The direct sum of Banach spaces with respect to a basis

by

WILLIAM J. DAVIS and DAVID W. DEAN (Columbus, Ohio)

§ 6. Introduction. Let $E$ be a Banach space of sequences such that if $(a_n) \in E$ and if $|a_n| \leq |a_m|$ for all $n$, then (i) $(a_n) \in E$ and (ii) $\|\langle a_n \rangle\| \leq \|\langle a_m \rangle\|$ (*) . In [13] A. Pelczyński introduced the notion of a countable sum of Banach spaces with respect to $E$. Specifically, if $(X_n)$ is a sequence of Banach spaces, then $\Sigma_{E} X_n$ is the space of sequences $(a_n)$ with $a_n \in X_n$ for each $n$, and with $\|\langle a_n \rangle\|_E$, where $\|a_n\|$ is the norm of $a_n$ in $X_n$. Defining a norm on this space by $\|\langle a_n \rangle\|_E = \|\langle |a_n| \rangle\|$, and using coordinate-wise addition and scalar multiplication, $\Sigma_{E} X_n$ is a Banach space.

Of particular interest is the case in which $X_n = E$ for each $n$, and $\Sigma_{E} E$ is isometric (isomorphically isomorphic) in a natural way to $E$. Examples of such spaces are $(\ell_0)$, $(\ell_p)$ $(1 \leq p < \infty)$, $(m)$. The Fréchet space $(s)$ is isomorphic to $\Sigma_{E} \ell_0$.

Using this isometry and other special properties of these examples, Pelczyński has penetrated deeply into their structures. In each of these examples, the sequence $(a_n)$, where $a_n = (\langle a_n \rangle)_{j=1}^{m}$, is a basis (or in the case of $(m)$, a generalized basis) satisfying (i) and (ii) above (*).

Let $q_j$ be a subsequence of the positive integers $N$ for each $j$ such that $\bigcup_{j} q_j = N$ and $q_j \cap q_k = \emptyset$ if $j \neq k$. Let $E(q_j) = \{x \in E \mid a_k = 0 \text{ if } k \notin q_j\}$.

If $E$ is $(q_1)$, $(q_2)$ $(1 \leq p < \infty)$, each $E(q_j)$ is isometric under the natural mapping $T$ such that $Tq_j = q_j$ where $q_j$ is the $j$-th element of $q_j$. Moreover, $E(q_j)$ is isometric to $\Sigma_{E} E(q_j)$ in a natural way. Another way to say this is as follows. Let $\tau : N \to N \times N$ be one to one and onto. Let $E_j = E(\{\{i \in N \mid \tau(i) = (j, k)\} \text{ for some } k\})$. Then $E$ is isometric to each $E_j$ and to $\Sigma_{E} E_j$. Moreover, the isometry $T$ of $E$ with $\Sigma_{E} E_j$ has the property that $Tq_j = q_k$ if $\tau(q) = (j, k)$, where $q_k$ is the $k$-th coordinate sequence in the $j$th copy of $E$. If every such $\tau$ induces an isometry of $E$ with $\Sigma_{E} E_j$, say that $E$ is dispersed. If every such $\tau$ induces an isomorphism $T$ such that $\|T\| < k$ where $k$ is independent of $\tau$, say the space is almost dispersed. If

(*) Every Banach space with a separable conjugate space has a generalized basis. Then every such space has a representation as a space of sequences.

(*) We shall say that such a basis or generalized basis is orthogonal [14].

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there is a \( \tau \) inducing an isomorphism of \( E \) with \( \Sigma_\infty E \), say \( E \) is weakly dispersed. The word induces means the following. If \( (a_{\alpha})_{\alpha} \in E_{\alpha} \) is a sequence in \( \Sigma_\infty E_{\alpha} \), \( (a_{\alpha})_{\alpha,\beta} \in E_{\beta} \), then this sequence is mapped to the sequence \( (a_{\alpha,\beta})_{\alpha,\beta} \) in \( E \).

With this structure Pełczyński ([13], see also [8], p. 398) has proved:

**Proposition 1.** Let \( E \) be weakly dispersed. If \( X \) is a complemented subspace of \( E \) and if \( E \) is isomorphic to a complemented subspace of \( X \), then \( E \) is isomorphic to \( X \).

