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On simultaneous extension of infinitely differentiable functions

by

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Introduction. The first result concerning the extension of differentiable functions is due to Lichtenstein [8], who has shown that any differentiable function on a closed set in R^3 whose boundary satisfies some conditions can be extended to a differentiable function in R^3. Whitney [15] has proved that any function $f(x) = f(x_1, \ldots, x_n)$ of class C^m on a closed subset of R^n can be extended to a function of class C^m in R^n. In the case $m = \infty$ this extension is not linear. Hestenes [6] has modified Whitney's proof, and has given also another method of extending the differentiable functions, which is applicable only when m is finite and the boundary has suitable properties. It is, however, sufficiently general to be of interest and the proof is relatively simple. The method used is a generalization of the reflection principle used by Lichtenstein [8] by the use of McShane's [9] lemma on the extension localisation.

Next, Mitjagin [10], and Ryll-Nardzewski (not published) have shown independently the existence of the linear continuous extension operator $L\colon C^\infty[a, b] \to C^\infty(R)$ $(a < b)$.

Seeley [14] has constructed a linear and continuous extension operator from the space of functions of class C^∞ on a half-space to the space of functions of class C^∞ on the whole R^n.

Recently Adams, Aronszajn and Smith (announced in [1]) obtained a complete result concerning the existence of extension, namely a necessary and sufficient condition for a convex domain D to possess the property of extension is that for a certain bounded cone C and for every $x \in D$ there be a congruent cone with a vertex x, contained in $\bar{D}$.

The first results concerning the linear extension operators of the continuous functions are contained in the papers of Borsuk [2] and Dugundji [3].

In this paper we prove: (1) the existence of a linear operator extending the functions of class C^∞ considered on the closed subset of R^n to the functions of class C^∞ on the whole R^n, (2) the existence of a linear operator extending the functions of class C^∞ defined on the closed subsets with the boundary satisfying some conditions of the n-dimensional manifold of class C^∞ to the functions of class C^∞ on the manifold.

Our method is similar to that of Hestenes: application of partition of unity and reduction to the case where the set under consideration is an n-dimensional cube. The linear extension operator for the cube is constructed by several applications of Ryll-Nardzewski's operator giving the extension in the one-dimensional case.

In Section 3 we give the application of extension operators. It is proved that every space of infinitely differentiable functions on a compact n-dimensional manifold of class C^∞ admitting partitions of unity of class C^∞ and possessing a cube division satisfying a certain condition is isomorphic with $C^\infty(I)$. This generalizes the earlier results of Mitjagin [10] and Grothendieck [4].

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1. Notation and terminology. R^n will denote, as usual, the n-dimensional real space; its elements are n-tuples $x = (x_1, \ldots, x_n)$. I^n will denote the unit cube in R^n, i.e.

$$I^n = \{x \in R^n: \ 0 \leqslant x_i \leqslant 1, \ i = 1, \ldots, n\}.$$

Throughout the paper, E will denote a linear topological space which will be assumed to be locally convex and sequentially complete ([1]).

Let A be an open subset of R^n: we shall say that the continuous function $f\colon A \to E$ is of class C^m $\big(f \in C^m_E(A)\big)$ for some integer m ($0 \leqslant m < < +\infty$) if there exist continuous partial derivatives $D^\mu f = D_1^{\mu_1} D_2^{\mu_2} \ldots D_n^{\mu_n} f$ for $|\mu| \leqslant m$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$; $|\mu| = \mu_1 + \mu_2 + \ldots + \mu_n$; μ_i is a non-negative integer and $D_i^{\mu_i}$ denotes Fréchet's derivative of order μ_i with respect to x_i. The function f is of class C^∞ on A $\big(f \in C^\infty_E(A)\big)$ if $f \in C^m_E(A)$ for every $m = 0, 1, 2, \ldots$ We shall write $C_E(A)$ instead of $C^0_E(A)$.

The topology in $C^m_E(A)$ is the topology of uniform convergence on the compact subsets of A in every pseudonorm in E.

In the case where A is a closure of an open subset of R^n we say that $f \in C^m_E(A)$ (m finite if all the derivatives $D^\mu f$ exist for $|\mu| \leqslant m$, are uniformly continuous in the interior of A and can be uniquely extended onto A. But since E is sequentially complete, it follows that if the function f from the open subset of R^n into E is uniformly continuous, then f has

([1]) A locally convex space is *sequentially complete* if for every sequence $\{u_n\}$, $n = 1, 2, \ldots$, of elements of this space the fact that $\sum_{n=1}^{\infty} \|u_n\| < +\infty$ for every pseudonorm implies that $\sum_{n=1}^{\infty} u_n$ exists and is an element of this space.

a unique extension to the closure of A. Thus $C^m_E(A)$ where A is a closure of an open subset of R^n denotes the linear topological space of the functions from A into E, which are uniformly continuous in the interior of A with all their derivatives. The function f is of class C^∞ on A $(f \in C^\infty_E(A))$ if $f \in C^m_E(A)$ for every $m = 0, 1, \ldots$

All theorems in this paper remain true also if we replace the notion of a Fréchet derivative and differentiability by weak Fréchet's derivative, i.e. if $g = D^\mu f$ means that for every $e^* \in E^*$ $e^* g$ is the value of a suitable Fréchet derivative of the scalar function $e^* f$.

A topological Hausdorff space X is said to be an *n-dimensional manifold of class C^p* ($p = 0, 1, \ldots, \infty$) if there exist a collection of pairs $(U_i, \varphi_i)_{i \in I}$ (which we shall call an *atlas*) satisfying some conditions (see Helgason [5]) where U_i is an open subset of X, φ_i is a homeomorphism from U_i onto an open subset of R^n, and $\{U_i\}_{i \in I}$ covers X.

