STUDIA MATHEMATICA, T. XXVIII. (1967)

The direct sum of Banach spaces with respect to a basis

by

WILLIAM J. DAVIS and DAVID W. DEAN (Columbus, Ohio)

§ 0. Introduction. Let E be a Banach space of sequences such that if $(a_n) \in E$ and if $|\beta_n| \leqslant |a_n|$ for all n, then (i) $(\beta_n) \in E$ and (ii) $\|(\beta_n)\| \leqslant \|(a_n)\|$ (1). In [13] A. Pełczyński introduced the notion of a countable sum of Banach spaces with respect to E. Specifically, if (X_n) is a sequence of Banach spaces, then $\Sigma_E X_n$ is the space of sequences (x_n) with $x_n \in X_n$ for each n, and with $(\|x_n\|) \in E$, where $\|x_n\|$ is the norm of x_n in X_n. Defining a norm on this space by $\|(x_n)\| = \|(\|x_n\|)\|_E$, and using coordinate-wise addition and scalar multiplication, $\Sigma_E X_n$ is a Banach space.

Of particular interest is the case in which $X_n = E$ for each n, and $\Sigma_E E$ is isometric (isometrically isomorphic) in a natural way to E. Examples of such spaces are (c_0), (l_p) $(l \leqslant p)$, (m). The Fréchet space (s) is isomorphic to $\Sigma_{(s)}(s)$.

Using this isometry and other special properties of these examples, Pełczyński has penetrated deeply into their structures. In each of these examples, the sequence (e_n), where $e_n = (\delta_{nj})_{j=1}^{\infty}$, is a basis (or in the case of (m), a generalized basis [1]) satisfying (i) and (ii) above (2).

Let σ_j be a subsequence of the positive integers N for each j such that $\bigcup \sigma_j = N$ and $\sigma_j \cap \sigma_k = \emptyset$ if $j \neq k$. Let $E(\sigma_j) = \{a \in E \mid a_k = 0 \text{ if } k \notin \sigma_j\}$. If E is (c_0), (l_p) $(1 \leqslant p \leqslant \infty)$, each $E(\sigma_j)$ is isometric under the natural mapping T such that $Te_{j_i} = e_i$ where j_i is the ith element of σ_j. Moreover, [13], E is isometric to $\Sigma_E E(\sigma_j)$ in a natural way. Another way to say this is as follows. Let τ: $N \to N \times N$ be one to one and onto. Let $E_j = E(\{n \mid \tau(n) = (j, k) \text{ for some } k\})$. Then E is isometric to each E_j and to $\Sigma_E E_j$. Moreover, the isometry T of E with $\Sigma_E E_j$ has the property that $Te_n = e_{jk}$ if $\tau(n) = (j, k)$, where e_{jk} is the kth coordinate sequence in the jth copy of E. If every such τ induces an isometry of E with $\Sigma_E E$, say that E is *dispersed*. If each τ induces an isomorphism T such that $\|T\|$, $\|T^{-1}\|$ $< k$ where k is independent of τ, say the space is *almost dispersed*. If

(1) Every Banach space with a w*-separable conjugate space has a generalized basis. Then every such space has a representation as a space of sequences.

(2) We shall say that such a basis or generalized basis is *orthogonal* [14].

there is a τ inducing an isomorphism of E with $\Sigma_E E$, say E is *weakly dispersed*. The word *induces* means the following. If $(a_{jk})_{j=1}^{\infty}$ is a sequence in $\Sigma_E E_j$, $(a_{jk})_{k=1}^{\infty}$ in E_j, then this sequence is mapped to the sequence $(a_{\tau^{-1}(j,k)})$ in E.

With this structure Pełczyński ([13], see also [8], p. 393) has proved:

PROPOSITION 1. *Let E be weakly dispersed. If X is a complemented subspace of E and if E is isomorphic to a complemented subspace of X, then E is isomorphic to X.*

The space $\Sigma_E E$ is at the center of this discussion. In § 1 it is shown that E is almost dispersed if and only if it is weakly dispersed and symmetric (defined to be the natural extension of Singer's symmetric condition on bases ([15], [16])). In § 2 the almost dispersed concept is characterized geometrically. Using this characterization, it is shown in § 4 that these spaces are isomorphic to (l_1), (c_0), (l_∞), or reflexive, and if the space is dispersed then isometric to (l_p) $(1 \leqslant p \leqslant \infty)$, or (c_0). Further it is shown that if (e_n) is a basis, then almost dispersed and perfectly homogeneous are the same. In § 3 it is noted that many of the properties established in [13] hold for these spaces.

