В частности, это касается C-суммирования последовательностей методами, непрерывными по Л. Влодарскому (см., например, [6]).

Не рассматривая вопрос о континуальности ограниченных полей суммирования таких методов, ограничимся в этом направлении следующей теоремой:

Теорема 3. *Если регулярный полунепрерывный метод суммирования T обладает свойством* (α) *и суммирует некоторую ограниченную расходящуюся последовательность, то он суммирует несепарабельное в пространстве m (а следовательно и несчетное) множество ограниченных последовательностей, расходящихся одновременно с любой их нетривиальной конечной линейной комбинацией.*

Действительно, если бы упомянутое в заключении теоремы множество последовательностей было сепарабельным, то можно было бы, взяв произвольную регулярную матрицу $\{a_{mn}\}$, ограниченно не совместную с методом T, выделить из нее регулярную подматрицу, которая была бы не слабее метода T ([1], теорема 8.5.2) и ограниченно не совместна с ним. А это противоречило бы полунепрерывному аналогу теоремы Мазура-Орлича, который, как известно, справедлив для полунепрерывных методов, обладающих свойством (α) ([4], стр. 242).

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КРАСНОЯРСКИЙ ГОСУДАРСТВЕННЫЙ ПЕДАГОГИЧЕСКИЙ ИНСТИТУТ, КРАСНОЯРСК

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Estimates for eigenfunctions

by

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0. Introduction. As is well known (cf. e.g. [8], vol. 1, p. 45) the Riemann-Lebesgue lemma says that if f is a periodic function ϵL_1 and $a_\nu = \int f(x)e^{-i\nu x}dx$ $(\nu = 0, \pm 1, \pm 2, \ldots)$ are the Fourier coefficients of f, then $a_\nu = o(1)$, $\nu \to \pm\infty$. We recall also that the proof follows by a density argument from the following facts: (i) The trivial fact that the functions $e^{i\nu x}$ are uniformly bounded which already implies $a_\nu = O(1)$. (ii) The fact that $a_\nu = o(1)$ in some dense subset of L_1, say L_2, in which case $\sum |a_\nu|^2 < \infty$, if we presuppose Parseval's formula, or the space of continuously differentiable functions, in which case $a_\nu = O(1/\nu)$, by partial integration.

What is the analogue of the Riemann-Lebesgue lemma for eigenfunction expansions? In this paper we attempt to answer this question for the case of the eigenvalue problem

$$Au = \lambda u \text{ in } \Omega \tag{0.1}$$

where A is any formally positive self-adjoint elliptic partial differential operator of order m, the essential (and very restrictive) assumption being that *the leading part A_m of A has constant coefficients*, and Ω is any domain of R^n, self-adjoint boundary conditions (say of the Dirichlet type) being imposed on the boundary of Ω. We assume further that the spectrum is *discrete*, so that there exists a complete set of eigenfunctions in the usual sense. (It is not clear to us what happens in the case of continuous spectrum, i.e. generalized eigenfunction expansions.) Then every f can be expanded in a series

$$f(x) = \sum_{\nu=1}^{\infty} f_\nu(x)$$

where

$$f_\nu(x) = \sum_r (f, \varphi_{\nu r})\varphi_{\nu r}(x) = \int_\Omega \sum_r \varphi_{\nu r}(x)\overline{\varphi_{\nu r}(y)}f(y)\,dy, \tag{0.2}$$

the summation being extended over an orthonormal basis (necessarily finite!) of eigenfunctions $\varphi_{\nu r}$ belonging to the νth eigenvalue λ_ν. We shall show that if $f \in L_1$ and vanishes off a compact subset of Ω, then the formula

$$(0.3)\qquad f_\nu(x) = o(\lambda_\nu^{(n-1)/m}), \quad \nu \to \infty,$$

holds uniformly on compact subsets of Ω (analogue of the Riemann-Lebesgue lemma). What the proof concerns it turns out that now step (ii) is the trivial one and the whole difficulty lies in step (i). Indeed, we shall show that any normalized solution of (0.1), regardless if it satisfies the boundary conditions or not, satisfies

$$(0.4)\qquad u(x) = O(\lambda_\nu^{(n-1)/2m}), \quad \lambda \to \infty,$$

uniformly on compact subsets of Ω *and in* u. From (0.4) and (0.2) easily follows with the aid of Schwarz' inequality

$$f_\nu(x) = O(\lambda_\nu^{(n-1)/m})$$

and therefore by a density argument (0.3) too. The proof of (0.4) again depends on an estimate of a certain fundamental solution. Similar results hold in L_p, $1 \leqslant p \leqslant 2$. For instance, for $p = 2$ we have

$$(0.5)\qquad f_\nu(x) = O(\lambda_\nu^{(n-1)/2m}), \quad v \to \infty,$$

whenever $f \in L_2$ and vanishes off a compact subset of Ω.