Let T be a subset of an n-dimensional manifold of class C^p, which either is open or is a closure of an open subset. A function $f\colon T \to E$ is said to be in $C^m_E(T)$ ($m \leqslant p$) if for every $x \in T$ there exists a chart (U_i, φ_i) with $x \in U_i$ such that $f \circ \varphi_i^{-1}$ is in $C^m_E\big(\varphi_i(U_i \cap T)\big)$.

We take that $f_n \to f$ $\big(f_n, f \in C^m_E(T)\big)$ in $C^m_E T$ if $f_n \circ \varphi_i^{-1} \to f \circ \varphi_i^{-1}$ in $C^m_E\big(\varphi_i(U_i \cap T)\big)$ for every $i \in I$.

Let S and T be the subsets of an n-dimensional manifold of class C^∞ such that S is a closed subset of T (in particular S and T can be subsets of R^n, because R^n is also an n-dimensional manifold of class C^∞).

The operator $L\colon C^m_E(S) \to C^m_E(T)$ is said to be a *linear extension operator* if the following conditions are satisfied

(i) Lf is an extension of f, i.e. $f(s) = Lf(s)$ for every s in S and every f in $C^m_E(S)$;

(ii) $L(c_1 f_1 + c_2 f_2) = c_1 Lf_1 + c_2 Lf_2$ for $f_k \in C^m_E(S)$ and $c_k \in R^1$, $k = 1, 2$;

(iii) L is continuous, i.e. for every compact subset Z of T, for every μ ($|\mu| \leqslant m$) and for every pseudonorm $\|.\|'$ in E there is a pseudonorm $\|.\|''$ in E and the constant $b > 0$ such that

$$\|Lf\|'_{Z,\mu} \leqslant b \sum_{|\nu| \leqslant m} \|f\|''_{\tilde{Z},\nu} \quad \text{where} \quad \tilde{Z} = Z \cap T.$$

An operator of linear extension $L\colon C^m_E(S) \to C^m_E(T)$ is called *regularly continuous* if condition (iii) can be replaced by the condition

(iii') $\|Lf\|_{Z,\mu} \leqslant b \sum_{|\nu| \leqslant m} \|f\|_{\tilde{Z},\nu}$ for every pseudonorm $\|.\|$ in E.

By this condition it follows that any regularly continuous extension operator preserves the classes C^p ($p = 1, 2, \ldots, m$), i.e. $LC^p_E(S) \subset C^p_E(T)$ for $p \leqslant m$.

The general definition of the linear extension operator for the linear subspace of $C_E(S)$ to the linear subspace of $C_E(T)$ in the case where T is a topological space and E is a linear topological space is given in the paper of Pełczyński [12].

We shall say that the manifold X of class C^p *admits partitions of unity* of class C^m $(m \leqslant p)$ if it is paracompact and for each locally finite covering $\{V_a\}$ (a ranging in some indexing set) of X there is a collection of functions ψ_a satisfying the conditions:

(1) $\psi_a\colon X \to R^1$ and is of class C^m on X for each a;

(2) $\psi_a(x) \geqslant 0$ for $x \in X$ and for each a;

(3) $\{x\colon \psi_a(x) \neq 0\} = \operatorname{supp}\psi_a \subset V_a$ for each a;

(4) $\sum\limits_a \psi_a(x) = 1$ for each $x \in X$.

The definition of paracompactness and the sufficient conditions for an n-dimensional manifold to admit partitions of unity can be found in Helgason [5].

A subset of R^n is said to be a *non-singular manifold of class* C^p $(p = 0, 1, \ldots, \infty)$ if for each of its elements $x = (x_1, \ldots, x_n)$ there exist functions $x_i = x_i(y_1, \ldots, y_{n-1})$ $(-\varepsilon \leqslant y_j \leqslant \varepsilon)$; $j = 1, \ldots, n-1)$, $i = 1, \ldots, n$, of class C^p whose matrix $\left(\dfrac{\partial x_i}{\partial y_j}\right)$ has rank $n-1$.

We shall say that a manifold B is *piecewise of class* C^p $(p = 0, 1, \ldots, \infty)$ if for every point $x' \in B$ there are an integer m $(1 \leqslant m \leqslant n)$ and a non-singular transformation

$$x_i = x_i(t_1, \ldots, t_n), \quad i = 1, \ldots, n, \tag{1}$$

of class C^p such that the points of $B \cap N$ are the totality of points in N determined by equations (1) and the relations:

$$t_j \geqslant 0 \ (j \leqslant m); \quad t_1 t_2 \ldots t_m = 0.$$

We shall say that a subset A of an n-dimensional manifold X of class C^∞ has a *boundary B piecewise of class* C^∞ if there is an atlas (U_i, φ_i) on X such that if $U_i \cap A \neq 0$ then φ_i $B \cap U_i$ is piecewise of class C^∞.

Let X be an n-dimensional manifold of class C^∞. By a *cube division* of X of class C^∞ we mean a pair consisting of a cube complex K and a homeomorphism π^* of K onto X with the following property. For each n-dimensional cube Q of K there is a chart (U_i, φ_i) in X such that $\pi^*(Q) \subset U_i$ and $\varphi_i\pi^*$ is of class C^∞.

2. Existence of linear extension operators.

THEOREM 1. *Assume that A is a closed subset of R^n, with a non-empty interior, whose boundary B is a non-singular manifold of class C^∞. Then there exists a linear and regularly continuous extension operator*

$$L\colon C_E^\infty(A) \to C_E^\infty(R^n).$$

Theorem 1 can be strengthened as follows:

THEOREM 2. *If A is a closed subset of R^n, with a non-empty interior, whose boundary B is a non-singular manifold piecewise of class C^∞, then there exists a linear and regularly continuous extension operator*

$$L\colon C_E^\infty(A) \to C_E^\infty(R^n).$$

The proofs of these theorems are based on two lemmas and two auxiliary propositions.

