain $F_T = 0$, and the representation (11) is unique for each $T \in A$.

THEOREM 4. *If B is a Banach algebra satisfying Conditions* 1 *and* 2, *then A is anti-isomorphic with the subalgebra of Y'^* consisting of those F for which*

(14) $$F_\varphi * F \in \pi[B] \quad \textit{for every } \varphi \in B,$$

where $\pi[B]$ is the image of B in the isomorphism of Proposition 2.

Proof. The operator T_F represented by (14) is a multiplier. Indeed, considering Proposition 2,

$$T_F(\varphi\psi) = (F_\varphi*F_\psi)*F = F_\varphi*(F_\psi*F) = \varphi T_F\psi, \quad \varphi, \psi \in B.$$

If $T_1\varphi = F_\varphi*F_1$ and $T_2\varphi = F_\varphi*F_2$ for every $\varphi \in B$, then $(T_1+T_2)\varphi = T_1\varphi+T_2\varphi = F_\varphi*F_1+F_\varphi*F_2$ and $T_1T_2\varphi = T_1(T_2\varphi) = (F_\varphi*F_2)*F_1$; hence the anti-isomorphism follows from the fact that $*$ is a multiplication in Y'^*.

§ 2. Let us suppose that there is a weak right identity $\{e_a\}$ in B. It will be shown that the algebra A of multipliers in this case is closely related to the factor space $B^{**}/Y^\perp$ with an Arens product.

LEMMA 2. *$Y^\perp$ is a closed two-sided ideal in B^{**}.*

Proof. It is obvious that $Y^\perp$ is linear and closed. If $F \in B^{**}$, $\Phi \in B^*$, $\varphi \in B$, then

$$[(F*\Phi)*\varphi]\psi = (F*\Phi)\varphi\psi = F(\Phi*\varphi\psi) = F[(\Phi*\varphi)*\psi]$$
$$= [F*(\Phi*\varphi)]\psi, \quad \psi \in B;$$

hence

$$(F*\Phi)*\varphi = F*(\Phi*\varphi).$$

If $F_1, F_2 \in B^{**}$, $\Phi \in B^*$, then

$$[(F_1*F_2)*\Phi]\psi = (F_1*F_2)(\Phi*\psi) = F_1[F_2*(\Phi*\psi)]$$
$$= F_1[(F_2*\Phi)*\psi] = [F_1*(F_2*\Phi)]\psi, \quad \psi \in B;$$

hence

$$(F_1*F_2)*\Phi = F_1*(F_2*\Phi).$$

Moreover, from

$$(F*\Phi)\varphi = F(\Phi*\varphi), \quad \Phi \in B^*, \varphi \in B,$$

it follows that $F \in Y^\perp$ if and only if $F*\Phi = 0$ for every $\Phi \in B^*$.

From these assertions it follows that $Y^\perp$ is a two-sided ideal. In fact, if $F \in Y^\perp$, then for every $G \in B^{**}$

$$(G*F)*\Phi = G*(F*\Phi) = 0, \quad \Phi \in B^*,$$

and

$$(F*G)*\Phi = F*(G*\Phi) = 0, \quad \Phi \in B^*, \quad \text{q.e.d.}$$

THEOREM 5. *If B is a Banach algebra with a weak right identity $\{e_a\}$, then Y'^* is isometric and isomorphic with $B^{**}/Y^\perp$.*

Proof. If $F \in B^{**}/Y^\perp$, then if follows from obvious inequalities that F is a continuous functional on Y also in the norm $\|\ldots\|'$ and $\|F\|' \leqslant \|F\|$.

Conversely, if $F \in Y'^*$, then

(15) $$|F(h*e_a)| \leqslant \|F\|'\|h*e_a\|' \leqslant \|F\|'\|h\|, \quad h \in Y',$$

and if Ψ: $\Psi(\varphi) = F(\Phi*\varphi)$, then from (7) it follows that $\Psi \in B^*$ and

$$\lim_a F([\Phi*\varphi]*e_a) = \lim_a F(\Phi*\varphi e_a) = \lim_a \Psi(\varphi e_a)$$
$$= \Psi(\varphi) = F(\Phi*\varphi), \quad \Phi \in B^*, \varphi \in B.$$

Thus $\lim_a F(h*e_a) = F(h)$ for every $h \in Y$. Moreover, taking into account formula (15), we have

$$|F(h)| \leqslant \|F\|'\|h\|, \quad h \in Y;$$

hence $F \in B^{**}/Y^\perp$ and $\|F\|' \geqslant \|F\|$.