§ 1. Symmetric spaces and dispersed generalized bases. Let P denote the set of permutations ϱ: $N \to N$. Suppose for each sequence $(a_i) \in E$ and each sequence (δ_i) such that $|\delta_i| \leqslant 1$ for each i that one has $(\delta_i a_i) \in E$. Following Singer [15], [16], define $|||(a_i)||| = \sup\{||(\delta_i a_{\varrho(i)})|| \mid |\delta_i| \leqslant 1$ for each i, $\varrho \in P\}$. If (e_n) is a basis for E, then $|||(a_i)||| = \sup\{\|\sum_1^n a_i e_{\varrho(i)}\| \mid n \in N, \varrho \in P\}$. If this new norm is equivalent to the old, say that the generalized basis (e_n) is *a symmetric system* for E. If $|||\ldots||| = ||\ldots||$, say that E is *symmetric*.

There is a close relation between a symmetric E and an almost dispersed E. For example, it is immediate that each symmetric generalized basis is orthogonal. If (e_n) is in fact a basis for E, then E is *collapsing*. That is:

PROPOSITION 2. *Let (e_n) be a basis for E which is a symmetric system and let (n_i) be a subsequence of N. Then the mapping $Te_{n_i} = e_i$ for each i induces an isomorphism of $E((n_i))$ onto E which is an isometry if E is symmetric.*

We remark here that if E is separable and the generalized basis (e_n) is orthogonal, then (e_n) is a basis [11].

Proof of Proposition 2. It is clear that for arbitrary sequences (a_i) of scalars, and (m_i), $(n_i) \subset N$, one has

$$\Big|\Big|\Big|\sum_{i=1}^{N} a_{n_i} e_{n_i}\Big|\Big|\Big| = \Big|\Big|\Big|\sum_{i=1}^{N} a_{n_i} e_{m_i}\Big|\Big|\Big|.$$

In particular, set $m_i = i$ for all i. Thus,

$$u_N = \sum_{i=1}^{N} a_{n_i} e_i$$

determines a Cauchy sequence in E such that

$$u_N \to u = \sum_{i=1}^{\infty} a_{n_i} e_i \quad \text{as} \quad N \to \infty.$$

Furthermore,

$$|||u||| = \Big|\Big|\Big|\sum_{i=1}^{\infty} a_{n_i} e_{n_i}\Big|\Big|\Big|.$$

If we take mutually disjoint sequences (n_i^j), $j = 1, 2, \ldots$, of positive integers, it is clear that a symmetric space E contains an infinite number of mutually "orthogonal" copies of itself, which might cause us to hope that $E \sim \Sigma E_j$, where $E_j = E((n_i^j))$. In section 4 we give a counterexample to this supposition. One does, however, obtain the following

THEOREM 1. *The sequence space E is almost dispersed if and only if (e_n) is a symmetric system and E is weakly dispersed.*

Proof. Let E be almost dispersed and $\varrho \in P$. Given any onto and one to one τ: $N \to N \times N$, there exists τ_1 such that $\tau^{-1} \circ \tau_1 = \varrho$. Let S and S_1 be the isomorphisms of E onto ΣE induced by τ and τ_1, respectively. Then $S^{-1} \circ S_1$ is an isomorphism of E onto itself satisfying $(S^{-1} \circ S_1) e_j = e_{\varrho(j)}$. Thus, from the definition of almost dispersed, there are constants c, $k > 0$, independent of ϱ, such that for every sequence $(a_i) \in E$ one has

$$c\|(a_i)\| \leqslant \sup_{|\delta_i| \leqslant 1, \varrho \in P} \|(\delta_i a_{\varrho(i)})\| \leqslant k\|(a_i)\|.$$

Conversely, assume that E is weakly dispersed and (e_n) is a symmetric system. Renorming with $|||\ldots|||$, E is symmetric and weakly dispersed. The latter condition guarantees a τ inducing an isomorphism of E with $\Sigma_E E$. Any other τ_1 is of the form $\tau \circ \varrho$ for some ϱ in P and the induced mappings S for ϱ and T for τ have composition $S \circ T$ with norm bounded independent of ϱ. Thus $T_1 = S \circ T$, the induced mapping for τ_1, has norm bounded independent of τ_1.