Let

$$I_i^n = \{x \in R^n\colon -\infty < x_j < +\infty \ (j \leqslant i);\ 0 \leqslant x_j \leqslant 1 \ (j > i)\}.$$

It is easy to see that $I_0^n = I^n$ and $I_n^n = R^n$.

Let B_i^+ denote the set of all $x \in R^n$ such that $-\infty < x_j < +\infty$ for $j < i$, $0 \leqslant x_i < +\infty$, $0 \leqslant x_j \leqslant 1$ for $j > i$ and let B_i^- denote the set of all $x \in R^n$ such that $-\infty < x_j < +\infty$ for $j < i$, $-\infty < x_i \leqslant 1$, $0 \leqslant x_j \leqslant 1$ for $j > i$ $(i = 1, \ldots, n-1)$.

Let us take

$$\mathscr{E}(B_i^+) = \{g \in C_E^\infty(B_i^+)\colon D^\mu g(x) = 0 \text{ for } x_i \geqslant 1 \text{ and } |\mu| = 0, 1, \ldots\}$$

and

$$\mathscr{E}(B_i^-) = \{g \in C_E^\infty(B_i^-)\colon D^\mu g(x) = 0 \text{ for } x_i \leqslant 0 \text{ and } |\mu| = 0, 1, \ldots\}.$$

LEMMA 1. *There exist linear and regularly continuous extension operators*

$$L_i^+\colon \mathscr{E}(B_i^+) \to C_E^\infty(I_i^n),$$

$$L_i^-\colon \mathscr{E}(B_i^-) \to C_E^\infty(I_i^n) \text{ for } \quad i = 1, \ldots, n-1.$$

Proof. Take an arbitrary entire function in R^1 $\varphi(t) = \sum\limits_{k=0}^{\infty} a_k t^k$ with the property $\varphi(2^m) = (-1)^m$ for $m = 0, 1, \ldots$ Such a function can be constructed (see Saks and Zygmund [13]).

We define the operator L_i^+ for the functions $g \in \mathscr{E}(B_i^+)$

$$L_i^+ g(x_1, \ldots, x_n) = \begin{cases} g(x_1, \ldots, x_n) & \text{for } \quad x_i \geqslant 0; \\ \sum\limits_{k=0}^{\infty} a_k g(x_1, \ldots, x_{i-1}, -2^k x_i, x_{i+1}, \ldots, x_n) & \text{for } \quad x_i < 0. \end{cases}$$

(a) The series $\sum\limits_{k=0}^{\infty} a_k D^\mu g(x_1, \ldots, x_{i-1}, -2^k x_i, x_{i+1}, \ldots, x_n)$ is absolutely and uniformly convergent on every compact set of $x \in I_i^n$ such that $x_i < 0$

in every pseudonorm in E, because

$$\sum_{k=0}^{\infty} a_k D^\mu [g(x_1, \ldots, x_{i-1}, -2^k x_i, x_{i+1}, \ldots, x_n)]$$

$$= \sum_{k=0}^{\infty} a_k (-1)^{\mu_i} 2^{k\mu_i} D^\mu g(x_1, \ldots, x_{i-1}, -2^k x_i, x_{i+1}, \ldots, x_n)$$

and the series $\sum_{k=0}^{\infty} |a_k| 2^{k\mu_i}$ is convergent for every $\mu_i = 0, 1, \ldots$

(b) $L_i^+ g \in C_E^\infty(I_i^n)$ if $g \in \mathscr{E}(B_i^+)$.

From (a) it follows that there exist derivatives

$$D^\mu L_i^+ g(x_1, \ldots, x_n) \quad \text{for all } x \in I_i^n \text{ such that } x_i < 0$$

and left-hand derivatives

$$D_-^\mu L_i^+ g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \quad \text{for} \quad |\mu| = 0, 1, \ldots$$

On the other hand, there exist derivatives

$$D^\mu L_i^+ g(x_1, \ldots, x_n) = D^\mu g(x_1, \ldots, x_n) \quad \text{for} \quad x \in I_i^n \text{ such that } x_i > 0,$$

and right-hand derivatives

$$D_+^\mu L_i^+ g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = D_+^\mu g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).$$

Thus we must show that

$$D_-^\mu L_i^+ g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = D_+^\mu g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$
$$\text{for} \quad |\mu| = 0, 1, \ldots$$

Accordingly let us take $x = (x_1, \ldots, x_n)$, $x^0 = (x_1^0, \ldots, x_{i-1}^0, 0, x_{i+1}^0, \ldots, x_n^0)$ and let $x \to x^0 - 0$, i.e. $x_i < 0$. Then we have

$$\lim_{x \to x^0 - 0} D^\mu L_i^+ g(x_1, \ldots, x_n)$$

$$= \lim_{x \to x^0 - 0} D^\mu \sum_{k=0}^{\infty} a_k g(x_1, \ldots, x_{i-1}, -2^k x_i, x_{i+1}, \ldots, x_n)$$

$$= \sum_{k=0}^{\infty} a_k (-1)^{\mu_i} 2^{k\mu_i} \lim_{x \to x^0 - 0} D^\mu g(x_1, \ldots, x_{i-1}, -2^k x_i, x_{i+1}, \ldots, x_n)$$

$$= (-1)^{\mu_i} D^\mu g(x^0) \sum_{k=0}^{\infty} a_k 2^{k\mu_i} = (-1)^{\mu_i} D^\mu g(x^0) \varphi(2^{\mu_i}) = D^\mu g(x^0).$$

(c) The operator L_i^+ is linear and regularly continuous. The first property is obvious, and the second one is a consequence of the estimation

$$\|L_i^+ g\|_{Z,\mu} = \sup_{x \in Z} \|D^\mu L_i^+ g(x)\| \leqslant \sum_{k=0}^{\infty} |a_k| 2^{k\mu_i} \|g\|_{\tilde{Z},\mu} = C \|g\|_{\tilde{Z},\mu}$$

with $\tilde{Z} = Z \cap B_i^+$, for every pseudonorm $\|.\|$ in E and for every compact subset Z of I_i^n.