In the special case $m = 2$ the above results (notably (0.4)) and the method are due to Minakshisundaram and Titchmarsh (cf. [7], chap. XVIII, in particular pp. 186-193; concerning the work of Minakshisundaram in this area, cf. also [1]). However, this case is particularly simple because the fundamental solution can be expressed in terms of Bessel functions while as we, in the general case, have to give direct estimates for it, which is our main new contribution. In the special case $m = 2$ one can also see that the exponent in (0.4) and consequently the one in (0.5) cannot be improved (cf. [7], pp. 192-193). However, the exponent in (0.3) can now be replaced by $(n-1)/2m$. It is conceivable that this will not be so if $m > 2$ so that $(n-1)/m$ is the best one can hope for in general. Cf. [3] where a similar phenomenon occurs.

Our method fails when the leading part A_m is not constant; this case would require a much more careful analysis of the fundamental solution. However, there is one instance of variable coefficients where (0.3) still holds true; this is the case of an expansion in spherical harmonics (Laplace series); thus Ω is now a manifold not immersed in R^n. A proof of this, based on the theory of Gegenbauer polynomials, was essentially given by Shapiro [6], p. 10. We give here a different proof making use of an idea similar to the one in [5], Section 3.

The plan of the article is as follows. We start in Section 1 by proving (0.4) in the special case $m = 2$. Then we extend the method successively in Section 2 and Section 3. In Section 4 we prove the estimate of the fundamental solution used in the previous sections. In Section 5 we give the applications to eigenfunction expansions. Finally, in Section 6 we treat the case of expansion in spherical harmonics, indeed in a somewhat more general framework.

1. An illustrative special case. Let us consider the equation

$$(1.1)\qquad -\Delta u = \lambda u, \quad |x| < 1,\ \lambda > 0,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator. We assume that $u \in L_2$ is normalized by

$$(1.2)\qquad \int_{|x|<1} |u|^2 dx = 1.$$

Without loss of generality we may assume (1) $\operatorname{grad} u \in L_2$ together with the estimate

$$(1.3)\qquad \int_{|x|<1} |\operatorname{grad} u|^2 dx \leqslant C^2 \lambda$$

with C independent of u.

If we set $v(x) = u(x/R)$, $R = \sqrt{\lambda}$, we get in place of (1.1):

$$(1.4)\qquad -\Delta v = v, \quad |x| < R,$$

and in place of (1.2) and (1.3):

$$(1.5)\qquad \int_{|x|<R} |v|^2 dx + \int_{|x|<R} |\operatorname{grad} v|^2 dx \leqslant CR^n.$$

Next we set $w(x) = \varphi(x)v(x)$ where $\varphi(x) = \varphi_0(x/R)$ and $\varphi_0(x)$ is any infinitely differentiable function such that $\varphi_0(x) = 1$ if $|x| \leqslant 1/3$ and $\varphi_0(x) = 0$ if $|x| \geqslant 2/3$. Then we have

$$(1.6)\qquad -\Delta w = w - 2\operatorname{grad}\varphi \cdot \operatorname{grad} v - \Delta\varphi v.$$

Let F be any fundamental solution of (1.4), i.e. $-\Delta F = F + \delta$, where δ is Dirac's function. Then holds

$$(1.7)\qquad u(0) = v(0) = w(0) = -\int_{\frac{R}{3}<|x|<\frac{2R}{3}} F(-x)(2\operatorname{grad}\varphi \cdot \operatorname{grad} v + \Delta\varphi v)\,dx.$$

(1) Because if necessary we can as well work with a smaller sphere, say $|x| < \frac{1}{2}$.

Now a particular fundamental solution is

$$F(x) = c_n \frac{Y_{(n-2)/2}(|x|)}{|x|^{(n-2)/2}}$$

where Y_μ is Bessel's function of the second kind of order μ and c_n is a suitable normalization constant. But

$$Y_\mu(t) = O\left(\frac{1}{\sqrt{t}}\right), \quad t \to \infty.$$

It follows that

$$\int\limits_{\frac{R}{3}<|x|<\frac{2R}{3}} |F(x)|^2 dx \leqslant C^2 R, \quad R \geqslant 1.$$