The following example shows that weakly dispersed does not imply almost dispersed. The space is isometric to (c_0). This makes it clear that the notion of dispersed depends on the basis chosen as well as the space itself.

EXAMPLE 1. Let (a_j) be the sequence $(1, 2, 1, 2, 4, 1, \ldots)$. Let E be the Banach space of sequences (β_j) such that $(a_j \beta_j) \in (c_0)$ with norm $\|(\beta_j)\| = \|(a_j \beta_j)\|$. The coordinate basis (e_n) is not symmetric since shifting a sequence may change its norm by large amounts. Let τ: $N \to N \times N$

be any one to one, onto mapping such that $a_n = a_j a_k$ if $\tau(n) = (j, k)$. Define $T: E \to \Sigma E$ by means of this τ. Thus, $T((\beta_n)) = (\beta_{jk})$ where $\beta_n = \beta_{jk}$ if $\tau(n) = (j, k)$. Therefore,

$$\|((\beta_{jk})_{k=1}^{\infty})_{j=1}^{\infty}\| = \|(\|(\beta_{jk})_{k=1}^{\infty}\|)_{j=1}^{\infty}\| = \sup_{1\leqslant j<\infty} (a_j \sup_{1\leqslant k<\infty} a_k|\beta_{jk}|)$$

$$= \sup_{1\leqslant j,k<\infty} a_j a_k |\beta_{jk}| = \sup_{1\leqslant n<\infty} a_n|\beta_n| = \|(\beta_n)\|.$$

Thus, T is an isometry. To see that T is onto, notice that all processes above are reversible. If follows from theorem 1 that (e_n) is not almost dispersed. This may be seen directly by choosing n such that $\tau(n) = (l, k)$ if $a_n = 1$ and $\tau(n) \neq (l, k)$ if $a_n > 1$. Then, the sequence $(1/na_n) \in E$, but $T((1/na_n)) \notin \Sigma E$ since $\|(\beta_{1k})\| = \infty$, where $T((1/na_n)) = (\beta_{jk})$.

§ 2. A geometric condition for a dispersed system. Basic to the concept of a dispersed or weakly dispersed system is the statement: If $\sigma_k = \{n \in N|\ \tau(n) = (k, i)$ for some $i\}$ and if x and y are elements of E such that $f_j(x) = 0$ if $j \notin \sigma_n$ and $f_j(y) = 0$ if $j \notin \sigma_m$, $n \neq m$ (i.e. if x and y are in different copies of E in ΣE), then $C\|x+y\| \leqslant \big\|\|x\|e_n + \|y\|e_m\big\| \leqslant \leqslant K\|x+y\|$. For a dispersed system, $C = K = 1$ and the system is symmetric, so $\|x+y\| = \big\|\|x\|e_i + \|y\|e_j\big\|$ for every $i \neq j$. The following proposition is useful in section 3 and in characterizing dispersed and almost dispersed systems using geometric conditions of the above type.

PROPOSITION 3. *Let E be almost dispersed and let $\varrho: N \to N$ be one to one. Then $X(\varrho(N)) = \{(a_i) \in E|\ a_i = 0$ if $i \notin \varrho(N)\}$ is isomorphic to E under an isomorphism S such that $s((a_i)) = (a_{\varrho(i)})$.*

Proof. Note first that $X(\varrho(N))$ is isometric to $X(\varrho(N), 1) = \{(x_i) \in \Sigma_E E|\ x_{ij} = 0$ if $j \neq 1$ or $i \notin \varrho(N)\}$, the isometry being $(x_i) \to (x_{i1}) \in X(\varrho(N), 1)$. Now suppose that $N - \varrho(N)$ is infinite. Let $\tau: N \to N\times N$ satisfy $\tau(\varrho(n)) = (n, 1)$ for each n. If T is the restriction to $X(\varrho(N))$ of the isomorphism of E onto $\Sigma_E E$ induced by τ and if $(a_i) \in X(\varrho(N))$, then $T((a_i)) = ((a_{jk})_{k=1}^{\infty})_{j=1}^{\infty}$ where $a_{jk} = 0$ unless $i = \varrho(n)$ and $\tau(i) = (j, k)$ where $k = 1$ and $j = n$, or $TX(\varrho(N)) = X((N, 1))$ which is isometric to E, letting $(x_n) \xrightarrow{I} (x_{n,1})$. The mapping S is $I^{-1}T$.