To construct the extension operator L_i^- let us take the transformation

$$\tau_i \colon B_i^- \to B_i^+, \quad \tau_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n).$$

The transformation τ_i induces a linear operator τ_i^*

$$\tau_i^* \colon \mathscr{E}(B_i^-) \to \mathscr{E}(B_i^+)$$

such that

$$\tau_i^* g(x_1, \ldots, x_n) = g\tau_i(x_1, \ldots, x_n) \quad \text{for} \quad g \in \mathscr{E}(B_i^-)$$

with the property $\|\tau_i^* g\|_{Z,\mu} = \|g\|_{\tau_i^{-1} Z,\mu}$ for every pseudonorm in E, every μ and every compact subset Z of B_i^+. Thus we can define the extension operator $L_i^- \colon \mathscr{E}(B_i^-) \to C_E^\infty(I_i^n)$ as follows:

$$L_i^- g = \tau_i^* L_i^+ \tau_i^* g \quad \text{for} \quad g \in \mathscr{E}(B_i^-).$$

It is easy to verify that L_i^- possesses all the required properties.

LEMMA 2. *There exist linear and regularly continuous extension operators*

$$L_i \colon C_E^\infty(I_{i-1}^n) \to C_E^\infty(I_i^n) \quad \textit{for} \quad i = 1, 2, \ldots, n.$$

Proof. Take the function

$$\psi(t) = \begin{cases} 0 & \text{for} \quad t \leqslant 0, \\ e^{1 - t^{-2}}(1 - e^{-(t-1)^{-2}}) & \text{for} \quad 0 < t < 1, \\ 1 & \text{for} \quad t \geqslant 1. \end{cases}$$

It is easy to show that $\psi \in C^\infty(R^1)$. Let λ denote the restriction of the function ψ to the interval $[0, 1]$ and let $\varkappa(t) = 1 - \lambda(t)$, $t \in [0, 1]$. Then we have $\lambda, \varkappa \in C^\infty[0, 1]$ and $\lambda(t) + \varkappa(t) = 1$ for every $t \in [0, 1]$. It is easy to see that the functions $\lambda(x_i) f(x_1, \ldots, x_n)$ and $\varkappa(x_i) f(x_1, \ldots, x_n)$ belong to $C_E^\infty(I_{i-1}^n)$ for every $f \in C_E^\infty(I_{i-1}^n)$. Moreover, all the derivatives with respect to x_i of the first function vanish for all x such that $x_i = 0$ and all the derivatives with respect to x_i of the second function vanish for all x such that $x_i = 1$. Thus it is possible to extend these functions by taking them equal to zero the first for $x_i < 0$, the second for $x_i > 1$. Hence we can consider that $\lambda f \in \mathscr{E}(B_i^-)$, $\varkappa f \in \mathscr{E}(B_i^+)$ and we can define the operator L_i as follows:

$$L_i f(x) = L_i^- (\lambda f)(x) + L_i^+ (\varkappa f)(x).$$

It is an operator of extension because if $x \in I^n_{i-1}$, then

$$L_i f(x) = L_i^-(\lambda f)(x) + L_i(\varkappa f)(x) = \lambda(x_i) f(x) + \varkappa(x_i) f(x) = f(x).$$

The operator L_i possesses the same properties as the operators L_i^- and L_i^+.

PROPOSITION 1. *Let Q denote an arbitrary n-dimensional closed interval. Then there exists a linear, regularly continuous extension operator*

$$L: C_E^\infty(Q) \to C_E^\infty(R^n).$$

Proof. (i) *The case where Q is bounded.* Let

$$Q = \{x \in R^n: a_i \leqslant x_i \leqslant b_i \ (-\infty < a_i < b_i < +\infty) \ i = 1, \ldots, n\}.$$

It is sufficient to prove that there exists an operator

$$L: C_E^\infty(I^n) \to C_E^\infty(R^n).$$

In fact: take $f \in C_E^m(Q)$ $(m = 0, 1, \ldots, \infty)$ and $\pi: I^n \to Q$

$$\pi(t_1, \ldots, t_n) = \big(a_1 t_1 + (1-t_1) b_1, \ldots, a_n t_n + (1-t_n) b_n\big);$$

then $f \circ \pi \in C_E^m(I)^n$ and we can define

$$L': C_E^\infty(Q) \to C_E^\infty(R^n),$$

$$L' f(x_1, \ldots, x_n) = L(f \circ \pi)\left(\frac{b_1 - x_1}{b_1 - a_1}, \ldots, \frac{b_n - x_n}{b_n - a_n}\right).$$

L' has the same properties as the operator L.

The operator

$$L: C_E^\infty(I^n) \to C_E^\infty(R^n)$$

can be defined as follows:

$$Lf = L_n L_{n-1} \ldots L_1 f \quad \text{for} \quad f \in C_E^\infty(I^n).$$

Evidently L is an extension operator and is well-defined, linear and regularly continuous as the composition of linear and regularly continuous extension operators.