Therefore we get from (1.7) in view of (1.5)

$$|u(0)| \leqslant C\Big(\int\limits_{\frac{R}{3}<|x|<\frac{2R}{3}} |F(x)|^2 dx\Big)^{1/2}\Big(\int R^{-2}|\operatorname{grad} v|^2 + \int R^{-4}|v|^2\Big)^{1/2}$$

$$\leqslant CR^{1/2}(R^{-1}R^{n/2}+R^{-2}R^{n/2}) \leqslant CR^{(n-1)/2} = C\lambda^{(n-1)/4}, \quad \lambda \geqslant 1.$$

(Here and in the sequel constants are denoted by one and the same letter C.) We have thus proved

THEOREM 1.1 (Minakshisundaram-Titchmarsh). *Any normalized solution of* (1.1) *satisfies*

$$|u(0)| \leqslant C\lambda^{(n-1)/4}, \quad \lambda \geqslant 1. \tag{1.8}$$

We note also the following

COROLLARY 1.1. *If v is a solution of* (1.4) *such that*

$$\int\limits_{|x|<R} |v(x)|^2 dx \leqslant CR^{1-\varepsilon}, \quad R \geqslant 1,\ \varepsilon > 0, \tag{1.9}$$

then $v = 0$.

It is readily seen that this is false if $\varepsilon = 0$.

Similar consequences may be obtained from the results of Section 2 and Section 3.

Remark 1.1. The same methods work also in L_p. We then find that (1.8) holds for any solution of (1.1) with the normalization condition (1.2) being replaced by

$$\int\limits_{|x|<1} |u|^p dx = 1.$$

2. Generalization I. We shall first extend the result and method of Section 1 to the equation

$$A_m u = \lambda u, \quad |x| < 1, \tag{2.1}$$

where A_m is any homogeneous of order m formally positive elliptic partial differential operator, i.e.

$$A_m = A_m(D) = \sum a_m D^m, \quad A_m(\xi) = \sum a_m \xi^m > 0 \quad \text{unless } \xi = 0. \tag{2.2}$$

Here and in the sequal we use D^m as the general mth order partial differentiation, $D^m = (-i)^m\left(\frac{\partial}{\partial x_1}\right)^{m_1}\ldots\left(\frac{\partial}{\partial x_n}\right)^{m_n}$ with $m = m_1+\ldots+m_n$; similarly ξ^m stands for $\xi_1^{m_1}\ldots\xi_n^{m_n}$.

We assume again that

$$\int\limits_{|x|<1} |u|^2 dx = 1.$$

Then we may assume without loss of generality that

$$\int\limits_{|x|<1} |D^k u|^2 dx \leqslant C^2\lambda^{2k/m}, \quad 0 \leqslant k \leqslant m. \tag{2.3}$$

If we set $v(x) = u(x/R)$, $R = \lambda^{1/m}$, we get

$$A_m v = v, \quad |x| < R \tag{2.4}$$

and

$$\int\limits_{|x|<R} |D^k v|^2 dx \leqslant C^2 R^n, \quad 0 \leqslant k \leqslant m. \tag{2.5}$$

Writing $w(x) = \varphi(x)v(x)$ with φ as in Section 1 we get

$$A_m w = w + \sum_{1\leqslant k\leqslant m} D^k\varphi A_m^{(k)} v \tag{2.6}$$

where $A_m^{(k)}$ stands for a partial differential operator of order $m-k$ (Leibnitz' formula).

If F is a fundamental solution of (2.4) we get (cf. (1.7))

$$u(0) = \sum_{1\leqslant k\leqslant m} \int\limits_{\frac{R}{3}<|x|<\frac{2R}{3}} F(-x) D^k\varphi A_m^{(k)} v\, dx. \tag{2.7}$$

We now use the following lemma which we prove later in Section 4:

LEMMA 2.1. *It is possible to choose F such that*

$$\int\limits_{R<|x|<2R} |F(x)|^2 dx \leqslant C^2 R, \quad R \geqslant 1, \tag{2.8}$$

where C is a constant. If the order m is sufficiently high ($m > n/2$), the same inequality (2.8) *holds also with the range of integration $|x| < R$.*

As in Section 1 we now get with the aid of (2.5), (2.7), (2.8), recalling also that $D^k\varphi$ is $O(R^{-k})$, the following estimate:

$$|u(0)| \leqslant C\Big(\int_{\frac{R}{2}<|x|<\frac{2R}{3}} |F(x)|^2dx\Big)^{1/2} \sum_{1\leqslant k\leqslant m} \sup|D^k\varphi| \Big(\int_{|x|<R} |A_m^{(k)}v|^2dx\Big)^{1/2}$$
$$\leqslant CR^{1/2}\sum_{1\leqslant k\leqslant m} R^{-k}R^{n/2} \leqslant CR^{(n-1)/2} = C\lambda^{(n-1)/2m}.$$

This proves

THEOREM 2.1. *Any normalized solution of* (2.1) *satisfies*

(2.9) $$|u(0)| \leqslant C\lambda^{(n-1)/2m}, \quad \lambda \geqslant 1.$$

3. Generalization II. Next we consider the equation

(3.1) $$Au = \lambda u, \quad |x|<1$$

where A is any partial differential operator with leading part A_m, where A_m is as in (2.2), i.e.