Next suppose that $N - \varrho(N)$ is finite. Embed $X(\varrho(N))$ as $X((\varrho(N), 1))$ in $\Sigma_E E$ and choose $\tau: N \to N\times N$. Then $\tau^{-1}((\varrho(N), 1)) = N_1$ has $N - N_1$ infinite and, by the above, $X(\tau^{-1}\varrho(N))$ is isomorphic to E, say under T, while τ^{-1} induces an isomorphism T^{-1} of $X((\varrho(N), 1))$ with $X(N_1)$. The mapping $TT_1^{-1}I = S$ is the one we seek, where I is the embedding $X(\varrho(N))$ to $X((\varrho(N), 1))$.

We remark that $\|S\|$, $\|S^{-1}\|$ are bounded independently of ϱ since $\|T\|$, $\|T_1^{-1}\|$ are so bounded.

COROLLARY 1. *Let E have orthogonal generalized basis (e_n). Then E is almost dispersed if and only if every one to one, into $\tau: N \to N\times N$ induces an isomorphism T from E into $\Sigma_E E$ such that $\|T\|$, $\|T^{-1}\|$ are bounded independently of τ.*

Proof. Sufficiency is immediate since the set of onto τ's is included. Thus let E be almost dispersed. Let $\tau: N \to N\times N$ be one to one and let $\tau_1: N \to N\times N$ be one to one and onto. Then $\tau_1^{-1}\circ\tau: N \to N$ is one to one. Let $\tau_1^{-1}\circ\tau = \varrho$ so $\tau = \tau_1\circ\varrho$. Let S and T_1 be the isomorphisms induced by τ and τ_1 respectively. Then $T = T_1\circ S$ is the desired isomorphism and is norm bounded independently of τ.

The geometric condition we are interested in is contained in the statement of the next theorem. For x in E let the *support of* x ($\operatorname{supp} x$) be $\{j|\ x_j \neq 0\}$. Say that x and u are *disjoint* if $\operatorname{supp} x \cap \operatorname{supp} u = \emptyset$.

THEOREM 2. *The following are equivalent.* (i) *E is almost dispersed.* (ii) *The generalized basis (e_n) is orthogonal and there exist constants $C, K > 0$ such that, if $x_1, \ldots, x_n, y_1, \ldots, y_n$ are elements of E satisfying $\operatorname{supp}(x_i) \cap \operatorname{supp}(x_j) = \emptyset = \operatorname{supp}(y_i) \cap \operatorname{supp}(y_j)$ whenever $i \neq j$, and such that $\|x_i\| = \|y_i\|$ $(i = 1, \ldots, n)$, then $C\|x_1 + \ldots + x_n\| \leqslant \|y_1 + \ldots + y_n\| \leqslant K\|x_1 + \ldots + x_n\|$.*

Proof. Suppose that E is almost dispersed. Let x_i, y_i be as in (ii) and let τ, $\tau_1: N \to N\times N$ be one to one, into maps such that $\tau(\operatorname{supp} x_i) \subset \subset ((i, j))$ and $\tau_1(\operatorname{supp} y_i) \subset ((i, j))$. Let T and T_1 be the isomorphisms induced by τ and τ_1 as guaranteed by corollary 1. Then there are constants $A, B > 0$, independent of τ, τ_1, such that

$$\begin{aligned}
\|x_1 + \ldots + x_n\| &\leqslant B\|Tx_1 + \ldots + Tx_n\| \\
&= B\big\|\|x_1\|e_1 + \ldots + \|x_n\|e_n\big\| \\
&= B\big\|\|y_1\|e_1 + \ldots + \|y_n\|e_n\big\| \\
&= B\|T_1y_1 + \ldots + T_1y_n\| \\
&\leqslant AB\|y_1 + y_2 + \ldots + y_n\| \\
&\leqslant AB^2\|T(y_1 + \ldots + y_n)\| \\
&= AB^2\big\|\|x_1\|e_1 + \ldots + \|x_n\|e_n\big\| \\
&\leqslant A^2B^2\|x_1 + x_2 + \ldots + x_n\|.
\end{aligned}$$