The space \( \Sigma_\infty E \) is at the center of this discussion. In § 1 it is shown that \( E \) is almost dispersed if and only if it is weakly dispersed and symmetric (defined to be the natural extension of Singer's symmetric condition on bases ([15], [16])). In § 2 the almost dispersed concept is characterized geometrically. Using this characterization, it is shown in § 4 that these spaces are isomorphic to \((e_i), (e_j), (e_k), (e_0)\), or reflexive, and if the space is dispersed then isometric to \((e_i) (1 \leq p \leq \infty) \) or \((e_0)\). Further it is shown that if \((e_0)\) is a basis, then almost dispersed and perfectly homogeneous are the same. In § 3 it is noted that many of the properties established in [13] hold for these spaces.

§ 1. Symmetric spaces and dispersed generalized bases. Let \( P \) denote the set of permutations \( q: N \rightarrow N \). Suppose for each sequence \((a_{\alpha})_{\alpha} \in E \) and each sequence \((\delta_{\alpha})_{\alpha} \) such that \(|\delta_{\alpha}| \leq 1 \) for each \( \alpha \) that one has \((\delta_{\alpha} a_{\alpha}) \in E \). Following Singer ([15], [16]), define \( ||(a_{\alpha})|| = \sup \{ ||(\delta_{\alpha} a_{\alpha})|| : |\delta_{\alpha}| \leq 1 \} \) for each \( \alpha, \gamma \in P \). If \((e_0)\) is a basis for \( E \), then \( ||(a_{\alpha})|| = \sup \{ ||\sum \alpha_a e_a|| \} \) for \( e_a \in E \).

If this norm is equivalent to the old, say that the generalized basis \((e_0)\) is a symmetric system for \( E \). If \( ||\ldots||_1 = \ldots = ||\ldots||_k \), say that \( E \) is symmetric.

There is a close relation between a symmetric \( E \) and an almost dispersed \( E \). For example, it is immediate that each symmetric generalized basis is orthogonal. If \((e_0)\) is in fact a basis for \( E \), then \( E \) is collapsing. That is:

**Proposition 2.** Let \((e_0)\) be a basis for \( E \) which is a symmetric system and let \((m_0) \subset N \) be a subsequence of \( N \). Then the mapping \( \tau_{m_0} = e_{m_0} \) induces an isomorphism of \( E[(m_0)] \) onto \( E \) which is an isometry if \( E \) is symmetric.

We remark here that if \( E \) is separable and the generalized basis \((e_0)\) is orthogonal, then \((e_0)\) is a basis ([11]).

**Proof of Proposition 2.** It is clear that for arbitrary sequences \((a_{\alpha})\) of scalars, and \((m_0), (n_0) \subset N \), one has\n
\[
\| \sum_{\alpha} a_{\alpha} e_{n_0} \| = \| \sum_{\alpha} a_{\alpha} e_{m_0} \|
\]

In particular, set \( m_0 = i \) for all \( i \). Thus,\n
\[
u_N = \sum_{i=1}^{N} a_{\alpha} e_i
\]

determines a Cauchy sequence in \( E \) such that\n
\[
u_N \rightarrow u = \sum_{i=1}^{\infty} a_{\alpha} e_i \quad \text{as} \quad N \rightarrow \infty.
\]

Furthermore,\n
\[
\| u \| = \| \sum_{i=1}^{\infty} a_{\alpha} e_{\alpha} \|.
\]

If we take mutually disjoint sequences \((a_{\alpha}^j)\), \( j = 1, 2, \ldots \), of positive integers, it is clear that a symmetric space \( E \) contains an infinite number of mutually "orthogonal" copies of itself, which might cause us to hope that \( E \) is \( \Sigma_\infty E \), where \( E = E[(a_{\alpha}^j)] \). In section 4 we give a counterexample to this supposition. One does, however, obtain the following.

**Theorem 1.** The sequence space \( E \) is almost dispersed if and only if \((e_0)\) is a symmetric system and \( E \) is weakly dispersed.