(3.2) $$A = A(x, D) = A_m(D) + \sum_{0\leqslant j\leqslant m-1} a_j(x)D^j$$

where the coefficients $a_j(x)$ are supposed to be "sufficiently" differentiable.

With $v(x) = u(x/R)$, $R = \lambda^{1/m}$, we get the equation (cf. (2.4))

(3.3) $$A_m v = v - \sum_{0\leqslant j\leqslant m-1} a_j(x/R)R^{j-m}D^jv$$

which again yields (cf. (2.7))

(3.4) $$u(0) = \sum_{1\leqslant k\leqslant m}\int_{\frac{R}{3}<|x|<\frac{2R}{3}} F(-x)D^k\varphi A_m^{(k)}v\,dx -$$
$$- \sum_{0\leqslant j\leqslant m-1}\int_{|x|<\frac{2R}{3}} F(-x)\varphi a_j(x/R)R^{j-m}D^jv\,dx.$$

We clearly may assume that (2.5) holds. Using the estimate obtained in Section 2 for the first term and estimating the second one in a similar fashion we obtain

$$|u(0)| \leqslant CR^{(n-1)/2} = C\lambda^{(n-1)/2m},$$

under the auxiliary assumption that $m > n/2$ (cf. lemma 2.1). However, by iterating (3.1) we can get as high order in A as we want to so this is no real restriction. Therefore holds

THEOREM 3.1. *Any normalized solution of* (3.1) *satisfies* (2.9).

Remark 3.1. If (3.2) would contain additional terms of order m, we would get $\lambda^{(n+1)/2m}$ in place of $\lambda^{(n-1)/2m}$ in (2.9). Therefore the present method is strictly limited to operators with constant leading part.

4. Proof of the estimate for the fundamental solution. We have yet to prove lemma 2.1. For this purpose we use a Fourier transform technique analogous to the one of [3] and [4]. We define F (formally) by

$$F(x) = (2\pi)^{-n}\int \frac{e^{ix\xi}}{A(\xi)-1}d\xi.$$

We write

$$F(x) = \sum_{j=0}^{N} F_j(x)$$

where

$$F_j(x) = (2\pi)^{-n}\int \frac{e^{ix\xi}\chi_j(\xi)}{A(\xi)-1}d\xi$$

and χ_j is a finite partition of unity, $\sum_{j=0}^{N}\chi_j(\xi) = 1$, with each χ_j infinitely differentiable, such that (i) the support of χ_0 is contained in the set where $A(\xi)<1$, (ii) the support of χ_j, $0<j<N$, is compact and, in the neighbourhood of it, it is possible to introduce new local coordinates such that $A(\xi)=1$ goes over to $\xi_1 = 0$, (iii) the support of χ_N is contained in $A(\xi)>1$. We shall treat each case separately.

$j=0$. There is no difficulty because F_0 is the inverse Fourier transform of an infinitely differentiable function with compact support, thus in particular it is square integrable.

$j=N$. Again there is no difficulty because integration by parts shows that x^rF_N is bounded when r is sufficiently large so that F_N too is square integrable over $|x|>1$. If $m>n/2$, we see directly that F_N is square integrable over the whole of R^n. This eventually leads to the second half of lemma 2.1.

$0<j<N$. Here arises the essential difficulty. We have the following inequality:

(4.1) $$\Big(\int_{R<|x|<2R}|F_j(x)|^2dx\Big)^{1/2} \leqslant CR^{-s}\Big(\int(1+|x|^2)^s|F_j(x)|^2dx\Big)^{1/2}$$
$$\leqslant CR^{-s}\Big(\int|(1-\Delta)^{s/2}\hat{F}_j(\xi)|^2d\xi\Big)^{1/2}, \quad R\geqslant 1,$$

for any $s \gtreqless 0$. Here powers of $1-\Delta$ are defined by

$$\widehat{(1-\Delta)^{s/2}\hat{T}} = (1+|x|^2)^{s/2}T$$

for any tempered distribution $\hat{T} = \hat{T}(\xi)$; T is the inverse Fourier transform of $\hat{T}$. Let us introduce the spaces H^s corresponding to the norms

$$\|\hat{T}\|_s = \left((2\pi)^{-n}\int|(1-\Delta)^{s/2}\hat{T}|^2 d\xi\right)^{1/2} = \left(\int(1+|x|^2)^s|T(x)|^2 dx\right)^{1/2}.$$

