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Schauder bases in compatible topologies

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1. Introduction. In a recent paper [5], Retherford raises the following question:

(1) If E is a vector space, $\mathcal{T}_1$, $\mathcal{T}_2$ are two compatible locally convex topologies (that is, give the same dual) and (b^i) is a Schauder basis for $E[\mathcal{T}_1]$, is (b^i) a Schauder basis for $E[\mathcal{T}_2]$?

The answer to the question is, in general, no. A counterexample is given in section 4 below. On the other hand, there are various conditions under which the answer to the question is yes. Arsove and Edwards [1] have shown that this is so if the Mackey topology of E is barrelled and McArthur [4] has shown it for the case in which E is weakly sequentially complete and the basis is unconditional. These results are stated precisely as theorems 1, 2 below. Theorem 3 is a generalization of theorem 2 (but *not* theorem 1) and answers question (1) under conditions which are not at all topological but rather are concerned with the type of basis. We should mention that theorems 1 and 2 are stated in a form slightly more general than the statement in the references quoted. There it is only asserted that a weak Schauder basis is a Schauder basis in the Mackey topology. However, the transition is quite straightforward and the argument is in fact contained in our proof of theorem 3.

It is interesting to note that Köthe's theory of sequence spaces, generally used to obtain counter-examples in the theory of locally convex spaces, is used here also as a means of obtaining a positive result about bases. Our theorem 3 is an almost immediate application of Proposition 2 due to Köthe. Section 3 gives a preliminary discussion of the relationship

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between sequence spaces and locally convex spaces with Schauder bases.

Finally, section 5 considers the relationship between the three theorems. First an example is given to show that theorem 3 is a proper generalization of theorem 2, and then examples are given to show that there is no relationship between theorems 1 and 3.

2. Notations and remarks. The symbol $E[\mathscr{T}]$ (or simply E if no confusion is likely) will indicate a locally convex space (always Hausdorff) with the topology, $\mathscr{T}$. A *$\mathscr{T}$-Schauder basis* for E is a sequence (b^i) of elements such that for each $x \in E$ there is a unique expansion $x = \sum_{i=1}^{\infty} x_i b^i$ where the x_i are scalars, the convergence is with respect to $\mathscr{T}$, and the mappings $x \to x_i$ are continuous.

If E, F are two vector spaces (real or complex) and there is a bilinear functional, $(x, y) \to \langle x, y\rangle$ on $E\times F$ such that for each $x \in E$ there is $y \in F$ such that $\langle x, y\rangle \neq 0$ and vice versa, then we say that the pair $\langle E, F\rangle$ is placed in *duality by the relation* $\langle x, y\rangle$. The most prevalent example is the case in which E is a locally convex space, F is the dual, E', and $\langle x, y\rangle = y(x)$. In general we have the two topologies, $\mathscr{T}_s(F, E)$, $\mathscr{T}_k(F, E)$ which are respectively the weak and Mackey topologies on E induced by F. The class of topologies finer than the weak and coarser than the Mackey are exactly those topologies $\mathscr{T}$ such that $(E[\mathscr{T}])' = F$. Such a topology is said to be *compatible* with the duality. The reader is referred to Köthe [3] for details.

If λ is a vector space of sequences, $x = (x_i)$ of real or complex numbers under coordinatewise arithmetic, we define

$$\lambda^{\times} = \Big\{u = (u_i) \Big| \sum_{i=1}^{\infty} |x_i u_i| < \infty \text{ for all } x = (x_i) \in \lambda\Big\};$$

$$\lambda^{\beta} = \Big\{u = (u_i) \Big| \sum_{i=1}^{\infty} x_i u_i \text{ is convergent for all } x = (x_i) \in \lambda\Big\}.$$

It is easy to see that if λ contains every sequence which is non-zero for only finitely many terms, then $\langle \lambda, \lambda^{\times}\rangle$ and $\langle \lambda, \lambda^{\beta}\rangle$ are placed in duality by the relation $\langle x, u\rangle = \sum_{i=1}^{\infty} x_i u_i$. The symbols e, e^n will refer respectively to the sequence for which every term is unity and the sequence whose nth term is unity and the rest are zero. We shall make use of the following particular vector spaces of sequences:

φ = the set of finitely non-zero sequences,

ω = the set of all sequences,

l^1 = the set of all absolutely summable sequences,

l^∞ = the set of all bounded sequences,

c_0 = the locally convex space of sequences which converge to zero equipped with the sup topology.