Therefore,

$$\frac{1}{AB}\|x_1 + \ldots + x_n\| \leqslant \|y_1 + \ldots + y_n\| \leqslant AB\|x_1 + \ldots + x_n\|.$$

If the system is dispersed, $A = B = 1$.

Now assume that (ii) is satisfied. It will be shown in § 4, independently of this argument, that if (e_n) is not a basis for E, then E is isomorphic to (m) in the natural way $(C_1\|(a_j)\|_E \leqslant \|(a_j)\|_m \leqslant K_1\|(a_j)\|_E$ for every $(a_j) \in E)$.

Thus we assume here that (e_n) is a basis for E. For any $(a_j)\in E$ condition (ii) promises that $C\|\sum_1^n a_j e_j\| \leqslant \|\sum_1^n a_j e_{\varrho(j)}\| \leqslant K\|\sum_1^n a_j e_j\|$ for every n and $\varrho\in P$. Thus (e_n) is a symmetric basis. Let σ_n be infinite, $\bigcup \sigma_n = N$, $\sigma_i \cap \sigma_j = \emptyset$ if $i\neq j$. For $x=(a_j)\in E$ let $x_{i_j} = a_j$ if $j\in\sigma_i$, and $x_{i_j}=0$ if $j\notin\sigma_i$. Let $\tau\colon N\to N\times N\in\tau\sigma_n = \{(n,k)|\, k\in N\}$. Since (e_n) is a basis, $x=\sum_1^\infty x_i$ and by (ii) $C\|\sum_1^n x_i\| \leqslant \|\sum_1^n \|x_i\| e_i\| \leqslant K\|\sum_1^n x_i\|$ for every n, or $C\|x\| \leqslant \|\sum_1^\infty \|x_i\| e_i\| \leqslant K\|x\|$ so that $(x_i)\in\Sigma_E E$ and the mapping T induced by τ is continuous. For any $(x_i)\in\Sigma_E E$ let $a_n = x_{ij}$ if $\tau^{-1}(i,j)=n$. Then $(a_n)\in E$ and $T\big((a_n)\big) = (x_i)$. By theorem 1 we conclude the argument.

Example 2. Using the geometric conditions we can now construct a space which is symmetric but not almost dispersed, and so not weakly dispersed. Similar such examples have been constructed by Singer [16].

Let (d) be the linear space of real sequences $a=(a_j)$ with norm

$$\|a\| = \sup_{\varrho\in P} \sum_1^\infty \frac{|a_{\varrho(j)}|}{j}.$$

Then (e_n) is a symmetric basis for (d) which is boundedly complete so that (d) is a conjugate space, and so it is not isomorphic to (c_0). (If (y_n) is biorthogonal to (e_n) and (f_n) is the natural basis for $l_1 = c_0^*$, then $(d)\sim[y_n]^*\sim[f_n]^* = (m)$ so that E is not separable; [7], p. 70). Further (e_n) converges weakly to 0 but not in norm as follows: Otherwise there exists f in $(d)^*$ and a sequence (e_{n_j}) such that $f(e_{n_j}) > \varepsilon$ for some $\varepsilon>0$ and some subsequence (n_j) of N. Then if $u = \sum_1^\infty \frac{1}{j} e_{n_j}$, $f(u)=\infty$. Thus (d) is not isomorphic to (l_1) ([7], p. 33). In a private communication J. R. Retherford has shown that (d) is not reflexive.