Let $F_j = F_{j0}+F_{j1}$ where F_{j0} and F_{j1} are arbitrary. Then (4.1) yields; for any s_0, s_1, to fix the ideas we assume $s_0 < s_1$,

$$\Big(\int_{R<|x|<2R}|F_j(x)|^2dx\Big)^{1/2} \leqslant CR^{-s_0}(\|\hat{F}_{j_0}\|_{s_0}+R^{s_0-s_1}\|\hat{F}_{j_1}\|_{s_1})$$

or, if we put (cf. e.g. [2], [4])

$$K(t, \hat{F}_j) = \inf(\|\hat{F}_{j0}\|_{s_0}+t\|\hat{F}_{j1}\|_{s_1}),$$

even

$$\Big(\int_{R<|x|<2R}|F_j(x)|^2dx\Big)^{1/2} \leqslant CR^{-s_0}K(R^{s_0-s_1}, \hat{F}_j).$$

Thus we see that

$$\Big(\int_{R<|x|<2R}|F_j(x)|^2dx\Big)^{1/2} \leqslant CR^{1/2}, \qquad R \geqslant 1,$$

holds, which would complete the proof, if we can show that

$$K(t, \hat{F}_j) \leqslant Ct^{\theta}, \qquad t \leqslant 1 \quad \text{with} \quad -\tfrac{1}{2} = s_0(1-\theta)+s_1\theta,$$

i.e. $\hat{F}_j \in H^{-1/2,\infty} = (H^{s_0}, H^{s_1})_{\theta,\infty}$ (cf. e.g. [2], [4]). To show this we remark that the spaces $H^{s,\infty}$ are invariant for changes of local coordinates. (This is immediate for H^s, s integer, from which the actual statement follows simply by interpolation.) Thus in view of (ii) it suffices to show that

$$\frac{1}{\xi_1} \in (H^{-1/2,\infty})_{\mathrm{loc}} \quad \text{or} \quad \log|\xi_1| \left(\text{suitable integral of } \frac{1}{\xi_1}\right) \in (H^{1/2,\infty})_{\mathrm{loc}}.$$

But the spaces $H^{s,\infty}$ $(0 < s < 1)$ are characterized by the condition

$$\left(\int|\hat{T}(\xi+h)-\hat{T}(\xi)|^2d\xi\right)^{1/2} \leqslant C|h|^{1/2}, \qquad \hat{T}\in L_2.$$

Therefore everything follows from the following elementary inequality

$$\int_{-\infty}^{\infty}\big|\log|\xi_1+h_1|-\log|\xi_1|\big|^2d\xi_1 \leqslant C_1|h_1|$$

which is immediate since we can reduce it to the case $h_1 = \pm 1$ by a change of variable.

Collecting together the information contained in each of the three above steps, we see that we have proved lemma 2.1.

Remark 4.1. The same method is applicable in L_p too, as long as $p < 2$. We then obtain analogous of theorem 2.1 and theorem 3.1 with the exponent $(n-1)/p$ (cf. remark 1.1). However, the results of the following sections can be deduced from the case $p = 2$ alone so we shall not carry out details.

5. Application to eigenfunction expansions. Consider now the eigenvalue problem

(5.1) $$Au = \lambda u \text{ in } \Omega$$

where A is the operator (3.2) and Ω any domain of R^n, self-adjoint boundary conditions being imposed on the boundary of Ω.

We assume that the spectrum is *discrete*. Then every f can be expanded in a series

$$f(x) = \sum_{\nu=1}^{\infty} f_\nu(x)$$

where

(5.2) $$f_\nu(x) = \sum_r (f, \varphi_{\nu r})\varphi_{\nu r}(x) = \int_\Omega \sum_r \varphi_{\nu r}(x)\overline{\varphi_{\nu r}(y)}f(y)\,dy$$

the summation being extended over an orthonormal basis of eigenfunctions $\varphi_{\nu r}$ belonging to the νth eigenvalue λ_ν, i.e. $\varphi_{\nu r}$ satisfies the eigenvalue equation

$$A\varphi_{\nu r} = \lambda_\nu\varphi_{\nu r} \text{ in } \Omega$$

together with the boundary conditions, and moreover the normalization condition

(5.3) $$\int_\Omega \varphi_{\nu r}(x)\overline{\varphi_{\nu s}}(x)\,dx = \begin{cases}1, & r = s,\\ 0, & r \neq s.\end{cases}$$

We shall prove the following

THEOREM 5.1. *Assume $f \in L_1$ and vanishes off a compact subset K of Ω. Then holds*

(5.4) $$f_\nu(x) = o(\lambda_\nu^{(n-1)/m}), \qquad \nu \to \infty,$$

uniformly on any compact subset K_1 of Ω.

