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Convolution approximation and shift approximation

by

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Part I. Convolution approximation

1. In this paper, the convolution

$$\int_0^t g(t-\tau)k(\tau)\,d\tau$$

will be denoted by gk (instead of the usual notation $g*k$). The ordinary product of two functions will be denoted, on writing explicitely the arguments, e. g. $g(t)k(t)$. Such a notation is also used in my book [2].

Let $C[0,T]$ $(0<T<\infty)$ be the class of real continuous functions on the interval $[0,T]$, $C_0[0,T]$ the subclass of $C[0,T]$ of functions which vanish at 0, and $C_0^\infty[0,T]$ the class of infinitely derivable functions in $[0,T]$ which vanish at 0 together with all their derivatives.

THEOREM I. *For any fixed $g\in C[0,T]$ which does not vanish identically in the right neighbourhood of 0, the set of convolutions gk with $k\in C_0^\infty[0,T]$ is dense in $C_0[0,T]$.*

This theorem is, in fact, due to Foiaş [1], who needed it to prove that the set of continuous functions is dense in the space of operators. However, Foiaş formulated it in a slightly different form: *For any fixed $g\in L^1[0,T]$, not vanishing almost everywhere at the neighbourhood of 0, the set of convolutions gk with $k\in C[0,T]$ is dense in $L^1[0,T]$.* Another formulation of Theorem I is given in the paper [3]. There it is proved that, *for any fixed $g\in C_0[0,T]$, non-vanishing in the neighbourhood of 0, the set of convolutions gk with absolutely continuous functions k is dense in $C[0,T]$.* The proof given in [3] can be used for Theorem I; the only needed modifications are the following: one assumes that $g\in C[0,T]$ (instead of $g\in C_0[0,T]$), $k_n, k\in C_0^\infty[0,T]$ (instead of $k_n, k\in AC$ (absolutely continuous functions)) and one has $k'\in C_0^\infty[0,T]$ (instead of $k'\in L^1[0,T]$).

Evidently, Theorem I is a little stronger than my earlier formulation in [3]. It is easy to see that it is also stronger than the formulation in Foiaş's paper [1]. In fact, let $g \in L^1[0, T]$ and let

$$h = \int_0^t g(\tau)\, d\tau.$$

By Theorem I, the set of convolutions hk', where $k \in C_0^\infty[0, T]$ is dense in $C_0[0, T]$. Since $hk' = gk$, we may also say that the set of convolutions gk where $k \in C_0^\infty[0, T]$ is dense in $C_0[0, T]$. A fortiori, the set of convolutions gk where $k \in C[0, T]$ is dense in $C_0[0, T]$. Since $C_0[0, T]$ is dense in $L^1[0, T]$, this set is dense in $L^1[0, T]$.

In the Part I of this paper we are going to strengthen Theorem I in three steps. In this way we shall obtain Theorems II, III and IV, each of them being stronger than the preceding one.

THEOREM II. *Given any fixed $g \in C[0, T]$ which does not vanish identically in the right neighbourhood of* 0, *the set of convolutions gk with $k \in C_0^\infty[0, T]$ is dense in $C_0^\infty[0, T]$.*

Proof. Let $f \in C_0^\infty[0, T]$ and let ε_n be a sequence of positive numbers, tending to 0 as $n \to \infty$. By Theorem I, there exists, for every positive integer n, a function $k_n \in C_0^\infty[0, T]$ such that

$$|gk_n^{(n)} - f^{(n)}| < T^{-n} \varepsilon_n.$$

This implies that

$$|gk_n^{(i)} - f^{(i)}| < T^{-i} \varepsilon_n \quad \text{for} \quad i = 0, 1, \ldots, n.$$

Thus, for any fixed i, the sequence $gk_n^{(i)}$ converges uniformly to $f^{(i)}$, as $n \to \infty$. This means that $gk_n \to f$ in the topology of $C_0^\infty[0, T]$, which proves Theorem II.

The fact that Theorem II is stronger than Theorem I follows from the remark that $C_0^\infty[0, T]$ is dense in $C_0[0, T]$.

Let $C[0, \infty)$ denote the class of continuous functions in the interval $[0, \infty)$ and $C_0^\infty[0, \infty)$ the class of indefinitely derivable functions in that interval. We say that a sequence of functions from the class $C[0, \infty)$ or from the class $C_0^\infty[0, \infty)$ is *convergent* in $C[0, \infty)$ or in $C_0^\infty[0, \infty)$ respectively, if the corresponding sequence of functions restricted to any bounded interval $[0, T]$ is convergent in the proper topology, of $C[0, T]$ or $C_0^\infty[0, T]$.