To see that (d) is not almost dispersed, fix k, m in N with $m>1$. Let $\beta_i = (\log mk)^{-1}$ if $j = m(i-1)+1, \ldots, mi$; 0 if $j\neq m(i-1)+1, \ldots, mi$. Then $\|\beta_i\| = (\log mk)^{-1}\sum_1^m \frac{1}{j}$ while $\|\sum_1^k \beta_i\| = \Big(\sum_1^{mk}\frac{1}{j}\Big)(\log mk)^{-1}$ which converges as $mk\to\infty$. Now

$$\Big\|\sum_{j=1}^k \|\beta_j\| e_j\Big\| = (\log mk)^{-1}\sum_{j=1}^m \frac{1}{j}\Big\|\sum_1^k e_n\Big\|$$

$$= \Big(\sum_1^m \frac{1}{j}\Big)\Big(\sum_1^k \frac{1}{n}\Big)(\log m + \log k)^{-1}.$$

For large k and very large m, $\|\sum_1^k \beta_j\|$ is near 1 while $\|\sum_1^k \|\beta_j\| e_j\|$ is greater than $\frac{1}{2}\sum_1^k \frac{1}{j}$. Thus the geometric condition is violated.

§ 3. Complemented subspaces of ΣE. In this section we prove the following theorem:

Theorem 3. *Let E be almost dispersed. If X is an infinite-dimensional complemented subspace of E, then X is isomorphic to E.*

The proof follows the lines of Pełczyński's argument for (c_0), (l_p) $(1\leqslant p<\infty)$, but does not depend on the special properties of the particular norms involved. One lemma is needed.

Lemma 1. *With E as in theorem 3, let (z_n) be a sequence in E with mutually disjoint supports and $z_n\neq 0$ for every n. Then $[z_n]$ is isomorphic to E and complemented in E. If E is dispersed then $[z_n]$ is isometric to E and there is a norm one projection onto $[z_n]$.*

Proof. We may assume $\|z_n\|=1$ (since $[z_n] = z_n/\|z_n\|$). From the geometric condition, one obtains

$$C\Big\|\sum_{j=1}^n a_j e_j\Big\| \leqslant \Big\|\sum_{j=1}^n a_j z_j\Big\| \leqslant K\Big\|\sum_{j=1}^n a_j e_j\Big\|$$

for every n, so the mapping $T\big(\Sigma a_j e_j\big) = \Sigma a_j z_j$ is an isomorphism of E onto $[z_n]$, which is an isometry if $C=K=1$.

To construct the projection, let $\tau\colon N\to N\times N$ be one-to-one so that $Sz_n\in E_n$, the nth copy of E in ΣE, where S is the isomorphism induced by τ. The span of Sz_n is one-dimensional, so there is a norm one projection of E_n onto Sz_n's span. By proposition 3 of [13], $[Sz_n]$ is complemented in ΣE, say under a projection π, and $\|\pi\|=1$ if $C=K=1$. The projection $S^{-1}\circ\pi\circ S$ from E onto $[z_n]$ has the desired properties.

Using lemma 1, we use Pełczyński's argument ([13], p. 214) to obtain:

Lemma 2. *If E is as in theorem 3, and if X is an infinite-dimensional subspace of E, then X contains an infinite-dimensional complemented subspace of E.*

The proof of theorem 3 is now an immediate consequence of proposition 1 with the aid of lemma 2.

Corollary 3. *Let E and F have almost dispersed bases. Then F can be embedded in E if and only if F is isomorphic to E.*

Proof. Let T be an embedding (isomorphism into) of F into E. Then TF contains a subspace Y which is complemented in E and isomorphic to E. Then $T^{-1}Y$ is complemented in F, and so is isomorphic to F by theorem 3. The other direction is just proposition 1.

We mention the following without proof, since the arguments are routine. If the unit vector system (e_n) for a B-space E is an orthogonal basis, then E^*, as a sequence space, has the coefficient functionals $(f_n) \subset E_*$ as an orthogonal system. Thus, if (X_n) is a sequence of B-spaces, $(\Sigma_{E_*} X_n^*)$ is a B-space, and in fact $(\Sigma_{E_*} X_n^*)$ is isomorphic to $(\Sigma_E X_n)^*$.

§ 4. In this section we characterize almost dispersed and dispersed B and B_0 spaces (³).

The next theorem and its corollary characterize almost dispered sequence spaces.

THEOREM 4. *Let E be almost dispersed. Then E is either isomorphic to (m), or (e_n) is an unconditional basis for E. If E is non-separable, the isomorphism with (m) is such that $Te_n = \delta_n$, where (δ_n) denotes the unit coordinate basis of (m).*

Proof. The fact that (e_n) is an unconditional basis in the separable case was proved by Kadeč and Pełczyński [11].