(ii) *The case where Q is unbounded.* Without loss of generality we can assume that

$$Q = \{x \in R^n: -\infty < x_i < +\infty \ (1 \leqslant i \leqslant t), \ 0 \leqslant x_i < +\infty \ (t < i \leqslant s), \ 0 \leqslant x_i \leqslant 1 \ (s < i \leqslant n)\}.$$

If $s = t$, then we define the operator $L: C_E^\infty(Q) \to C_E^\infty(R^n)$ as follows:

$$Lf = L_n L_{n-1} \ldots L_{s+1} f, \quad 1 \leqslant s < n.$$

The case where $s = t = n$ is trivial. If $s > t$, then we take

$$Lf = L_n \ldots L_{s+1} L'_s \ldots L'_{t+1} f$$

where

$$L'_{t+i} g(x_1, \ldots, x_n) = \begin{cases} g(x_1, \ldots, x_n) & \text{for} \quad x_{t+i} \geqslant 0, \\ L_{t+i} r_{t+i} g(x_1, \ldots, x_n) & \text{for} \quad x_{t+i} < 0 \end{cases}$$

$$i = 1, \ldots, s-t,$$

$r_{t+i} g$ denoting the restriction of g to the set I^n_{t+i-1}.

PROPOSITION 2 (cf. [6]). *Let X be an n-dimensional paracompact manifold of class C^∞, admitting partitions of unity of class C^∞. Assume that T is a closed subset of X and for every $x \in T$ there exists an open neighbourhood N_x in X and a linear regularly continuous exetension operator*

$$L_x: C_E^\infty(\overline{N}_x \cap T) \to C_E^\infty(\overline{N}_x).$$

Then there exists a linear regularly continuous extension operator

$$L: C_E^\infty(T) \to C_E^\infty(X).$$

Proof. The family of neighbourhoods $\{N_x\}_{x \in T}$ covers the set T. Let $N_0 = X - T$; then N_0 is open and the sets N_0, N_x, $x \in T$, cover X. X is paracompact, and so we can find a locally finite open covering U_0, U_x, $x \in T$, such that $U_0 \subset N_0$, $U_x \subset N_x$ for $x \in T$. Since X admits partitions of unity, there exist functions $\varphi_0, \varphi_x, x \in T$, of class C^∞ such that $\operatorname{supp}\varphi_0 \subset U_0$ and $\operatorname{supp}\varphi_x \subset U_x$ for every $x \in T$. We define the extension operator L

$$Lf(y) = \sum_{x \in T} \varphi_x(y) L_x f(y).$$

The sum is finite for every $y \in X$. If $y \in T$, then it belongs to the finite number of sets of the covering. Let them be the sets $U_{x^1}, \ldots, U_{x^s}$. From the assumption it follows that

$$L_{x^i} f(y) = f(y) \quad \text{for} \quad i = 1, \ldots, s;$$

moreover, $\sum_{i=1}^{s} \varphi_{x^i}(y) = 1$. Hence, $Lf(y) = f(y)$. The operator L is linear because it is the sum of linear operators.

To show its regular continuity, we use the following estimation. Let Z denote a compact subset of X. Since the set $Z \cap \bigcup_{x \in T} U_x$ is also compact, it intersects only a finite number of members of the covering. Let them be $U_{x^1}, \ldots, U_{x^s}$; then

$$Lf(y) = \sum_{i=1}^{s} \varphi_{x^i}(y) L_{x^i} f(y) \quad \text{for} \quad y \in Z.$$

Let

$$K_i^\nu = \sup_{y\in Z\cap U_{x^i}} |D^\nu \varphi_{x^i}(y)| \quad \text{and} \quad K = \max_{\substack{|\nu|\leqslant|\mu| \\ 1\leqslant i\leqslant s}} K_i^\nu.$$

By Leibniz's formula,

$$\|Lf\|_{Z,\mu} = \sup_{y\in Z} \Big\|D^\mu\Big(\sum_{i=1}^{s}\varphi_{x^i}(y)L_{x^i}(f(y)\Big)\Big\|$$

$$\leqslant \sup_{y\in Z}\sum_{i=1}^{s}\sum_{|\nu|\leqslant|\mu|}\binom{\mu}{\nu}K_i^{\mu-\nu}\|D^\nu L_{x^i}f(y)\|$$

$$\leqslant K\sum_{i=1}^{s}\sum_{|\nu|\leqslant|\mu|}\binom{\mu}{\nu}\|L_{x^i}f\|_{Z\cap U_{x^i},\nu}$$

for every pseudonorm $\|.\|$ in E.

Proposition 2 can be generalized to the case of vector bundles, to be stated as follows (we use the terminology of Lang [7]).

Let X be a manifold of class C^p and let $\pi\colon E\to X$ be a vector boundle over X with a Banach space $\mathscr{E}$ as a fibre. Assume that Y is a closed subset of X and let $S(Y)$ denote the space of all continuous sections of the boundle over Y (i.e. $\xi\in S(Y)$ iff $\xi\colon Y\to E$ and $\xi\circ\pi^{-1} = id_E$). If for every $y\in Y$ there exist an open neighbourhood N_y of y and a linear extension operator $L_y\colon S(N_y\cap Y)\to S(N_y)$, then there exists a linear extension operator $L\colon S(Y)\to S(X)$.

This extension is given by $L\xi(x) = \sum_{y\in Y}\varphi_y(x)L_y\xi(x)$, where $\{\varphi_y\}$ is a partition of unity for $\{N_y\}$. It is well-defined because every $L_y\xi(x)$ is an element of $\pi^{-1}(x)$, which is homeomorphic with $\mathscr{E}$, and the summation is understood in the space $\mathscr{E}$ (cf. Lang [7], p. 51).