THEOREM III. *For any fixed $g \in C[0, \infty)$ which does not vanish identically in the right neighbourhood of* 0, *the set of convolutions gk with $k \in C_0^\infty[0, \infty)$ is dense in $C_0^\infty[0, \infty)$.*

Proof. Let $f \in C_0^\infty[0, \infty)$. By Theorem II, there is, for any positive integer p, a sequence of functions $k_{pn} \in C_0^\infty[0, \infty)$ such that $|gk_{pn}^{(i)} - f^{(i)}| < \varepsilon_i$ in $[0, p]$, where $0 < \varepsilon_{in} \to 0$ as $n \to \infty$. This implies that the diagonal sequence $gk_{nn}^{(i)}$ converges to $f^{(i)}$ uniformly in every interval $[0, T]$ $(0 < T < \infty)$, which proves Theorem III.

Evidently, Theorem III reduces to Theorem II, when restricting the considered functions from $[0, \infty)$ to $[0, T]$.

2. Let M_+ denote the space of all operators $a = p/q$ (see [2]), where $p, q \in C[0, \infty)$ and q does not vanish identically in any right neighbourhood of 0. We say that a sequence $a_n \in M_+$ *converges* in M_+, if there is a function $q \in C[0, \infty)$ non-vanishing identically in any neighbourhood of 0, such that all the operational products $a_n q$ are functions of class $C[0, \infty)$ and $a_n q$ converges almost uniformly in $C[0, \infty)$ (i. e., uniformly in every bounded interval $[0, T]$). We say that an operator $a \in M_+$ *does not vanish* in the right neighbourhood of 0, if it is of the form p/q, where both p and q are functions which do not vanish identically in the right neighbourhood of 0 (see [4]).

THEOREM IV. *For any fixed operator $g \in M_+$ which does not vanish in the right neighbourhood of* 0, *the set of elements gk, where $k \in C_0^\infty[0, \infty)$, is dense in $C_0^\infty[0, \infty)$.*

Proof. Let $f \in C_0^\infty[0, \infty)$. There is a function $q \in C[0, \infty)$, non-vanishing identically in the right neighbourhood of 0, such that $gq \in C[0, \infty)$; evidently the function gq does not vanish either identically in the right neighbourhood of 0. Thus, by Theorem III, there are functions $k_n \in C_0^\infty[0, \infty)$ such that the sequence $(gq)k_n$, i. e., $g(qk_n)$, converges to f in the topology of $C_0^\infty[0, \infty)$. Since $qk_n \in C_0^\infty[0, \infty)$, Theorem IV is proved.

In order to see that Theorem IV is stronger than Theorem III, it suffices to observe that $C[0, \infty)$ is a subset of M_+.

Part II. Shift approximation

3. Let S be a linear subspace of M_+ containing $C_0^\infty[0, \infty)$, with the following properties:

1° S is a locally convex topological space such that every sequence $f_n \in C_0^\infty[0, \infty)$ which converges in $C_0^\infty[0, \infty)$ converges also in S to the same limit; moreover, every sequence $f_n \in S$ which converges in S, converges also in M_+ to the same limit; finally, we assume that $C_0^\infty[0, \infty)$ is dense in S;

2° If $f \in S$, then $h^\lambda f \in S$ (h shift-operator) for every $\lambda \geqslant 0$. In the topology of S, $h^\lambda f$ is a continuous function of λ in the interval $0 \leqslant \lambda < \infty$;

3° There is a family of semi-norms $\|f\|_a$ with $a \in A$ such that: for any $a \in A$ there is a number $\lambda_a > 0$ such that $\lambda > \lambda_a$ implies $\|h^\lambda f\|_a = 0$ for every $f \in S$.

We are going to give a few examples of the space S.

(i) *Space* $C_0^\infty[0,\infty)$. Here, we have

$$h^\lambda f = \begin{cases} f(t-\lambda) & \text{for} \quad t \geqslant \lambda, \\ 0 & \text{for} \quad 0 \leqslant t < \lambda. \end{cases} \tag{1}$$