**Proof.** Let \( E \) be almost dispersed and \( \gamma P \). Given any \( \alpha \) and \( \beta \), there exists \( \tau_\alpha \) such that \( \tau_\alpha \gamma \cap \tau_\beta = \emptyset \). Let \( S \), \( S_\alpha \), \( S_\beta \), be the isomorphisms of \( E \) onto \( \Sigma_\infty E \) induced by \( \gamma \) and \( \tau_\alpha \), respectively. Then \( S^{-1} S_\alpha \beta \) is an isomorphism of \( E \) onto itself satisfying \( (S^{-1} S_\alpha \beta)_{\gamma} = \alpha_{\beta} \). Thus, from the definition of almost dispersed, there are constants \( c, d \) such that for every sequence \((a_{\alpha})\) \( E \) one has\n
\[
\| a_{\alpha} \| \leq \sup \{ \| a_{\alpha} + \beta \| : \| a_{\alpha} \| \}
\]

Conversely, assume that \( E \) is weakly dispersed and \((e_0)\) is a symmetric system. Renorming with \( ||\ldots||_1 \) is symmetric and weakly dispersed. The latter condition guarantees a \( \tau \) inducing an isomorphism of \( E \) with \( \Sigma_\infty E \). Any other \( \tau_\alpha \) is of the form \( \gamma P \) for some \( \varrho \) in \( P \) and the induced mappings \( S \), \( \varrho \), and \( T \) for \( \gamma \) have composition \( S \circ T \) with norm bounded independent of \( \varrho \). Thus \( T_\alpha \), \( S \circ T \), the induced mapping for \( \tau_\alpha \), has norm bounded independent of \( \tau_\alpha \).

The following example shows that weakly dispersed does not imply almost dispersed. The space is isometric to \((e_0)\). This makes it clear that the notion of dispersed depends on the basis chosen as well as the space itself.

**Example 1.** Let \((a_{\alpha})\) be the sequence \((1, 2, 1, 2, 4, 1, \ldots) \). Let \( E \) be the Banach space of sequences \((\beta_{\alpha})\) such that \((\alpha_{\beta} \beta_{\alpha})_{\gamma} = \alpha_{\beta} \) with norm \( \| \beta_{\alpha} \| = \| \alpha_{\beta} \| \). The coordinate basis \((e_0)\) is not symmetric since shifting a sequence may change its norm by large amounts. Let \( \tau : N \rightarrow N \times N \)
be any one to one, onto mapping such that $a_n = a_0 a_{n}$ if $\tau(n) = (j, k)$, Define $T: E \to \Sigma E$ by means of this $\tau$. Thus, $T(\beta_0) = \beta_0$ where $\beta_n = \beta_{0}$ if $\tau(n) = (j, k)$. Therefore,
\[
\left\| \left( \begin{array}{c} \beta_0 \\ \beta_j \\ \beta_k \\ \vdots \\ \beta_n \\
\end{array} \right) \right\|_E = \left\| \left( \begin{array}{c} \beta_0 \\ \beta_j \\ \beta_k \\ \vdots \\ \beta_n \\
\end{array} \right) \right\|_\Sigma E = \sup_{1 \leq i \leq \infty, 1 \leq j \leq \infty} a_i a_j a_k = \sup_{1 \leq i \leq \infty, 1 \leq j \leq \infty} a_i a_j a_k = \sup_{1 \leq i \leq \infty, 1 \leq j \leq \infty} a_i a_j a_k = ||\beta_0||.
\]
Thus, $T$ is an isometry. To see that $T$ is onto, notice that all processes above are reversible. If follows from theorem 1 that $(a_0)$ is not almost dispersed. This may be seen directly by choosing $s$ such that $\tau(s) = (l, k)$ if $a_0 = 1$ and $\tau(s) \neq (l, k)$ if $a_0 > 1$. Then, the sequence $(1/n_a)_E E$, but $T((1/n_a)_E E)$ since $||\beta_0|| = \infty$, where $T((1/n_a)_E E) = (\beta_0)$.

§ 2. A geometric condition for a dispersed system. Basic to the concept of a dispersed or weakly dispersed system is the statement: If $a_k = (n \cdot \chi(n) = (k, i)$ for some $i$) and if $x$ and $y$ are elements of $E$ such that $f_i(x) = 0$ if $j \neq i$ and $f_j(y) = 0$ if $j \neq m$ (i.e., if $x$ and $y$ are in different copies of $E$ in $\Sigma E$), then $0 \leq ||x + y|| \leq ||x|| + ||y||$, $||x + y|| \leq ||x|| + ||y||$. For a dispersed system, $C = K = 1$ and the system is symmetric, so $||x + y|| = ||x|| + ||y||$ for every $x \neq y$. The following proposition is useful in section 3 and in characterizing dispersed and almost dispersed systems using geometric conditions of the above type.