Proof. By Cauchy's inequality we get from (5.2) when $x \in K_1$

(5.5) $$|f_\nu(x)| \leqslant \sup_{K\cup K_1}\sum_r|\varphi_{\nu r}(x)|^2\int_K|f(x)|\,dx.$$

We claim that

(5.6) $$\sum_r|\varphi_{\nu r}(x)|^2 = O(\lambda_\nu^{(n-1)/m})$$

uniformly on any compact part K_2 of Ω. To this end let us apply theorem 3.1 to

$$u(x) = \sum_r \varphi_{\nu r}(x)\overline{\varphi_{\nu r}(x_0)}, \qquad x_0 \in K_2.$$

We get

$$\sum_r |\varphi_{\nu r}(x_0)|^2 = |u(x_0)| \leqslant C\lambda_r^{(n-1)/2m}\Big(\int_\Omega |u(x)|^2 dx\Big)^{1/2}.$$

But in view of (5.3)

$$\int |u(x)|^2 dx = \sum_{r,s} \varphi_{\nu r}(x_0)\overline{\varphi_{\nu s}(x_0)} \int \varphi_{\nu r}(x)\overline{\varphi_{\nu s}(x)}\, dx = \sum_r |\varphi_{\nu r}(x_0)|^2$$

and (5.6) readily follows. Now (5.5) and (5.6) together imply (5.4) except for an "O" in place of the "o". But a density argument, as explained in the Introduction, at once gives the improvement to "o". The proof is complete.

Remark 5.1. We note also that if $f \in L_2$ and vanishes off a compact subset K of Ω then

$$f_\nu(x) = O(\lambda_\nu^{(n-1)/2m}), \qquad \nu \to \infty,$$

uniformly on any compact subset K_1 of Ω. Indeed, since f_ν satisfies equation (5.1), we only have to involve theorem 3.1 directly. Similarly if $f \in L_p$, $1 < p < 2$, and vanishes off K we obtain by *interpolation*

$$f_\nu(x) = O(\lambda_\nu^{(n-1)/pm}),$$

uniformly in K_1.

6. On the expansion in spherical harmonics. This section is essentially independent of the rest of the article.

We start by considering a somewhat more general situation. We consider any compact infinitely differentiable manifold Ω of dimension n upon which acts transitively a Lie group G, and an elliptic partial differential operator A on Ω which is left invariant by the actions of G. We choose also an invariant measure dx on Ω assuming that A becomes self-adjoint with respect to the associated scalar product $\int u\bar{v}\, dx$. Then the eigenvalue equation

$$Au = \lambda u \text{ in } \Omega$$

has a denumerable number of eigenvalues λ_ν and choosing an orthonormal basis $\varphi_{\nu r}$ of eigenfunctions for each ν we get a complete set of eigenfunctions. Every f can be expanded in a series

$$f(x) = \sum_{\nu=1}^{\infty} f_\nu(x) \quad \text{with} \quad f_\nu(x) = \int \sum_r \varphi_{\nu r}(x)\overline{\varphi_{\nu r}(y)} f(y)\, dy$$

as in Section 5. We now observe (cf. [5], Section 3) that

$$\sum_r |\varphi_{\nu r}(x)|^2, \qquad \nu \text{ fixed},$$

is independent of x. Thus integrating over Ω and using the normalization of $\varphi_{\nu r}$ we get

$$\sum_r |\varphi_{\nu r}(x)|^2 = \frac{M_\nu}{V} \tag{6.1}$$

where M_ν is the *multiplicity* of the eigenvalue λ_ν and V the *volume* of Ω, $V = \int_\Omega dx$. Using now (6.1) in place of theorem 3.1 in the proof of theorem 5.1 we get

THEOREM 6.1. *Assume $f \in L_1$. Then holds uniformly*

$$f_\nu(x) = o(M_\nu). \tag{6.2}$$

We consider some simple special cases.