For A, we can take the set of pairs $a = (p, r)$ of integers p, r $(p \geqslant 0, r \geqslant 1)$ and then let

$$\|f\|_{(p,r)} = \max_{0 \leqslant t \leqslant r} |f^{(p)}(t)|.$$

(ii) *Space* $C_0^p[0,\infty)$. The elements of this space are functions in $[0,\infty)$, derivable up to the order p in that interval, and vanishing together with these derivatives at 0. A sequence $f_n \in C_0^p[0,\infty)$ is said to *converge* in $C_0^p[0,\infty)$, if for any $i = 0, \ldots, p$, the sequence $f_n^{(i)}$ converges uniformly in every interval $[0, T]$ $(0 < T < \infty)$. Formula (1) holds also in the actual case. For A, we can take $a = (i, r)$ of integers i, p $(0 \leqslant i \leqslant p, r \geqslant 1)$ and then let

$$\|f\|_{(i,r)} = \max_{0 \leqslant t \leqslant r} |f^{(i)}(t)|.$$

(iii) *Space* $C_0[0,\infty)$. Its elements are continuous functions in $[0,\infty)$, vanishing at 0. This is a particular case of the preceding example (with $p = 0$). Actually, for A we can take the set of all positive integers $a = r$ and let

$$\|f\|_r = \max_{0 \leqslant t \leqslant r} |f(t)|.$$

(iv) *Space* $L^p[0,\infty)$, $p \geqslant 1$. The elements of this space are functions which are locally p-integrable in $[0,\infty)$, i. e. p-integrable on every bounded interval $[0, T]$. Formula (1) holds also in the present case. For A, we can take the set of all positive integers and let

$$\|f\|_r = \sqrt[p]{\int_0^r |f(\tau)|^p d\tau}.$$

(v) *Space* D'_+. As elements of this space we take the distributions whose support lies on $[0,\infty)$. It turns out to say that these elements are distributions defined on the whole line $(-\infty,\infty)$ and vanish in $(-\infty, 0)$. In particular, continuous functions in $(-\infty,\infty)$, vanishing in $(-\infty, 0)$, are distributions. In order to imbed $C_0^\infty[0,\infty)$ into D'_+, we extend the definition of $f \in C_0^\infty[0,\infty)$ onto the negative part of the real axis, assuming that f vanishes on that part. Evidently, formula (1) makes sense in the case of D'_+. For A, we can take the set of all smooth (infinitely derivable) functions a of bounded support (vanishing outside a bounded interval). Then we let for $f \in D'_+$,

$$\|f\|_a = \Big| \int_{-\infty}^{\infty} f(t)\, a(t)\, dt \Big|.$$

(vi) *Space* M_q. The elements of this space are operators (elements of M_+) which can be represented in the form $f = p/q$, where $p, q \in C[0,\infty)$ and q does not vanish identically in the right neighbourhood of 0. The space M_q is thus determined by the function q. For A, we can take the set of positive integers and let

$$\|f\|_r = \max_{0 \leqslant t \leqslant r} |qf|$$

(qf is a continuous function).

We have evidently $C_0^\infty[0,\infty) \subset C_0^p[0,\infty) \subset C_0[0,\infty) \subset L^p[0,\infty) \subset D'_+$. We also have $D'_+ \subset M_q$, provided we take for q a function of class $C_0^\infty[0,\infty)$; then D'_+ is a proper subset of M_q.

THEOREM V. *For every operator $g \in S$ which does not vanish in the right neighbourhood of 0, the set of elements of the form*

$$\lambda_1 h^{\tau_1} g + \ldots + \lambda_n h^{\tau_n} g, \tag{2}$$

where λ_i and τ_i are real numbers, $\tau_i > 0$, and h is the shift-operator, is dense in S.

Proof. Let us consider the integral

$$\int_0^\infty h^\tau g k(\tau)\, d\tau \tag{3}$$

where $k \in C_0^\infty[0,\infty)$. Remark that, in the interpretations (i)-(v), this integral can be written in the form

$$\int_0^t g(t-\tau) k(\tau)\, d\tau,$$

because of formula (1). Thus it equals the convolution gk. We shall show that it equals gk also in the general case. In fact, the integral

$$\int_0^b h^\tau g k(\tau)\, d\tau \tag{4}$$

is defined for every finite $b > 0$, since its integrand is continuous. The value of (3) is to be considered as the limit of (4), as $b \to \infty$. The existence of that limit follows from the inequality

$$\Big\| \int_{\tau_1}^{\tau_2} h^\tau g k(\tau)\, d\tau \Big\|_a \leqslant \int_{\tau_1}^{\tau_2} \|h^\tau g\|_a |k(\tau)|\, d\tau \quad (\tau_1 < \tau_2)$$

and from the fact that $\|h^\tau g\|_a = 0$ for $\lambda_a < \tau_1 \leqslant \tau \leqslant \tau_2$. On the other hand, by 1°, the limit (3) can be also considered in the operational sense, and so we see that it equals

$$g\int_0^\infty h^\tau k(\tau)\,d\tau = gk \in S$$

(see e. g. formula (9.1), p. 337, of [2]).