3. Bases and sequence spaces. If E is a locally convex space with a Schauder basis, then there is a natural correspondence between E and a vector space of sequences λ. In order to apply the Köthe theory of sequence spaces, however, one must consider the connection between the topological dual, E', and the Köthe α-dual, λ^x. As we shall see below there is a natural correspondence between E' and a subset of the Köthe β-dual, λ^β. Unfortunately this subset is, in general, proper and can remain so even if $\lambda^\times = \lambda^\beta$. In the case in which this subset is all of λ^x, the connection between types of bases and sequence space concepts is quite interesting and will be discussed in a forthcoming paper. For our present purposes, we shall only require that $\lambda^\times = \lambda^\beta$.

Suppose, then, that the sequence (b^i) is a Schauder basis for E. We shall say that (b^i) is an *unconditional basis* if the convergence of each expansion is unconditional. We shall say that (b^i) is a *bounded multiplier basis* if

$$\sum_{i=1}^{\infty} x_i b^i \in E \text{ and } |y_i| \leqslant |x_i| \text{ for all } i \quad \text{implies} \quad \sum_{i=1}^{\infty} y_i b_i \in E$$

(cf. Day [2], p. 58).

Let $f^i \in E'$ be defined by $f^i(b^j) = \delta_{ij}$. We define the following vector spaces of sequences:

$$\lambda = \Big\{x = (x_i) \Big| \sum_{i=1}^{\infty} x_i b^i \in E\Big\},$$

$$\mu = \Big\{u = (u_i) \Big| \sum_{i=1}^{\infty} u_i f^i \in E'[\mathscr{T}_s(E, E')]\Big\}.$$

We say, following Köthe ([3], p. 409), that a sequence space λ is *normal* if $x = (x_i) \in \lambda$ and $|y_i| \leqslant |x_i|$ for all i implies $y = (y_i) \in \lambda$. Finally, we consider the mappings η: $\lambda_0 \to E$, λ': $E' \to \mu$ defined by

$$\eta(x_i) = \sum_{i=1}^{\infty} x_i b^i, \qquad \eta'\Big(\sum_{i=1}^{\infty} u_i f^i\Big) = (u_i).$$

Connecting these concepts, we have

PROPOSITION 1.

(i) *(b^i) is a bounded multiplier basis if and only if λ is normal.*

(ii) *If λ is normal, then $\lambda^\beta = \lambda^\times$.*

(iii) $\mu \subset \lambda^\beta$.

(iv) *The maps η, η' are linear homeomorphisms onto when λ, μ, E, E' are equipped with their weak topologies from μ, λ, E', E, respectively.*

Proof. The first and third follow immediately from the definitions and the last is easily proved with the observations that $\langle \eta(x), u\rangle = \langle x, \eta'(u)\rangle$ for all $x \in \lambda$, $u \in E'$.

For the second assertion, we note that in any case $\lambda^\times \subset \lambda^\beta$. On the other hand, if λ is normal and $u = (u_i) \in \lambda^\beta$, $x = (x_i) \in \lambda$. Let y be defined by

$$y_i = \begin{cases} |x_i u_i|/u_i & \text{if} \quad u_i \neq 0, \\ 0 & \text{if} \quad u_i = 0. \end{cases}$$

Then $|y_i| \leqslant |x_i|$ so $y \in \lambda$. Hence

$$\sum_{i=1}^{\infty} |x_i u_i| = \sum_{i=1}^{\infty} y_i u_i$$

and since $u \in \lambda^\beta$, the two series are convergent. Thus $u \in \lambda^\times$.

