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An Approximation Theorem and its applications in Operational Calculus

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Let $g(t)$ be a given real valued function which vanishes identically for $t<0$, is continuous for $0 \leqslant t < \infty$ and does not vanish identically in any right neighbourhood $0 < t < \delta$ of the origine. We consider finite sums of the form

$$\sigma(t) = \sum_{i=0}^{m} \lambda_i g(t-t_i), \tag{1}$$

where λ_i and t_i are real numbers and m is an integer. We shall show that the set of all sums $\sigma(t)$ is dense in the space $C[a, b]$ of continuous functions. More exactly, we shall prove the following

THEOREM. *Given any function $f(t)$, continuous in a bounded and closed interval $a \leqslant t \leqslant b$, and a positive number ε, there is an integer m and real numbers λ_i, t_i such that $|\sigma(t)-f(t)| < \varepsilon$ for $a \leqslant t \leqslant b$. Moreover, if $f(a) = 0$, then all the numbers t_i can be chosen non less than a.*

Before we give the proof, we are going to show some applications of the above theorem.

First application. We are in a position to obtain easily the Titchmarsh theorem on convolution:

Let f and g be continuous in $[0, T]$, and let g do not vanish identically in any right neighbourhood of the origin. If

$$\int_0^x f(x-t)g(t)\,dt = 0$$

in $[0, T]$, then $f(x) = 0$ in $[0, T]$.

In fact, we complete first the definition of g to the whole real axis of t so as to have g continuous for $0 \leqslant t < \infty$ and vanishing for $-\infty < t < 0$. By Theorem, given any $\varepsilon > 0$, there is a function of the form (1) with $t_i \geqslant 0$ such that $|\sigma(t)-t| < \varepsilon$ in $[0, T]$. Thus, we have for $0 \leqslant x \leqslant T$,

$$(2) \qquad \Big|\int_0^x f(x-t)\,t\,dt\Big| \leqslant \varepsilon\int_0^x |f(x-t)|\,dt + \Big|\int_0^x f(x-t)\,\sigma(t)\,dt\Big|$$

$$\leqslant \varepsilon\int_0^T |f(t)|\,dt + \sum_{i=0}^m |\lambda_i|\cdot\Big|\int_0^x f(x-t)\,g(t-t_i)\,dt.$$

If $0 \leqslant x \leqslant t_i$, then the integral

$$\int_0^x f(x-t)\,g(t-t_i)\,dt$$

equals to 0, for the integrand is null. If $t_i < x \leqslant T$, then the last integral equals to

$$\int_{t_i}^x f(x-t)\,g(t-t_i)\,dt = \int_0^{x-t_i} f(x-t_i-u)\,g(u)\,du = 0,$$

in view of the hypothesis. Thus the sum on the right side of (2) vanishes identically for $0 \leqslant x \leqslant T$. Since ε can be chosen arbitrarily, it follows that

$$\int_0^x f(x-t)\,t\,dt = 0 \quad \text{for} \quad 0 \leqslant x \leqslant T.$$

This can also be written

$$\int_0^x (x-t)f(t)\,dt = 0,$$

which implies, when differentiating twice, that $f(x) = 0$ for $0 \leqslant x \leqslant T$. And this was our assertion.

Second application. Let C be the ring of continuous functions in $0 \leqslant t < \infty$ with convolution as multiplication. It follows from the Titchmarsh theorem that C has no divisors of zero. It can be therefore completed to a quotient field M. The elements of this field are called *operators*. One says that a sequence of operators $a_n \in M$ converges to $a \in M$, if there is a function $q \in C$ such that $qa_n \in C$, $qa \in C$ and qa_n converges to qa almost uniformly, i.e., uniformly in each finite interval $[0, T]$. Now, Foiaş [2] has proved that C is dense in M. We shall show that a slightly stronger statement follows from our Theorem. In fact, let a be a given operator. This operator can be represented in the form f/g, where $f, g \in C$, $f(0) = 0$, and g does not vanish identically in any neighbourhood of the origin. The sum (1) can be written in the form

$$\sigma = \sum_{i=0}^m \lambda_i h^{t_i}\cdot g = kg,$$

where h is the shift-operator [3]. It follows from our Theorem that there is a sequence of sums σ_n with $t_i \geqslant 0$ which converges to f almost uniformly in $[0, \infty)$. This means that $k_n \to f/g$, where k_n and σ_n are related as k and σ. Thus we have proved that *the set of polynomials*

$$k = \sum_{i=0}^m \lambda_i h^{t_i}$$

(*with non negative* t_i) *is dense in* M. Hence the theorem of Foiaş follows immediately, on remarking that any given operator a can also be represented in the form $l^2 f/g$, where f and g satisfy the same requirement as before, and $l^2 = \{t\}$. Then $k_n \to f/g$ implies $l^2 k_n \to a$, where $l^2 k_n$ are evidently continuous functions.