In the non-separable case, (e_n) cannot be a basis, so there exists $x \in E$ such that for any finite set $\sigma \subset N$

$$\Big\|x - \sum_{n\in\sigma} f_n(x) e_n\Big\| \geqslant d > 0.$$

By orthogonality,

$$\Big\|\sum_{n\in\sigma} f_n(x) e_n\Big\| \leqslant \|x\|.$$

If $\sum_1^\infty f_n(x) e_n$ converged it would converge to x which it does not. Thus there exists an $\varepsilon > 0$ and $0 = n_0 < n_1 < \ldots$ such that

$$w_j = \sum_{n=n_{j-1}+1}^{n_j} f_n(x) e_n$$

satisfies $\|w_j\| \geqslant \varepsilon$. For any $\alpha = (a_j) \epsilon (m)$, define $\beta_{jn} = a_j$ if $n_{j-1} < n \leqslant n_j$, and denote by $\sum a_j w_j$ the sequence $(\beta_{jn} f_n(x)) \epsilon E$. Using orthogonality,

$$\varepsilon \|\alpha\|_{(m)} \leqslant \Big\|\sum a_j w_j\Big\| \leqslant \|x\| \, \|\alpha\|_{(m)}.$$

Now, let $u_j = \|w_j\|^{-1} w_j$, and let $\tau\colon N \to N \times N$ be defined by $\tau(k) = (j, k)$ if $n_{j-1} < k \leqslant n_j$. Let v_j be the element u_j in the jth copy of E in ΣE. By proposition 2, E is isomorphic to $\Sigma E(\sigma_j)$, where $E(\sigma_j) = \mathrm{sp}(e_{n_{j-1}+1}, \ldots \ldots, e_{n_j})$ in the jth copy of E in ΣE. Therefore, $y = (y_n) \epsilon E$ if and only if $(y_j v_j) \epsilon \Sigma E(\sigma_j)$, which occurs if and only if $\Sigma y_j u_j \epsilon E$. Therefore, (u_j) is equivalent to (e_n). This proves $E \sim (m)$ since (u_j) is equivalent to (δ_{nj}) in (m).

(³) Certain of these results were obtained simultaneously by the referee and the authors.

COROLLARY 3. *If E is a separable, non-reflexive, almost dispersed sequence space, then $E \sim (c_0)$ or $E \sim (l_1)$.*

Proof. A non-reflexive space with an unconditional basis (e_n) contains a copy of (c_0) or of (l_1) [10]. By corollary 2, E is isomorphic to either (c_0) or (l_1).

A. Pełczyński and W. Ruckle have pointed out, in private communications, that if E is dispersed and separable, then it is a P-space as in [5]. Therefore it is isometric to (c_0) or (l_p), $1 \leqslant p < \infty$.

Finally if E is non-normable and a B_0-space (Fréchet space) with basis (e_n), and almost dispersed, then it is isomorphic to (s) by theorem 7 of [4].

The following proposition was observed by the referee. A basis (x_n) for a Banach space x is called *perfectly homogeneous* [3] if every sequence of the form

$$z_n = \sum_{p_n+1}^{p_{n+1}} a_k x_k$$

with $0 < c_1 \leqslant \|z_n\| \leqslant c_2 < \infty$ is a basis for its closed span, $[z_n]$, equivalent to the basis (x_n) (such a sequence is called a *block basic sequence*). That is, the mapping $\sum a_i x_i \to \sum a_i z_i$ is an isomorphism of X onto $[z_n]$.

PROPOSITION 4. *The sequence space E is almost dispersed if and only if the unit vector basis (e_n) is perfectly homogeneous.*

Proof. The *only if* part is clear from the proof of lemma 1. Thus assume that (e_n) is perfectly homogeneous. Such a basis is symmetric [16], so we may assume in fact that

$$\|(a_j)\| = \sup_{|\delta_i|\leqslant 1, \varrho\epsilon P} \Big\|\sum \delta_i a_i e_{\varrho(i)}\Big\|.$$