In addition to the above considerations, we shall make use of the following rather deep result of Köthe ([3], p. 417):

PROPOSITION 2. *If λ is normal, $x \in \lambda$, $x^n = (x_1, \ldots, x_n, 0, 0, \ldots)$, then the sequence (x^n) converges to x in the topology, $\mathscr{T}_k(\lambda^\times, \lambda)$.*

4. Bases in compatible topologies. First we show that the answer to (1) is, in general, negative.

Let $\lambda = \{x = (x_i) \mid (x_i) \text{ is eventually constant}\} = \{ae + x \mid a \text{ is a scalar}, x \in \varphi\}$. Clearly $\lambda^\times = l^1$ and the sequence (e^n) is a Schauder basis for λ in the topology, $\mathscr{T}_s(\lambda^\times, \lambda)$. We show that (e^n) is not a $\mathscr{T}_k(\lambda^\times, \lambda)$-Schauder basis for λ. Let $u^n = e^n - e^{n+1}$ be considered as an element of $\lambda^\times$. Then, although for each m, $\lim_n \langle e^n, u^m\rangle = \langle e, u^m\rangle$, the convergence is not uniform with respect to m. In fact, $\langle e^n, u^n\rangle - \langle e, u^n\rangle = 1$ for all n. Hence we need only show that the convex circled hull A of the set $\{u^n\}$ is $\mathscr{T}_s(\lambda, \lambda^\times)$-relatively compact. Let S be the closed unit ball in l^1. Then by the definition of convex circled hull we have

$$A = \Big\{a = (a_i) \,\Big|\, a = \sum_{j=1}^{\infty} \beta_j u^j,\ \beta = (\beta_j) \in S \cap \varphi\Big\}.$$

Hence A is a coordinatewise bounded subset of ω and since ω is a Montel space in the topology $\mathscr{T}_s(\varphi, \omega)$ (the Tychonov theorem), A is $\mathscr{T}_s(\varphi, \omega)$-relatively compact. Furthermore, if $ae + x \in \lambda$ and $a \in A$, then

$$(2) \qquad \langle ae + x, a\rangle = \alpha\langle e, a\rangle + \langle x, a\rangle = \alpha \sum_{j=1}^{\infty} \beta_j \langle e, u^j\rangle + \langle x, a\rangle = \langle x, a\rangle.$$

Therefore, if (a^ν) is a net in A, there is a cofinal subnet (call it again (a^ν)) and an element $a^0 \in \omega$ such that if $ae + x \in \lambda$, then, in view of (2), $\lim_\nu \langle ae + x, a^\nu\rangle = \lim_\nu \langle x, a^\nu\rangle = \langle x, a^0\rangle$.

We are finished then if we can show that $a^0 \in \lambda^\times = l^1$ and that (2) remains valid if a is replaced by a^0, for this means that a^0 is a $\mathscr{T}_s(\lambda, \lambda^\times)$-limit point of the orginal net. Now by the above characterization of A, it follows that $A \subset 2S$ so $a^\nu \in 2S$ for all ν. Since a^0 is the coordinatewise limit of the net (a^ν), we may conclude that $a^0 \in 2S \subset l_1 = \lambda^\times$. Furthermore, by the same characterization, we may write,

$$a^\nu = \sum_{j=1}^{\infty} \beta_j^\nu u^j, \qquad \beta^\nu = (\beta_j^\nu) \in S \cap \varphi.$$

Hence, $a_i^\nu = \beta_i^\nu - \beta_{i-1}^\nu$ ($\beta_0^\nu = 0$) so $\beta_i^\nu = a_1^\nu + \ldots + a_i^\nu$ and since (a^ν) is coordinatewise convergent to (a^0), we may define $\beta_i^0 = \lim_\nu \beta_i^\nu = a_1^0 + \ldots + a_i^0$. Therefore, (β^ν) is a net in S which is coordinatewise convergent to $\beta^0 = (\beta_i^0)$ so $\beta^0 \in S \subset l^1$. Hence, $\lim_i \beta_i^0 = 0$. Therefore,

$$\begin{aligned} \langle ae + x, a^0\rangle &= \alpha\langle e, a^0\rangle + \langle x, a^0\rangle = \alpha \sum_{i=1}^{\infty} a_i^0 + \langle x, a^0\rangle \\ &= \alpha \lim \beta_i^0 + \langle x, a^0\rangle = \langle x, a^0\rangle. \end{aligned}$$

Hence a^0 satisfies (2) and the counterexample is established.