Proof of Theorem 1. Let $x'\in B$. Since B is a non-singular manifold of class C^∞, it is representable near x' by functions

$$x_i = x_i(y_1, \ldots, y_{n-1}) \quad (-\varepsilon\leqslant y_j\leqslant\varepsilon;\ j = 1, \ldots, n-1),$$

$i = 1, \ldots, n$, of class C^∞, whose matrix $(\partial x_i/\partial y_j)$ has rank $n-1$. Let b_i be the direction cosines of the inner normal to B at x'. By the use of the equations $x_i = x_i(y_1, \ldots, y_{n-1})+b_iy_n$ for $y_n\geqslant 0$ one obtains a representation of a neighbourhood of x'. We have the determinant

$$\begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \ldots & \dfrac{\partial x_1}{\partial y_{n-1}} & b_1 \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial x_n}{\partial y_1} & \ldots & \dfrac{\partial x_n}{\partial y_{n-1}} & b_n \end{vmatrix} \neq 0$$

in a neighbourhood U of x'. Hence, there exist inverse functions y_j of class C^∞ such that

$$y_j = y_j(x_1, \ldots, x_n), \quad j = 1, \ldots, n.$$

The function

$$g(y_1, \ldots, y_n) = f\big(x_1(y_1, \ldots, y_{n-1})+b_1y_n, \ldots, x_n(y_1, \ldots, y_{n-1})+b_ny_n\big)$$

is of class C^∞ on the set

$$Q = \{(y_1, \ldots, y_n)\colon -\varepsilon\leqslant y_j\leqslant\varepsilon\ j = 1, \ldots, n-1,\ y_n\geqslant 0\}$$

for $f\in C^\infty(A)$. By Proposition 1 there exists a linear and regularly continuous extension operator

$$L'\colon C_E^\infty(Q)\to C_E^\infty(R^n).$$

Hence we can define the extension operator

$$L_{x'}\colon C_E^\infty(\overline{U\cap A})\to C_E^\infty(\overline{U})$$

as follows:

$$L_{x'}f(x_1, \ldots, x_n) = L'g\big(y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)\big).$$

We have shown that there exists locally a linear and regularly continuous extension operator ; thus by Proposition 2 it follows that the hypothesis is true.

Proof of Theorem 2. Again by Proposition 2 we can restrict ourselves to a neighbourhood N of a point $x'\in B$. By virtue of the definition of a manifold piecewise of class C^∞ we can assume that the points of $\overline{B\cap N}$ are given by relations of the form

$$0\leqslant x_i\leqslant\varepsilon\ (i\leqslant m), \quad x_1\ldots x_m = 0; \quad -\varepsilon\leqslant x_i\leqslant\varepsilon\ (i>m).$$

Consider first the case where the set $\overline{N\cap A}$ consists of the points x such that

(2) $$0\leqslant x_i\leqslant\varepsilon\ (i\leqslant m), \quad -\varepsilon\leqslant x_i\leqslant\varepsilon \quad (i>m).$$

By Proposition 1 there exists a linear and regularly continuous extension operator

$$L_x\colon C_E^\infty(\overline{N\cap A})\to C_E^\infty(\overline{N}).$$

Now consider the other case, where the points of N determined by (2) belong to $N-N\cap A$. In this case let

$$A_i = \{x\in\overline{N}\colon -\varepsilon\leqslant x_i\leqslant 0\} \quad \text{for} \quad i\leqslant m.$$

Then $N\cap A = A_1\cup\ldots\cup A_m$. Let $f\in C_E^\infty(\overline{N\cap A})$; we shall define the mappings $f_1, \ldots, f_m$ of class C^∞ on $\overline{N}$ with $f = f_1+\ldots+f_m$ on $A_1\cap\ldots\cap A_m$.

Take $g_1 = f$ on the set A_1 (i.e. g_1 is a restriction of f to the set A_1) and use the auxiliary extension operator L_1 from Lemma 2. Let $f_1 = L_1 g_1$ on the set $\overline{N}$. Let us suppose that we have constructed f_{i-1} $(i > 1)$ and take $g_i = f - f_1 - \ldots - f_{i-1}$ on A_i; then we can use the auxiliary operator L_i (see Lemma 2) to g_i. Let $f_i = L_i g_i$ on $\overline{N}$. The function $g_i(x) = 0$ for $x \in A_i \cap (A_1 \cup \ldots \cup A_{i-1})$ because $f = f_1 + \ldots + f_{i-1}$ on $A_1 \cup \ldots \cup A_{i-1}$ by virtue of the inductive assumption and since $f_i(x) = 0$ on this set. On the other hand, for $x \in A_i$,

$$f(x) = g_i(x) = f(x) - f_1(x) - \ldots - f_{i-1}(x).$$

Hence we put $Lf = f_1 + \ldots + f_m$.

It is easy to see that the operator L defined in this way satisfies all the required conditions.

THEOREM 3. *Let X be an n-dimensional manifold of class C^∞ paracompact and admitting partitions of unity of class C^∞. Assume that A is a closed subset of X with a non-empty interior and a boundary B piecewise of class C^∞. Then there exists a linear and regularly continuous extension operator $L\colon C_E^\infty(A) \to C_E^\infty(X)$.*

Proof. Let us take $p \in B$ and the chart (U_i, φ_i) where $p \in U_i$. By Proposition 2 it is sufficient to show that there are a neighbourhood W of p in X, $W \subset U_i$, and a linear regularly continuous extension operator L_p: $C_E^\infty(\overline{W \cap A}) \to C_E^\infty(\overline{W})$. Let $\varphi_i(p) = x$. Since $\varphi_i(B \cap U_i)$ is piecewise of class C^∞, we prove by the same method as in Theorem 1 that there exist a neighbourhood N of x, $N \subset \varphi_i(U_i)$ and a linear regularly continuous extension operator L_x': $C_E^\infty\big(\overline{N \cap \varphi_i(B \cap U_i)}\big) \to C_E^\infty(\overline{N})$. Take $\varphi_i^{-1}(N) = W$. Then there exists a linear regularly continuous extension operator L_p: $C_E^\infty(\overline{W \cap A}) \to C_E^\infty(\overline{W})$ defined as follows:

$$L_p f(q) = L_x' f \circ \varphi_i^{-1}\big(\varphi_i(q)\big).$$

The operator L_p possesses obviously the same properties as the operator L_x'.