Then the mappings $S_n((a_i)) = 0$ if $i \leqslant n$, a_{i-n} if $i \geqslant n+1$, are isometries. We shall prove that there are positive constants K_1, K_2 such that given a block basis sequence (z_n) with $\|z_n\| = 1$ for every n then $K_1\|\sum a^i e_i\| \leqslant \leqslant \|\sum a_i z_i\| \leqslant K_2\|\sum a_i e_i\|$. Using a technique found in [13], p. 215, it is then easy to show that the geometric condition is satisfied. Assume that such K_2 exists. Then there is a sequence of isomorphisms (T_n) such that $(T_n e_j)$ is a block basic sequence for each n and $\|T_n e_j\| = 1$ for all n, j, and for each n there is an element

$$\sum_1^{k_n} a_{jn} e_j = x_n$$

having norm $\leqslant 1$ such that $\|T_n x_n\| > n$. Let c_n be the largest integer in

$$\bigcup_{1}^{k_n} \operatorname{supp}(T_n e_j).$$

Then let $w_j = T_1 e_j$ for $j = 1, \ldots, k_1$ while $w_{k_1+\ldots+k_n+j} = S_{c_1+\ldots+c_n} T_{n+1} e_j$, $j = k_n+1, \ldots, k_{n+1}$. Then (w_j) is a block basic sequence such that $\|w_j\| = 1$ for all j and

$$\Big\| \sum_{j=k_1+\ldots+k_n+1}^{k_{n+1}} a_{j(n+1)} e_j \Big\| \leqslant 1$$

but

$$\Big\| \sum_{j=k_1+\ldots+k_n+1}^{k_{n+1}} a_{j(n+1)} w_j \Big\| > n.$$

In a similar way one shows that K_1 exists.

The following problems arise naturally:

PROBLEM 1. *Pełczyński has conjectured that the only B-spaces with perfectly homogeneous bases are isomorphic to (c_0) or (l_p) $(1 \leqslant p < \infty)$. The only* remaining part of this problem is: *If E is separable, almost dispersed and reflexive, is E isomorphic to some (l_p) $(1 < p < \infty)$?*

PROBLEM 2. *A wide class of complemented subspaces of (m) is known which contains subspaces isomorphic to (m),* [9]. *Are all complemented subspaces of (m) isomorphic to (m)?*

PROBLEM 3. *Does proposition 2 remain valid in the non-separable case?*

References

[1] M. G. Arsove and R. E. Edwards, *Generalized bases in topological linear spaces*, Studia Math. 19 (1960), p. 95-113.

[2] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[3] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), p. 151-164.

[4] C. Bessaga, A. Pełczyński and S. Rolewicz, *Some properties of the space (s)*, Colloq. Math. 7 (1959), p. 45-51.

[5] F. Bohnenblust, *An axiomatic characterization of L_p spaces*, Duke Math. J. 6 (1940), p. 627-640.

[6] W. J. Davis, *Dual generalized bases in linear topological spaces*, Proc. A. M. S. 17 (1966), p. 1057-1063.

[7] M. M. Day, *Normed linear spaces*, New York 1962.

[8] D. W. Dean, *Projections in certain continuous function spaces*, Canad. J. Math. 14 (1962), p. 385-401.

[9] — *Subspaces of $C(H)$ which are direct factors of $C(H)$*, Proc. A. M. S. 16 (1965), p. 237-242.

[10] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. Math. 52 (1950) p. 518-527.

[11] M. Kadeč and A. Pełczyński, *Basic sequences, biorthogonal systems and norming sets in Banach and Fréchet spaces*, Studia Math. 25 (1965), p. 297-323.

[12] A. Markushevich, *Sur les bases (au sens large) dans les espaces linéaires*, Dokl. Akad. Nauk S. S. S. R. 41 (1943), p. 227-229.

[13] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), p. 209-228.

[14] I. Singer, *On a theorem of I. M. Gelfand*, Uspekhi Mat. Nauk 17 (1962, p. 169-176 (Russian).

[15] — *On Banach spaces with symmetric bases*, Revue de Math. Pures. et Appl., Acad. R. P. R. 6 (1961), p. 159-166 (Russian).

[16] — *Some characterizations of symmetric bases in Banach spaces*, Bull. Acad. Pol. Sci. 10 (1962), p. 185-192.

Reçu par la Rédaction le 25. 5. 1966