Example 6.1. If $\Omega = T^n$ = torus of dimension n, G = translations, $A = -\Delta$ (Laplace operator), we get expansion in ordinary multiple Fourier series, by choosing $\varphi_{\nu r}(x) = e^{ihx} = e^{i(h_1x_1+\ldots+h_nx_n)}$, $h_1^2+\ldots+h_n^2 = \lambda_\nu$, h_j integer, so that

$$f_\nu(x) = \sum_r a_h e^{ihx}$$

where a_h are the ordinary Fourier coefficients. From additive number theory (Waring's problem) it is known that $M_\nu = O(\lambda_\nu^{(n-1)/2})$. Thus in this case (6.2) can be replaced by our previous $f_\nu(x) = O(\lambda_\nu^{(n-1)/m})$ with $m = 2$. However, this has a little interest because the same result follows in a trivial manner already from the usual Riemann-Lebesgue lemma for multiple Fourier series, $a_h = O(1)$.

Example 6.2. Let Ω and G be as in example 6.1 but let A be any elliptic operator with constant coefficients, which is then automatically invariant for G. Then we may combine the present results with those of Section 5 to obtain a result in a different sense. Indeed, by (6.1) and (5.6) we get $M_\nu = O(\lambda_\nu^{(n-1)/m})$. Now M_ν, as above in example 6.1, can be interpreted as the number of integer solutions of $A(h) = \lambda = \lambda_\nu$. Thus we have obtained a (rough) estimate of the number of solutions of this "Diophantine" problem. *We do not know whether this has any interest from the point of view of additive number theory.*

Example 6.3. Finally, we come now to our main case, let Ω = unit sphere in R^{n+1}, G = rotations, $A = -\Delta$ (Laplace-Beltrami operator). We are thus dealing with expansion in spherical harmonics (Laplace series). The eigenfunctions (spherical harmonics) belonging to λ_ν correspond to harmonic polynomials of degree ν in R^{n+1}. Using this it is easy to see that $M_\nu = O(\nu^{n-1})$ and, since moreover $\lambda_\nu = \nu(\nu+n-1)$, we again get $f_\nu(x) = O(\lambda_\nu^{(n-1)/m})$ with $m = 2$. As already noted in the Introduction this was proved by V. Shapiro [6] by a different method. Thus we have at least one (non-trivial) instance of variable coefficients when we get the same estimate as in the case of constant coefficients (Section 5).

Let us finally indicate some would-be consequences of our Riemann-Lebesgue lemma in the case of a compact manifold. We *assume* now that

(6.3) $$\sum_r |\varphi_{\nu r}(x)|^2 = O(\lambda_\nu^{(n-1)/m})$$

holds uniformly over Ω, the operator A may or may not be invariant under G; but, strictly speaking, our only example is the case considered in example 6.3 (spherical harmonics). Then we have of course

(6.4) $$f_\nu(x) = O(\lambda_\nu^{(n-1)/m}), \quad \nu\to\infty,$$

uniformly, if $f\in L_1$. But a similar argument gives also

$$f_\nu(x) = O(\lambda_\nu^{(n-1)/2m}), \quad \nu\to\infty,$$

uniformly, if $f\in L_2$ (cf. remark 5.1). Noting that the mapping $f\to f_\nu$ is a projection in L_2 we see that

$$\sup|f_\nu(x)| \leqslant C\lambda_\nu^{(n-1)/2m}\Big(\int_\Omega |f_\nu(x)|^2dx\Big)^{1/2}$$

with C independent of ν (and f). Taking the sum, we get

(6.5) $$\Bigg(\sum_\nu\bigg(\frac{\sup_\Omega|f_\nu(x)|}{\lambda_\nu^{(n-1)/2m}}\bigg)^2\Bigg)^{1/2} \leqslant C\Big(\int_\Omega |f(x)|^2dx\Big)^{1/2} < \infty \quad \text{if} \quad f\in L_2.$$

This may be considered as a form of Parseval's formula. By *interpolation* we get from (6.4) and (6.5):

(6.6) $$\Bigg(\sum_\nu\bigg(\frac{\sup_\Omega|f_\nu(x)|}{\lambda_\nu^{(n-1)/pm}}\bigg)^q\Bigg)^{1/q} \leqslant C\Big(\int_\Omega |f(x)|^pdx\Big)^{1/p} < \infty \quad \text{if} \quad f\in L_p,$$

$$\frac{1}{p}+\frac{1}{q} = 1, \quad 1<p<2, \quad 2<q<\infty,$$

which is a form of Hausdorff-Young theorem (cf. [8], vol. 2, p. 101).