3. Isomorphism of spaces of infinitely differentiable functions on manifolds.

THEOREM 4. *Let X be a compact n-dimensional manifold of class C^∞ admitting partitions of unity of class C^∞. Assume that there is a cube division (π^*, K) for X such that if $V_1, \ldots, V_p$ denote all inverse images of n-dimensional cubes in complex K, then* $\mathrm{bd}(V_1 \cup \ldots \cup V_i)$ $(i = 1, \ldots \ldots p-1)$ *is a manifold piecewise of class C^∞. Then*

$$C^\infty(X) \simeq C^\infty(I).$$

The proof makes use of some auxiliary notions and properties. Consider the intervals $I = [0, 1]$, $\tilde{I} = [0, 1)$, $\underset{\sim}{I} = (0, 1]$, $J = (0, 1)$. Let $n \geqslant 1$ be a fixed integer and let $D, D_1, D_2 \ldots$ denote a product $I^\alpha \times \tilde{I}^\beta \times \underset{\sim}{I}^\gamma \times J^\delta$ with $0 \leqslant \alpha, \beta, \gamma, \delta$; $\alpha + \beta + \gamma + \delta = n$, $\mathscr{E}(D) = C^\infty(\overline{D}/\overline{D} - D)$, the space of functions of class C^∞ on $\overline{D}$ vanishing with all their derivatives on $\overline{D} - D$.

In the sequel we shall use the notions of tensor product. The general definition of tensor product of linear topological spaces can be found in Grothendieck [4]. In the case where M is a subspace of $C^\infty(X)$ and N is a subspace of $C^\infty(Y)$ the tensor product $M \hat{\otimes} N$ may be equivalently defined as the closed subspace of $C^\infty(X \times Y)$, which is spanned on all the functions $\{f(x) \cdot g(y)\}$, $f \in M$, $g \in N$.

We have:

3.1. $\mathscr{E}(D_1) \hat{\otimes} \mathscr{E}(D_2) \simeq \mathscr{E}(D_1 \times D_2)$.

This proposition is a consequence of a more general theorem (Grothendieck [4], II part, Theorem 13).

3.2. $\mathscr{E}(I) \hat{\otimes} \mathscr{E}(I) \simeq \mathscr{E}(I)$.

Since $\mathscr{E}(I) \cong m(n^\alpha)$ (Mitiagin [10]), where $m(n^\alpha)$ denotes the space of all real sequences $x = \{\xi_n\}$, such that $|x|_\alpha = \sup_n n^\alpha |\xi_n| < +\infty$ $(\alpha = 1, 2, \ldots)$ and $m(n^\alpha) \hat{\otimes} m(n^\alpha) \simeq m(n^\alpha)$; so we obtain the statement of 3.2.

3.3. $\mathscr{E}(I) \simeq \mathscr{E}(\underset{\sim}{I}) \simeq \mathscr{E}(J)$.

The proof makes use of the following properties:

(1) $\mathscr{E}(\underset{\sim}{I}) \simeq \mathscr{E}(\tilde{I})$,

(2) $\mathscr{E}(I) \simeq \mathscr{E}(I) \times \mathscr{E}(I)$,

(3) $\mathscr{E}(I) \simeq \mathscr{E}(I) \times \mathscr{E}(\tilde{I})$,

(4) $\mathscr{E}(J) \simeq \mathscr{E}(J) \times \mathscr{E}(\tilde{I})$,

(5) $\mathscr{E}(\underset{\sim}{I}) \simeq \mathscr{E}(\underset{\sim}{I}) \times \mathscr{E}(\underset{\sim}{I})$,

(6) $\mathscr{E}(\underset{\sim}{I}) \simeq \mathscr{E}(I) \times \mathscr{E}(J)$.

Property (1) is obvious.

(2). Since $\mathscr{E}(I) \simeq m(n^\alpha)$, the isomorphism T is given as follows:

$$T(\{\xi_n\}, \{\eta_n\}) = (\xi_1, \eta_1, \xi_2, \eta_2, \ldots).$$

The isomorphism in (3), (4), (5) can be defined as follows:

$$Tf = (f_1, f_2), \quad \text{where} \quad f_1(t) = f(\tfrac{1}{2}t), \quad f_2(t) = (f - L\tilde{f})\big(\tfrac{1}{2}(t+1)\big)$$

for $t \in [0, 1]$, $\tilde{f}$ denoting restriction of f to the interval $[0, \frac{1}{2}]$ and L denoting Ryll-Nardzewski's extension operator from $C^\infty[0, \frac{1}{2}]$ to $C^\infty(R)$. It is easy to verify that $D^k Lg(1) = 0$ if $D^k g(0) = 0$ for $g \in C^\infty[0, \frac{1}{2}]$ and $k = 0, 1, \ldots$