Proposition 3. Let $E$ be almost dispersed and let $\varphi: N \to N$ be one to one. Then $X(\varphi(N))$ is isometric to $X(\varphi(N), 1) = ((a_0)_E E, a_0 = 0$ if $\varphi(N)$ is isometric to $E$ under an isomorphism $S$ such that $a(k) = a_0(k)$.

Proof. Note first that $X(\varphi(N))$ is isometric to $X(\varphi(N), 1) = ((a_0)_E E, a_0 = 0$ if $\varphi(N)$ is isometric to $E$ under an isomorphism $S$ such that $a(k) = a_0(k)$. Now suppose that $N \neq \varphi(N)$ is infinite. Let $T: N \to N \times N$ satisfy $\tau\varphi(n) = (n, 1)$ for each $n$. If $T$ is the restriction to $X(\varphi(N))$ of the isomorphism of $E$ onto $\Sigma E$ by $\tau$ and $(a_k)_E E$, then $T(\varphi(n)) = ((a_0)_E E, \varphi(n) = 0$ unless $i = \varphi(n)$ and $\tau(i) = (j, k)$ where $k = 1$ and $j = n$, or $T(\varphi(n)) = (X(N), 1)$, which is isometric to $E$, letting $\varphi(n) = (a_0)_E E$.

Next suppose that $N \neq \varphi(N)$ is finite. Embed $X(\varphi(N))$ as $X(\varphi(N), 1)$ in $\Sigma E$ and choose $\tau: N \to N \times N$. Then $1(\varphi(N), 1) = \varphi(N)$ is $N \times N$, infinite and, by the above, $X(\varphi(N))$ is isometric to $E$, say under $T$, while $\varphi^{-1}$ induces an isomorphism $T^{-1}$ of $X(\varphi(N), 1)$ with $X(N)$. The mapping $T^{-1} = S$ is the one we seek, where $S$ is the embedding $X(\varphi(N))$ to $X(\varphi(N), 1)$.

We remark that $||S||$, $||S^{-1}||$ are bounded independently of $\varphi$ since $||T||$, $||T^{-1}||$ are so bounded.

Corollary 1. Let $E$ have orthogonal generalized basis $(a_k)$. Then $E$ is almost dispersed if and only if for every one to one, onto $\tau: N \to N \times N$ induces an isomorphism $T$ from $E$ into $\Sigma E$ such that $||T||$, $||T^{-1}||$ are bounded independently of $\tau$.

Proof. Sufficiency is immediate since the set of onto $\tau$ is included. Thus let $E$ be almost dispersed. Let $\tau: N \to N \times N$ be one to one and let $\tau_1: N \to N \times N$ be one to one and onto. Then $\tau_1 \circ \tau: N \to N$ is one to one. Let $\tau_1 \circ \tau = e$ so $\tau = \tau_1 \circ e$. Let $S$ and $T_1 \circ \tau$ be the isomorphisms induced by $\tau$ and $\tau_1$, respectively. Then $T_1 \circ \tau$ is the desired isomorphism and is norm bounded independently of $\tau$.

The geometric condition we are interested in is contained in the statement of the next theorem. For $x \in E$ let the support of $x$ (support) be $\{a \neq 0\}$. Say that $x$ and $y$ are disjoint if support $\cap$ support $\neq \varnothing$.

Theorem 2. The following are equivalent. (i) $E$ is almost dispersed. (ii) The generalized basis $(a_k)$ is orthogonal and there exist constants $C, K > 0$ such that, if $a_1, \ldots, a_k, y_1, \ldots, y_n$ are elements of $E$ satisfying support $(a_k) \cap$ support $(y_1) \cap \ldots \cap$ support $(y_n)$ whenever $i \neq j$, and such that $||a_1|| = ||a_k||$ (i.e., 1, \ldots, n), then $C||a_1 + + \ldots + a_n|| \leq ||y_1 + + \ldots + y_n|| \leq K||a_1 + + \ldots + a_n||$.