We now give three theorems which answer question (1) positively under various restrictions. Let E be a locally convex space with dual E' and let $\mathscr{T}_1, \mathscr{T}_2$ be two topologies which are compatible with the duality.

THEOREM 1 ([1], Theorem 11, p. 109). *If $E[\mathscr{T}_k(E', E)]$ is barrelled, then every $\mathscr{T}_1$-Schauder basis is a $\mathscr{T}_2$-Schauder basis.*

THEOREM 2 ([4], cor. 2). *If $E[\mathscr{T}_s(E', E)]$ is sequentially complete, then every unconditional $\mathscr{T}_1$-basis is an unconditional $\mathscr{T}_2$-basis.*

THEOREM 3. *Every $\mathscr{T}_1$-bounded multiplier basis is a $\mathscr{T}_2$-bounded multiplier basis.*

Proof. Let (b^i) be a $\mathscr{T}_1$-bounded multiplier basis. Clearly, since $\mathscr{T}_1$ is compatible, (b^i) is a weak Schauder basis. Suppose that the weak limit of $\sum_{i=1}^{n} x_i b^i$ is an element x of E, and $|y_i| \leqslant |x_i|$. Then x is the $\mathscr{T}_1$-limit of some series $\sum_{i=1}^{n} x_i' b^i$ and since the linear functionals are the same, $x_i = x_i'$. Hence $|y_i| \leqslant |x_i'|$, so $\sum_{i=1}^{n} y_i b^i$ is $\mathscr{T}_1$-convergent and hence weakly convergent to an element of E. Thus (b^i) is a weak bounded multiplier basis and we can apply the definitions and results of section 3 to this case. In particular, we note that from proposition 2, if $x = (x_i) \in \lambda$ and $x^n = (x_i, \ldots, x_n, 0, 0, \ldots)$, then the sequence (x_n) is $\mathscr{T}_k(\lambda^\times, \lambda)$-convergent to x. But $\mu \subset \lambda^\beta = \lambda^\times$ so every convex $\mathscr{T}_s(\lambda, \mu)$-compact set is $\mathscr{T}_s(\lambda, \lambda^\times)$-compact. Hence $\mathscr{T}_k(\lambda^\times, \lambda)$ is finer than $\mathscr{T}_k(\mu, \lambda)$ so the sequence (x^n) is $\mathscr{T}_k(\mu, \lambda)$ convergent to x.

Now suppose that $x \in E$, x is the $\mathscr{T}_s(E', E)$ limit of $\sum_{i=1}^{n} x_i e^i$, and A is a convex $\mathscr{T}_s(E, E')$-compact subset of E'. Since η' is weakly continuous, $\eta'(A)$ is a convex $\mathscr{T}_s(\lambda, \mu)$-compact set. Also $\eta^{-1}(x) = (x_i)$ is in λ, so x^n is convergent to $\eta^{-1}(x)$, uniformly on $\eta'(A)$. Hence, $\eta(x^n) = \sum_{i=1}^{n} x_i b^i$ is uniformly convergent to x on A. This shows that (b^i) is a $\mathscr{T}_k(E', E)$-Schauder basis, and since $\mathscr{T}_k(E', E)$ is finer than $\mathscr{T}_2$, (b^i) is a $\mathscr{T}_2$-Schauder basis. Finally, by an argument similar to one given at the beginning of this proof, it follows that (b^i) is a $\mathscr{T}_2$-bounded multiplier basis.

5. Comparison of results. In the first place, we see that theorem 2 follows immediately from theorem 3. In fact, it follows from the definitions and a remark of Day ([2], p. 59) that if $E[\mathscr{T}_s(E', E)]$ is sequentially complete, then every unconditional basis is a bounded multiplier basis.

On the other hand, the sequences (e^n) form a bounded multiplier basis for c_0, but this space is not weakly sequentially complete.

Finally we note that theorems 1, 3 are not comparable. The sequence (e^n) form a bounded multiplier basis for l^∞ with the topology, $\mathscr{T}_s(l^1, l^\infty)$, but the Mackey topology is not barrelled; and, the space l^1 is a Banach space and thus barrelled and (u^n), where $u^1 = e'$, $u^n = e^{n-1} - e^n$, $n \geq 2$, forms a conditional basis for l^1 and hence is not a bounded multiplier basis.

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