THEOREM 6. *If B is a Banach algebra with a weak right identity $\{e_a\}$, then there is an isometric isomorphism between the multiplier algebra A and a certain closed subalgebra of $B^{**}/Y^\perp$ with an Arens product.*

Proof. We have only to show the isometry. If $T\in A$ is represented by $F\in Y'^{*}$, then from (11) follows

(16) $$\Phi(T\varphi) = (F_{\varphi} * F)\Phi \equiv F(\Phi * \varphi), \quad F\in B^{**}, \ \Phi\in B^{*}, \ \varphi\in B;$$

hence

$$\|T\| = \sup_{\|\Phi\|=1, \|\varphi\|=1} |\Phi(T\varphi)| \leqslant \sup_{\|\Phi * \varphi\|=1} |F(\Phi * \varphi)| \leqslant \|F\|$$

since $\|\Phi * \varphi\| \leqslant \|\Phi\| \|\varphi\|$.

On the other hand, if $h\in Y$ and $\|h\| = 1$, then

(17) $$|h(Te_a)| \leqslant \|T\|$$

since $Y \subseteq B^{*}$. Moreover, from (16) it follows that $h(Te_a) = F(h * e_a)$; hence (17) implies

$$|F(h)| = \lim_{a} |F(h * e_a)| = \lim_{a} |h(Te_a)| \leqslant \|T\|, \quad h\in Y, \ \|h\| = 1;$$

thus $\|F\| \leqslant \|T\|$.

COROLLARY. *The following assertions are equivalent*:

(a) $A = B^{**}/Y^{\perp}$;

(b) $\pi[B]$ *is a right ideal in* $B^{**}/Y^{\perp}$;

(c) $\pi[B]$ *is a right ideal in* B^{**}.

References

[1] C. E. Rickart, *General theory of Banach algebras*, D. Van Nostrand C. Inc. 1960.

[2] P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pac. J. Math. 11 (1961), p. 847-870.

[3] R. E. Edwards, *Endomorphisms of function-spaces*, Pac. J. Math. 14 (1964), p. 31-48.

[4] — *Approximation by convolution*, Pac. J. Math. 15 (1965), p. 85-95.

[5] A. Figa-Talamanca, *Translation-invariant operators in L^p-spaces*, Duke Math. J. 32 (1965), p. 495-501.

[6] L. Hörmander, *Translation-invariant operators in L^p*, Acta Math. 104 (1960), p. 93-104.

[7] L. Máté, *Multiplier operators and quotient algebra*, Bull. Ac. Polonaise 13 (1965), p. 523-526.

[8] — *On the factor theory of commutative Banach algebras*, Magy. Tud. Akad. Mat. Kut. Int. Közl. 9 (1964), p. 359-364.

[9] — *Embedding multiplier operators of a Banach algebra B into its second conjugate space B^{**}*, Bull. Ac. Polonaise 13 (1965), p. 809-812.

[10] J. D. Stafney, *Arens multiplication and convolution*, Pac. J. Math. 14 (1964), p. 1423-1448.

POLYTECHNICAL UNIVERSITY OF BUDAPEST

Reçu par la Rédaction le 17. 12. 1965

STUDIA MATHEMATICA, T. XXVIII. (1967)

Conditions de non existence des solutions de l'équation différentielle des opérateurs de J. Mikusiński

par

B. STANKOVIĆ (Novi Sad)

Soit K le corps des opérateurs de J. Mikusiński [3]. Nous considérons l'équation différentielle

(1) $$x'(\lambda) + \frac{u(\lambda)}{v} x(\lambda) = 0$$

où $v(t)$ est une fonction continue sur $[0, \infty)$ et $u(\lambda) = \{u(\lambda, t)\}$, $u(\lambda, t)$ est une fonction continue sur le domaine D, D: $a \leqslant \lambda \leqslant \beta$, $0 \leqslant t < \infty$.

L'idée essentielle dans cet article est d'utiliser la théorie bien connue de l'intégrale de Laplace pour obtenir des conditions de non existence des solutions de l'équation citée. La même méthode peut être utilisée pour obtenir les théorèmes d'existence [4].

Remarquons que le problème d'existence des solutions de l'équation de la forme

$$\sum_{i=0}^{n} a_i x^{(i)}(\lambda) = 0,$$

où a_i sont des polynômes par rapport à l'opérateur différentiel s, a été résolu par J. Mikusiński [4].

J'ai aussi établi [5, 6] des résultats qui concernent les équations dont les coefficients sont des éléments de certains sous-ensembles de K.

Dans la suite on va utiliser les notations suivantes. Domaines:

$$D: a \leqslant \lambda \leqslant \beta, \ 0 \leqslant t < \infty; \qquad D_n: a \leqslant \lambda \leqslant \beta, \ 0 \leqslant t < n.$$

Fonctions:

$$f_n = \{f_n(t)\} = \begin{cases} f(t) & \text{pour} \quad 0 \leqslant t < n, \\ 0 & \text{pour} \quad t \geqslant n, \end{cases}$$

$$f_\infty = \{f_\infty(t)\} = \begin{cases} 0 & \text{pour} \quad 0 \leqslant t < n, \\ f(t) & \text{pour} \quad t \geqslant n, \end{cases}$$