Proof. Suppose that $E$ is almost dispersed. Let $a_1, a_k$ be as in (ii) and let $\tau_1: N \to N \times N$ be one to one, into maps such that $\tau_1$ support $a_i \subset$ support $(y_1) \cap \ldots \cap$ support $(y_n)$ whenever $i \neq j$. Then $S$ and $T_1 \circ \tau$ be the isomorphisms induced by $\tau_1$ and $\tau_1$ as guaranteed by corollary 1. There are constants $A, B > 0$, independent of $\tau, \tau_1$, such that

$||a_1 + + \ldots + a_n|| \leq B||T_1 a_1 + + + + T_1 a_n||$

$||a_1 + + + + a_n|| \leq B||a_1 + + + + a_n||$

Now assume that (ii) is true. It will be shown in § 4, independently of this argument, that if $(a_k)$ is not a basis for $B_1$, then $E$ is isomorphic to $(m)$ in the natural way $(G_{n}(a_k))_{E} \subset (||a_k||)_{E} \subset K_{n}(a_k)_{E}$ for every $(a_k) \neq B_1$.
Thus we assume here that \((e_n)\) is a basis for \(E\). For any \((a_i)\) \(\in E\) condition (ii) promises that \(C|\sum \frac{a_i}{\|a_i\|} | \leq K |\sum \frac{a_i e_{\|a_i\|}}{\|a_i\|}\|\|\|\text{ for every } n \text{ and } a \in E\). Thus \((e_n)\) is a symmetric basis. Let \(a_n = \sum a_i\|a_i\|\), \(i \neq j\). If \(a_n = 0\) then \(\|a_n\| = \|a\|\). For \(x = (a_i) e_j\) let \(\alpha_i = a_i 0\) \(\|a_i\|\text{ and } \alpha_i = 00\) \(\|a_i\|\text{ if } j \neq i\). Let \(r : N \rightarrow N \times N \times E_{\|a\|} = \{(i, j, k) : k \leq N\}.\) Since \((e_n)\) is a basis, \(x = \sum \frac{a_i}{\|a_i\|} e_n\) and by condition (ii) \(C|\sum \frac{a_i}{\|a_i\|} | \leq K \sum \frac{1}{\|a_i\|}\|a\|\|\|\text{ for every } n\), or \(C|\|a\|\| \leq K \sum \frac{1}{\|a_i\|}\|a\|\|\|\text{ so that } \|a_i\| e_n E\text{ and the mapping } T \text{ induced by } r \text{ is continuous.}\) For any \((a_i) e_n E\) let \(a_n = a_i\) if \(i = j\), \(a_i = 0\) if \(i \neq j\). Then \((a_n) E\) and \(T((a_n)) = (a_n)\). By theorem 1 we conclude the argument.

**Example 2.** Using the geometric conditions we can now construct a space which is symmetric but not almost dispersed, and so not weakly dispersed. Similar examples have been constructed by Singer (16).

Let \(d\) be the linear space of real sequences \(a = (a_i)\) with norm

\[
\|a\| = \sup_{e_j} \sum_{i=1}^{\infty} \frac{|a_i|}{j}.
\]

Then \((e_n)\) is a symmetric basis for \(d\) which is boundedly complete so that \(d\) is a conjugate space, and so it is not isomorphic to \((e_n)\). If \((y_n)\) is biorthogonal to \((e_n)\) and \((f_i)\) is the natural basis for \(l_1 = e_n^*\) then \((d,e_n) \sim (f_i)\) \(= 0\) so that \(d\) is not separable (7), p. 79). Further \((e_n)\) converges weakly to 0 but not in norm as follows: Otherwise there exists \(f \in (d)^*\) and a sequence \((f_n)\) such that \(|f_n| > \epsilon > 0\) for some \(\epsilon > 0\) and some subsequence \((y_m)\) of \(N\). Then \(u = \sum_{j=1}^{\infty} \frac{1}{j} e_n f_n, f(u) = \infty\). Thus \((d)\) is not isomorphic to \((l_1)\) (7), p. 33). In a private communication J. R. Retherford has shown that \((d)\) is not reflexive.

To see that \((d)\) is not almost dispersed, fix \(k, m \in N\) with \(m > 1\). Let \(\beta_i = (\log mk)^{-1}\) if \(j = m(i-1)+1, \ldots, m, \beta_i = 0\) if \(j \neq 0\) \((m)\). Then \(\|b_i\| = (\log mk)^{-1} m^{-1} \sum_{j=1}^{\infty} \beta_j\) while \(\|b_i\| = (\sum_{j=1}^{\infty} m^{-1}) (\log mk)^{-1}\) which converges as \(m \rightarrow \infty\). Now

\[
\|b_i\| = (\log mk)^{-1} m^{-1} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} e_k.
\]

For large \(k\) and very large \(a_i\), \(\|b_i\|\) is near 1 while \(\|b_i\|\) is greater than \(\frac{1}{2} \sum \frac{1}{j}\). Thus the geometric condition is violated.