It is worth mentionning that Boehme [1] deduced from the theorem of Foiaş the following interesting property of operators:

Every sequence of operators $a_1, a_2, \ldots$ can be represented in the form

$$\frac{p_1}{q}, \frac{p_2}{q}, \ldots \qquad (p_n, q \in C)$$

with a common denominator q.

The proof of our Theorem will be based on a lemma which is a modification of a lemma of Foiaş. In the sequel, the convolution

$$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

will be denoted by $f * g$.

LEMMA. *Let f and g be continuous functions in $[0, T]$ such that $f(0) = g(0) = 0$, g non vanishing identically in the neighbourhood of 0. Then there exist a sequence of functions k_n, absolutely continuous in $[0, T]$, such that $k_n(0) = 0$ and $g * k_n$ converges to f uniformly in $[0, T]$.*

Proof of lemma. We consider the set of continuous functions in $[0, T]$ as a Banach space $C[0, T]$ with the norm

$$\|f\| = \max_{0 \leqslant t \leqslant T} |f(t)|.$$

Assume that there is no sequence k_n with the required property. Then the distance of f from the set of all functions $g * k$ with absolutely continuous k and $k(0) = 0$ is positive. By the Banach-Hahn theorem, there exists a linear functional F on $C[0, T]$ such that $F(f) = 1$ and $F(g * k) = 0$ for every absolutely continuous function k such that $k(0) = 0$. It follows from a known representation theorem of F. Riesz that every functional F on $C[0, T]$ is of the form

$$F(f) = \int_0^T f(t)\,dh(T-t),$$

where h is a function of bounded variation in $[0,T]$ such that $h(0)=0$. Therefore, this function can be chosen so that

$$(3)\qquad \int_0^T f(t)\,dh(T-t)=1, \quad \text{and} \quad \int_0^T (g*k)(t)\,dh(T-t)=0$$

for every absolutely continuous k with $k(0)=0$. The last integral can also be written in the form

$$\int_0^T h(T-t)\,d(g*k)(t) = h(T-t)(g*k')(t)\,dt = \big(h*(g*k')\big)(T)$$
$$= \big(k'*(g*h)\big)(T) = \int_0^T k'(T-t)(g*h)(t)\,dt.$$

Since the derivative k' can be any integrable function, it follows that $g*h=0$ in $[0,T]$ and, by Titchmarsh's theorem, $h=0$ in $[0,T]$. This contradicts (3) and, therefore, proves Lemma.

Proof of theorem. Assume first that $f(0)=0$ and $a=0$, $b=T$. By Lemma, there is a continuous (even absolutely) function k with $k(0)=0$ such that $|g*k-f|<\varepsilon/2$ in $[0,T]$. By the mean value theorem we have, for $t_i=iT/n$,

$$g*k=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} g(t-\tau)k(\tau)\,d\tau = \frac{1}{n}\sum_{n=0}^{n-1} g(t-\tau_i)k(\tau_i)$$

for properly chosen τ_i satisfying $t_i\leqslant\tau_i\leqslant t_{i+1}$. Hence, for $0\leqslant t\leqslant T$,

$$(4)\qquad \Big|g*k-\frac{1}{n}\sum_{i=0}^{n-1} g(t-t_i)k(t_i)\Big|$$
$$\leqslant \frac{1}{n}\sum_{i=0}^{n-1}\big[|g(t-\tau_i)-g(t-t_i)|\cdot|k(t_i)|+|g(t-\tau_i)|\cdot|k(\tau_i)-k(t_i)|\big].$$

Since the functions g and k are continuous, the expression in the brackets becomes less that $\varepsilon/2$ for n sufficiently large, say for $n>n_0$. Consequently, the whole expression (4) becomes less than $\varepsilon/2$ for $n>n_0$ and, therefore,

$$\Big|\frac{1}{n}\sum_{i=0}^{n-1} g(t-t_i)k(t_i)-f(t)\Big|<\varepsilon \quad \text{in} \quad 0\leqslant t\leqslant T$$

for $n>n_0$. This proves Theorem in case when $f(0)=0$ and $a=0$. If $f(0)\neq 0$, we take a point t_0 at which $g(t_0)\neq 0$ and apply the preceding result to the function

$$f_1(t)=f(t)-\frac{f(0)}{g(t_0)}g(t+t_0)$$

which evidently has the property $f_1(0)=0$. Also the restriction to $a=0$ is easy to be removed by a proper translation. Thus the proof is complete.

Remark. In fact, our Theorem is very elementary, since its formulation need even not the notion of an integral. Nevertheless, it implies, as we saw, important theorems of Analysis. Therefore it would be very interesting to find an elementary proof of it. To this end, the proof produced above could yield perhaps some help, for it gives some more information on the numbers λ_i and t_i than Theorem in our preceding formulation.

References

[1] T. K. Boehme, *On sequences of continuous functions and convolution*, Studia Mathematica 25 (1965), p. 333-335.

[2] C. Foiaş, *Approximation des opérateurs de J. Mikusiński par des fonctions continues*, ibidem 21 (1961), p. 73-74.

[3] J. Mikusiński, *Operational Calculus*, Pergamon Press 1959.

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