Let

$$w_n(h^{1/n}) = \lambda_1 h^{1/n} + \lambda_2 h^{2/n} + \ldots + \lambda_{n^2} h^n,$$

where

$$\lambda_i = \int_{(i-1)/n}^{i/n} k(\tau)\,d\tau \qquad (n = 1, 2, \ldots, n^2).$$

We have

$$\|gk - w_n(h^{1/n})g\|_a \leqslant \Big\| \int_0^n h^\tau gk(\tau)\,d\tau - w_n(h^{1/n})g \Big\|_a + \Big\| \int_n^\infty hgk(\tau)\,d\tau \Big\|_a.$$

For $n > \lambda_a$ the last integral vanishes, so we can write

$$(5) \qquad \|gk - w_n(h^{1/n})g\|_a \leqslant \Big\| \sum_{i=1}^{n^2} \int_{(i-1)/n}^{i/n} (h^\tau g - h^{1/n} g)\,k(\tau)\,d\tau \Big\| \leqslant \sum_{i=1}^{n^2} \int_{(i-1)/n}^{i/n} \|h^\tau g - h^{1/n} g\|_a \cdot |k(\tau)|\,d\tau.$$

For $\tau \geqslant (i-1)/n > \lambda_a$ we have $\|h^\tau g - h^{1/n} g\|_a \leqslant \|h^\tau g\|_a + \|h^{1/n} g\|_a = 0$, it therefore suffices to consider, in (5), only expressions

$$(6) \qquad \|h^\tau g - h^{1/n} g\|_a$$

with τ and i/n belonging to the bounded interval $0 \leqslant \tau \leqslant \lambda_a + 1$. Since the function $h^\lambda g$ is supposed continuous, expression (6) becomes less than any given $\varepsilon > 0$ if $0 \leqslant (i-1)/n \leqslant \tau \leqslant i/n \leqslant \lambda_a + 1$ and n is sufficiently large, say $n > n_0$. Thus we obtain, for $n > n_0$,

$$\|gk - w_n(h^{1/n})g\|_a \leqslant \varepsilon \int_0^{\lambda_a+1} |k(\tau)|\,d\tau.$$

This proves that $w_n(h^{1/n})g \to gk$ in the topology of S. We can also say that the set of elements $w(h^{1/n})g$, where w are polynomials with real coefficients, is dense in the set of convolutions gk with respect to the topology of S. Since the set of convolutions gk is dense in $C_0^\infty[0, \infty)$, by Theorem IV, and the last set is dense in S, by hypothesis, the set of elements $w(h^{1/n})g$ is dense in S, which proves Theorem V.

4. We are now going to discuss some particular cases of Theorem V.

If S is one of the spaces $C_0^\infty[0, \infty)$, $C_0^p[0, \infty)$ or $C_0[0, \infty)$, then the hypothesis that the operator g does not vanish in the right neighbourhood of 0 means that the function g (actually g is a function) does not vanish identically in this neighbourhood. Theorem V says that, for any fixed g with that property, the set of elements

$$(7) \qquad \lambda_1 g(t-\tau_1) + \ldots + \lambda_n g(t-\tau_n)$$

with $\tau_i > 0$ is dense in the considered space.

This implies, in particular, that *every function $f \in C[0, \infty)$ (not necessarily vanishing at 0) can be approximated almost uniformly in $[0, \infty)$ by sums (7) with $g \in C_0[0, \infty)$, where at most the number τ_1 is negative, all others τ_i $(i > 1)$ being positive.*

In fact, there is a point $t_1 > 0$ such that $g(t_1) \neq 0$. The function

$$f(t) - \frac{f(t_1)}{g(t_1)}\, g(t+t_1)$$

belongs evidently to $C_0[0, \infty)$ and can be therefore approximated almost uniformly by sums

$$\lambda_2 g(t-\tau_2) + \ldots + \lambda_n g(t-\tau_n).$$

Hence, our assertion follows, on taking $\lambda_1 = f(t_1)/g(t_1)$ and $\tau_1 = -t_1$. If we restrict the functions to a bounded interval $[0, T]$, then we obtain a theorem proved in [3].

In a similar way we can show that *every function $f \in C^p[0, \infty)$ can be approximated almost uniformly together with their derivatives up to the order p by sums (7), where at most $p+1$ numbers $\tau_1, \ldots, \tau_{p+1}$ are negative.*

If $S = L^p[0, \infty)$ $(p \geqslant 1)$, then the assumption that g does not vanish in the right neighbourhood of 0 means that there is no right neighbourhood of 0 in which the function g vanishes almost everywhere. Theorem V says that the set of elements (7) is dense in $L^p[0, \infty)$. If we restrict the functions to a bounded interval $[0, T]$, then we obtain a theorem proved by Skórnik in [6].