**§ 3. Complemented subspaces of \(\Sigma E\).** In this section we prove the following theorem:

**Theorem 3.** Let \(E\) be almost dispersed. If \(X\) is an infinite-dimensional complemented subspace of \(E\), then \(X\) is isomorphic to \(E\).

The proof follows the lines of Pelczyski’s argument for \((e_n)\), \((f)\) (1 < \(p < \infty\)), but does not depend on the special properties of the particular norms involved. One lemma is needed.

**Lemma 1.** With \(E\) as in theorem 3, let \((x_n)\) be a sequence in \(E\) with mutually disjoint supports and \(x_n = 0\) for every \(n\). Then \((x_n)\) is isomorphic to \(E\) complemented in \(E\). If \(E\) is dispersed then \((x_n)\) is isometric to \(E\) and there is a norm one projection onto \((x_n)\).

**Proof.** We may assume \(\|x_n\| = 1\) (since \(\|x_n\| = \|x_n\|\)). From the geometric condition, one obtains

\[
C\|\sum \frac{a_i}{\|a_i\|} \| \leq K \sum \frac{1}{\|a_i\|}\|a\|\|\leq K \sum \frac{1}{\|a_i\|}\|a\|\|\text{ for every } n, \text{ so that the mapping } T[\Sigma a_n e_n = \Sigma a_n e_n] \text{ is an isomorphism of } E \text{ onto } (e_n) \text{ which is an isometry if } C = K - 1.
\]

To construct the projection, let \(r : N \rightarrow N \times N \times E_{\|a\|} = \{(i, j, k) : k \leq N\}.\) The span of \(S_{x_n} E_n\) is one-dimensional, so there is a norm one projection onto \(S_{x_n} E_n\). By proposition 3 of (13), \((S_{x_n} E_n)\) is complemented in \(\Sigma E_n\) under a projection \(\pi\), and \(\|\pi\| = 1\) if \(C = K - 1\). The projection \(\pi^{-1} \cap \pi 0\) from \(E\) onto \((e_n)\) has the desired properties.

**Lemma 2.** If \(E\) is as in theorem 3, and if \(X\) is an infinite-dimensional subspace of \(E\), then \(X\) contains an infinite-dimensional complemented subspace of \(E\).

The proof of theorem 3 is now an immediate consequence of proposition 1 with the aid of lemma 2.

**Corollary 3.** Let \(E\) and \(F\) have almost dispersed bases. Then \(F\) can be embedded in \(E\) and only if \(F\) is isomorphic to \(E\).

**Proof.** Let \(T\) be an embedding (isomorphism into) of \(F\) into \(E\). Then \(TF\) contains a subspace \(Y\) which is complemented in \(E\) and isomorphic to \(E\). Then \(T^{-1} Y\) is complemented in \(F\), and so is isomorphic to \(F\) by theorem 3. The other direction is just proposition 1.
We mention the following without proof, since the arguments are routine. If the unit vector system \((s_n)\) for a \(B\)-space \(E\) is an orthogonal basis, then \(E^*\), as a sequence space, has the coefficient functionals \((f_n) \subset E^*\) as an orthogonal system. Thus, if \((X_n)\) is a sequence of \(B\)-spaces, \((\Sigma_n X_n^*)\) is a \(B\)-space, and in fact \((\Sigma_n X_n^*)^*\) is isomorphic to \((\Sigma_n X_n^*)\).

\[ \text{§ 4. In this section we characterize almost dispersed and dispersed spaces.} \]

The next theorem and its corollary characterize almost dispersed sequence spaces.

**Theorem 4.** Let \(E^d\) be almost dispersed. Then \(E\) is either isomorphic to \((m)\), or \((s_n)\) is an unconditional basis for \(E\). If \(E\) is non-separable, the isomorphism with \((m)\) is such that \(T_{s_n} = \delta_{n}\), where \(\delta_n\) denotes the unit coordinate basis of \((m)\).

**Proof.** The fact that \((s_n)\) is an unconditional basis in the separable case was proved by Kadec and Pelczynski [11].