To prove (6) we take $f_1(t) = f\left(\frac{1}{2}(t+1)\right), f_2(t) = (f - L'\tilde{f})\left(\frac{1}{2}t\right)$ for $t \in [0, 1]$, $\tilde{f}$ is as above, L' denoting the extension operator from $C^\infty[\frac{1}{2}, 1]$ to $C^\infty(R)$ such that $D^k L'g(0) = 0$ for every $k = 0, 1, \ldots$ For instance, $L'f(t) = \varphi(t) Lf(t)$, where $\varphi(t)$ is a function of class $C^\infty \varphi(t) = 0$ for $t \leqslant 0$; $\varphi(t) = 1$ for $t \geqslant \frac{1}{2}$ and L is an arbitrary extension operator from $C^\infty[\frac{1}{2}, 1]$ to $C^\infty(R)$. Using the same method as in [11] we get

$$\mathscr{E}(\underset{\sim}{I}) \simeq \mathscr{E}(I)\times\mathscr{E}(J) \simeq \mathscr{E}(I)\times\mathscr{E}(I)\times\mathscr{E}(J) \simeq \mathscr{E}(I)\times\mathscr{E}(\tilde{I}) \simeq \mathscr{E}(I)$$

and

$$\begin{aligned}\mathscr{E}(J) &\simeq \mathscr{E}(J)\times\mathscr{E}(\tilde{I}) \simeq \mathscr{E}(J)\times\mathscr{E}(\tilde{I})\times\mathscr{E}(\tilde{I}) \simeq \mathscr{E}(J)\times\mathscr{E}(I)\times\mathscr{E}(J)\times\mathscr{E}(\tilde{I})\\ &\simeq \left(\mathscr{E}(J)\times\mathscr{E}(I)\right)\times\left(\mathscr{E}(I)\times\mathscr{E}(J)\right)\times\left(\mathscr{E}(I)\times\mathscr{E}(\tilde{I})\right) \simeq \mathscr{E}(\underset{\sim}{I})\times\mathscr{E}(I)\times\mathscr{E}(I)\\ &\simeq \mathscr{E}(I)\times\mathscr{E}(I) \simeq \mathscr{E}(I).\end{aligned}$$

LEMMA. *Let X be as in Theorem 4. Then*

(A) $$C^\infty(X) \simeq C^\infty(V_1)\times C^\infty(V_2/V_2 \cap V_1)\times \ldots$$
$$\ldots \times C^\infty\left(V_{p-1}/V_{p-1} \cap (V_1 \cup \ldots \cup V_{p-2})\right)\times C^\infty(X/V_1 \cup \ldots \cup V_{p-1}).$$

Proof of Lemma. We shall show

(B) $$C^\infty(X/V_1 \cup \ldots \cup V_i)$$
$$\simeq C^\infty\left(V_{i+1}/V_{i+1} \cap (V_1 \cup \ldots \cup V_i)\right)\times C^\infty(X/V_1 \cup \ldots \cup V_{i+1})$$

for $i = 0, 1, \ldots p-2$.

By $C^\infty(X/V_1 \cup \ldots \cup V_i)$ and $C^\infty\left(V_{i+1}/V_{i+1} \cap (V_1 \cup \ldots \cup V_i)\right)$ for $i = 0$ we mean $C^\infty(X)$ and $C^\infty(V_1)$, respectively.

Since $\mathrm{bd}(V_1 \cup \ldots \cup V_{i+1})$ for $i = 0, 1, \ldots, p-1$ are manifolds piecewise of class C^∞, there exists by Theorem 3 linear and continuous extension operator L_{i+1}: $C^\infty(V_1 \cup \ldots \cup V_{i+1}) \to C^\infty(X)$. Let $r_{i+1}f$ denote a restriction of f to the set $V_1 \cup \ldots \cup V_{i+1}$ for $f \in C^\infty(X)$. Then f can be represented as follows:

$$f = L_{i+1}r_{i+1}f - (f - L_{i+1}r_{i+1}f)$$

where $L_{i+1}r_{i+1}f \in L_{i+1}C^\infty(V_1 \cup \ldots \cup V_{i+1})$ and

$$(f - L_{i+1}r_{i+1}f) \in C^\infty(X/V_1 \cup \ldots \cup V_{i+1}).$$

Since $L_{i+1}C^\infty(V_1 \cup \ldots \cup V_{i+1}) \simeq C^\infty(V_1 \cup \ldots \cup V_{i+1})$, (B) is obtained. To get (A) we use (B) step by step for $i = 0, 1, \ldots p-2$.

Proof of Theorem 4. Let us remark that $V_{i+1} \cap (V_1 \cup \ldots \cup V_i)$ $(i = 1, 2, \ldots, p-1)$ is a sum of some $(n-1)$-dimensional faces of the cube V_{i+1} because according to the assumption of Theorem 4 $\mathrm{bd}(V_1 \cup \ldots \cup V_{i+1})$ $(i = 0, 1, \ldots p-1)$ is a manifold. Hence

$$C^\infty\left(V_{i+1}/V_{i+1} \cap (V_1 \cup \ldots \cup V_i)\right) \simeq \mathscr{E}(D) \quad (i = 0, 1, \ldots, p-1),$$

where $\bar{D} = \pi^{*-1}(V_{i+1})$. From 3.1 follows that

$$\mathscr{E}(D) \simeq \mathscr{E}(I)^\alpha \hat{\otimes} \mathscr{E}(\underset{\sim}{I})^\beta \hat{\otimes} \mathscr{E}(\tilde{I})^\gamma \hat{\otimes} \mathscr{E}(J)^\delta$$

with $0 \leqslant \alpha, \beta, \gamma, \delta$; $\alpha+\beta+\gamma+\delta = n$. So by 3.3. and 3.2 we obtain $\mathscr{E}(D) \simeq \mathscr{E}(I)$. Using the lemma we have

$$C^\infty(X) \simeq \mathscr{E}(I)\times \ldots \times\mathscr{E}(I),$$

and according to property (2) in 3.3 we have $C^\infty(X) \simeq \mathscr{E}(I) = C^\infty(I)$.

PROBLEM 1. *Does every compact n-dimensional differentiable manifold possess a cube division satisfying the conditions of theorem 4?*

PROBLEM 2. *Is the hypothesis of theorem 4 true when instead of a cube division for the manifold there is a simplicial division?*

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