If $S = D'_+$, then Theorem V says that *every distribution from D'_+ can be approximated distributionally by sums (7) with any other distribution g from D'_+ which does not vanish in the right neighbourhood of 0, and positive numbers τ_i.* In particular, it can be approximated by sums with the delta distribution: $\lambda_1 \delta(t-\tau_1) + \ldots + \lambda_n \delta(t-\tau_n)$. It might to seem, at first, more paradoxal that the delta distribution $\delta(t)$ can be approximated by sums (7) with positive τ_i and with an arbitrary given function $g \in C_0^\infty[0, \infty)$.

Finally, it follows from Theorem V that *every operator $f \in M_+$ can be approximated operationally by sums (2) with any other operator $g \in M_+$ which does not vanish in the right neighbourhood of 0.* In fact, there exists a function $q \in C[0, \infty)$ which does not vanish identically in the right neighbourhood of 0 such that $f = p_1/q$ and $g = p_1/q$, where $p_1 p_2 \in C[0, \infty)$. Let M_q be the set of all operators which can be represented in the form p/q ($p \in C[0, \infty)$). By Theorem V, f can be approximated by sums (2) in the topology of M_q. But every sequence which converges in the topology of M_q also converges operationally, which proves our assertion. Taking in particular $g = 1$, we see that every operator from M_+ can be approximated by polynomials $\lambda_1 h^{\tau_1} + \ldots + \lambda_n h^{\tau_n}$ of the shift-operator with positive τ_i.

5. We have considered, so far, functions, distributions and operators defined on the one-dimensional real space R. However, all the theorems can also be interpreted in the Euclidean space R^m of any number of dimensions. Then by an interval $[0, T]$ we understand the set of points $t = (t_1, \ldots, t_m)$ whose coordinates satisfy the inequalities $0 \leqslant t_i \leqslant T_i$, where $T = (T_1, \ldots, T_m)$. Similarly, the interval $[0, \infty)$ means the set of points t with $t_i \geqslant 0$. By the convolution

$$\int_0^t g(t-\tau) k(\tau) d\tau$$

we understand an integral stretched on the set $0 \leqslant \tau_i \leqslant t_i$ ($i = 1, \ldots, m$). The proof of Theorem I is based on the Titchmarsh theorem, which holds for any number of dimensions (see [5]). This theorem permits to introduce the class of m-dimensional operators $a = p/q$, where q does not vanish identically in the m-dimensional right neighbourhood of 0. Then all the preceding considerations remain true in the new, more general, interpretation.

References

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On generalized topological divisors of zero in m-convex locally convex algebras

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By a *topological algebra* we mean in this paper a topological linear space together with an associative jointly continuous multiplication. An element x of a topological algebra A, $x \neq 0$, will be called a *left* (*right*) *topological divisor of zero* if there exists a non-void subset $P \subset A$ such that zero is not in the closure $\overline{P}$ of P but $0 \in \overline{xP}$ ($0 \in \overline{Px}$). Here, as usual, $UV = \{xy: x \in U, y \in V\}$. An element $x \in A$ is called a *topological divisor of zero in A* if it is both a right and a left topological divisor of zero. It is a classical fact of the theory of Banach algebras, due to Šilov [3] (for algebras without a unit, see [5]) that a complex Banach algebra either possesses topological divisors of zero or is isomorphically homeomorphic to the field of complex numbers. The same holds for locally bounded algebras — a class more general than the class of Banach algebras [5]. Here we investigate the problem for another generalization of Banach algebras, namely for the class of locally convex multiplicatively convex topological algebras (shortly, we shall call them *m-convex algebras* throughout this paper). An m-convex algebra is a topological algebra (over complexes) with a basis for neighbourhoods of the origin consisting of sets $\{U\}$ which are convex, symmetric and idempotent, i. e. such that $UU \subset U$. Or, which is equivalent, it is a locally convex algebra with the topology given by means of family $\mathscr{P}$ of submultiplicative pseudonorms:

$$\|xy\| \leqslant \|x\| \|y\| \tag{1}$$

and, in the case where the algebra in question possesses a unit e,

$$\|e\| = 1 \tag{2}$$

for each $\|\cdot\| \in \mathscr{P}$. We may assume that $\mathscr{P}$ consists of all continuous pseudonorms satisfying (1) and (2) in the case where there is a unit element. The theory of these algebras was created by Arens [1] and Michael [2].

The statement that an m-convex algebra either possesses topological divisors of zero or is isomorphically homeomorphic to the field of