We contrast this with a previous result (cf. [5], Section 3, cf. also [6] for the case of spherical harmonics), now again under the assumption that A is invariant under G:

(6.7) $$\sum_\nu \sup_\Omega |f_\nu(x)| < \infty \quad \text{if} \quad f\in W_2^{n/2,1}$$

where $W_p^{s,q}$ denote Sobolev spaces of fractional order (cf. e.g. [2], [4], [5]). This is an analogue of a theorem of Bernstein (cf. [8], vol. 1, pp. 240-241). By interpolation we get from (6.7) and (6.4):

(6.8) $$\Bigg(\sum_\nu\bigg(\frac{\sup_\Omega|f_\nu(x)|}{\lambda_\nu^{(n-1)/pm}}\bigg)^q\Bigg)^{1/q} < \infty \quad \text{if} \quad f\in W_2^{s,q},$$

$$s = n\Big(\frac{1}{q}-\frac{1}{2}\Big), \quad \frac{1}{p}+\frac{1}{q} = 1, \quad 2<p<\infty, \quad 1<q<2.$$

This again is an analogue of a theorem of Szász (cf. [8], vol. 1, p. 243) and extends somewhat a previous generalization of this theorem (cf. [5], Section 3).

We may sum up the above results (6.4)-(6.8): we have

$$\Bigg(\sum_\nu\bigg(\frac{\sup_\Omega|f_\nu(x)|}{\lambda_\nu^{(n-1)/pm}}\bigg)^q\Bigg)^{1/q} < \infty, \quad \frac{1}{p}+\frac{1}{q} = 1$$

provided

$f\in L_1$	if	$p=1$	(Riemann-Lebesgue),
$f\in L_p$	if	$1<p<2$	(Hausdorff-Young),
$f\in L_2$	if	$p=2$	(Parseval),
$f\in W_2^{n(1/q-1/2),q}$	if	$2<p<\infty$	(Szász),
$f\in W_2^{n/2,1}$	if	$p=\infty$	(Bernstein),

all this thus under the sole assumption of (6.3) along with the assumption of invariance of A under G.

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Über die Methode des „schnellsten Falles" für das Minimumproblem von Funktionalen in Hilbertschen Räumen

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Auf einem reellen Hilbertschen Raum sei ein reelles, differenzierbares Funktional $\Phi(x)$ erklärt. Unter einem Gradienten von Φ versteht man bekanntlich dasjenige Element: $\operatorname{grad}\Phi(x)$, welches die Gleichung $\Phi'(x,h) = \big(\operatorname{grad}\Phi(x), h\big)$ identisch in $h\in H$ erfüllt. Die in dieser Note betrachtete Methode besteht in folgendem: man nimmt an, daß die Gleichung

$$\frac{dx(t)}{dt} = -\operatorname{grad}\Phi\big(x(t)\big) \tag{I}$$

eine Lösung für alle $t \geqslant 0$ besitzt; dann konvergiert unter gewissen Voraussetzungen $\Phi\big(x(t)\big)$ gegen sein Minimumwert $\inf\Phi(x)$, oder auch $x(t)$ gegen Element x^*, mit $\Phi(x^*) = \inf\Phi(x)$. Auch der Fall, wenn die Gleichung (I) nur annäherungsweise erfüllt ist, wird behandelt.

Die Idee dieser Methode rührt schon vom A. Cauchy her, und wurde von anderen Verfassern verallgemeinert (s. [1] und [2]).

Satz 1. *Sei $\Phi(x)$ ein auf H erklärtes, konvexes, stetiges von unten beschränktes Funktional, dessen Gradient stetig ist. Die abstrakte Funktion $x(t)$ erfülle die Gleichung* (I). *Dann gilt:*

$$\lim\Phi\big(x(t)\big) = \inf\Phi(x).$$

Beweis erfolgt in folgenden Schritten:

Lemma 1. *Erfüllen die Elemente x_1, x_2 die Ungleichung $\Phi(x_2) < \Phi(x_1)$, so gilt:*

$$\|\Phi(x_1)-\Phi(x_2)\| \leqslant \|\operatorname{grad}\Phi(x_1)\|\cdot\|x_1-x_2\|.$$

Aus der Konvexität von Φ ergibt sich nämlich

$$\Phi\big(x_1+\varepsilon(x_2-x_1)\big) \leqslant \Phi(x_1)+\varepsilon[\Phi(x_2)-\Phi(x_1)] \quad \text{für} \quad 0<\varepsilon<1$$

also:

$$\frac{1}{\varepsilon}\{\Phi\big(x_1+\varepsilon(x_2-x_1)\big)-\Phi(x_1)\} \leqslant \Phi(x_2)-\Phi(x_1)$$