In the non-separable case, \((s_n)\) cannot be a basis, so there exists \(\sigma E\) such that for any finite set \(\sigma \subset \mathbb{N}\)

\[ \|x - \sum_{n \in \sigma} f_n(x)e_n\| \geq d > 0. \]

By orthogonality,

\[ \|\sum_{n \in \sigma} f_n(x)e_n\| \leq \|x\|. \]

If \(\sum f_n(x)e_n\) converges to \(x\) which it does not. Thus there exists an \(\varepsilon > 0\) and \(0 = n_0 < n_1 < \cdots\) such that

\[ v_\sigma = \sum_{n = n_0}^{n_\sigma} f_n(x)e_n \]

satisfies \(\|v_\sigma\| \geq \varepsilon\). For any \(\alpha = (a_\sigma)(x)\), define \(\beta_{\sigma} = a_i\) if \(n_{i-1} < \sigma < n_i\), and denote by \(\Sigma a_\sigma\) the sequence \((\beta_{\sigma}f_{\sigma}(x))\). Using orthogonality,

\[ \|a_\sigma\| \leq \sum_{\sigma \in \sigma} \|a_\sigma\| \leq \|v_\sigma\| \leq \|v_\sigma\|. \]

Now, let \(u_\sigma = \|a_\sigma\|^*v_\sigma\) and let \(\tau : N \to N \times \mathbb{N}\) be defined by \(\tau(k) = (j, k)\) if \(n_{j-1} < k < n_j\). Let \(v_j\) be the element \(u_j\) in the \(j\)th copy of \(E\) in \(\Sigma E\). By proposition 2, \(E\) is isomorphic to \(\Sigma E\), where \(E(\alpha) = (v_\sigma)_{\sigma = 1}^{\infty} \cdots \).

Therefore, \(y_{\tau} = (y_{\sigma})_{\sigma = 1}^{\infty} \cdots \).

Then the mappings \(S_n(\alpha) = 0\) if \(i < n\), \(\alpha_{n}\), if \(i = n+1\), are isometries. We shall prove that there are positive constants \(K_1, K_2\) such that given a block basis sequence \((\alpha_n)\) with \(\|\alpha_n\| = 1\) for every \(n\) and \(K_1\|\sum \alpha_{n}\| \leq \sum \|\alpha_{n}\| \leq K_2\|\sum \alpha_{n}\|\).

A technique found in [13], p. 215, it is then easy to show that the geometric condition is satisfied. Assume that such \(K_1\) exists. Then there is a sequence of isomorphisms \((T_n)\) such that \((T_n\alpha)\) is a block basic sequence for each \(n\) and \(\|T_n\alpha\| = 1\) for all \(n\), \(j\), and for each \(\alpha\) there is an element

\[ \sum_{i=1}^{n} a_{ij} j_i = \alpha_n. \]
having norm \( \leq 1 \) such that \( \| T_n x_n \| > n \). Let \( c_n \) be the largest integer in
\[
\bigcup_{j} \text{supp}(T_n x_j).
\]
Then let \( w_j = T_n x_j \) for \( j = 1, \ldots, k_j \) while \( w_{k_1} + \ldots + w_{k_{j+1}} = \sum_{i=1}^{k_{j+1}} c_i x_i \), given
\( j = k_{j-1} + 1, \ldots, k_j \). Then \( (w_j) \) is a block basic sequence such that \( \| w_j \| = 1 \)
for all \( j \) and
\[
\left\| \sum_{i=k_{j-1} + 1}^{k_j} c_i x_i \right\| \leq 1
\]
but
\[
\left\| \sum_{i=k_{j-1} + 1}^{k_j} c_i x_i \right\| > n.
\]
In a similar way one shows that \( K_1 \) exists.

The following problems arise naturally:

**Problem 1.** Pelczyński has conjectured that the only \( B \)-spaces with perfectly homogeneous bases are isomorphic to \( (c_0) \) or \( (l_p) \) \((1 \leq p < \infty)\). The only remaining part of this problem is: If \( E \) is separable, almost dispersed and reflexive, is \( E \) isomorphic to some \( (l_p) \) \((1 < p < \infty)\)?

**Problem 2.** A wide class of complemented subspaces of \( (m) \) is known which contains subspaces isomorphic to \( (m) \), [9]. Are all complemented subspaces of \( (m) \) isomorphic to \( (m) \)?

**Problem 3.** Does proposition 2 remain valid in the non-separable case?

References