The proof of the existence of the limit space $(\mathscr{C}_N, \tau_N)$ parallels the proof of Theorem 5.

THEOREM 11. *$\mathfrak{J}\tau_N x$ iff $\mathfrak{J}$ N-converges to x.*

Proof. $\mathfrak{J}\tau_N x$ implies that there exists $p \in N$ such that $x \in \mathscr{C}_p$ and $\mathfrak{N}_p(x) \leqslant \mathfrak{J}$. $p \in N$ implies p is a unit in S; $\mathfrak{N}_p(x) \leqslant \mathfrak{J}$ iff $p\,\mathfrak{J}$ converges to px in $\mathscr{C}$.

The operator $e^{cs}\colon \mathscr{C}' \to \mathscr{C}'$ is defined by:

$$e^{cs}\{f(t)\} = \begin{cases} 0 & \text{for} \quad t < -c, \\ f(t+c) & \text{for} \quad -c \leqslant t. \end{cases}$$

(A derivation of the operator e^{cs} may be found in [4], Part III, Chap. 2). From the definition of e^{cs} one notes $e^{cs}D \subset D$, hence $e^{cs} \in S$. If $\mathfrak{J}$ N-converges to x, then there exists $p \in \mathscr{C}^*$, p a unit in S, such that $p\mathfrak{J}$ converges to px in $\mathscr{C}'_c$ for some $c \geqslant 0$. If $p\mathfrak{J}$ converges to px in $\mathscr{C}'_c$ then $e^{-cs}p\mathfrak{J}$ converges to $e^{-cs}px$ in $\mathscr{C}'_0$, hence in $\mathscr{C}_1$. Since $e^{-cs}p \in N$, the proof is complete.

Two questions arise in connection with this section:

(a) It was established that $\mathscr{C}_N \subset S$. It is suspected, but as yet unverified, that $\mathscr{C}_N = S$.

(b) Let T_N denote the topology induced on $\mathscr{C}_N$ by the S-topology. Let σ denote the Limitierung induced on $\mathscr{C}_N$ by the topology T_N. Theorem 10 and Theorem 11 show that $\sigma \leqslant \tau_N$. Is T_N the finest topology on $\mathscr{C}_N$ with this property?

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Strong differentials in L^p

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CHAPTER I

This chapter contains the statement of the main results. Chapters II-VI contain their proofs. Chapter VII contains some additional remarks.

In what follows the function $f(x) = f(x_1, x_2, \ldots, x_n)$ is defined in the n-dimensional unit cube: $0 \leqslant x_j \leqslant 1, j = 1, \ldots, n$, and is of the class L^p there, $1 \leqslant p < \infty$. We assume once for all that $n \geqslant 2$.

Definition 1. *The function f has at a point x a k-th differential in L^p* — for brevity, a (k, p) *differential* — if there is a polynomial $P(t) = P(t_1, \ldots, t_n)$ of degree k or less such that

$$\left(\frac{1}{|Q|}\int\limits_Q |f(x+t)-P(t)|^p dt\right)^{1/p} = o(h^k), \quad h \to 0, \tag{1.1}$$

where Q is an n-dimensional cube containing the origin and of edge h.

The purpose of this paper is to investigate the connections between this differential and certain other notions of differential. In [3] a connection between this and what may be thought of as the partial (k, p) differential is discussed. The main theorem of [3] is:

THEOREM A. *If f has a (k, p) differential at each point of a set E, then for any integer m satisfying $1 \leqslant m < n$ the function f has a (k, p) differential almost everywhere in E with respect to the variable $x' = (x_1, x_2, \ldots, x_m)$.*

Actually what we shall need here is the following result, also proved in [3], of which Theorem A is a simple consequence.

THEOREM A'. *Let $x' = (x_1, \ldots, x_m)$, $x'' = (x_{m+1}, \ldots, x_n)$ and let $f(x) = f(x_1 \ldots, x_n) = f(x', x'')$ be non-negative and integrable over the unit cube Q^0. Let a be any positive number and let Q and I denote respectively arbitrary n-dimensional and m-dimensional cubes with edge h. If at each point $x = (x', x'')$ of a set $E \subset Q^0$ we have*

$$\int\limits_Q f(\xi)\,d\xi = o(h^{n+a}), \quad h \to 0,$$

for cubes containing x, then at almost all points of E

$$\int_I f(\xi', x'')\,d\xi' = o(h^{m+a}), \quad h \to 0,$$

for cubes I containing x'.

Many of the ideas and techniques of [3] are used in this paper and in most cases we refer to [3] rather than repeat the argument here. A part of the argument, however, which recurs constantly in what follows deserves a restatement. It is the following theorem of Calderón and Zygmund [1].

If $f(x)$, $x \in Q^0$, has a (k,p) differential, $k \geqslant 0$, $p \geqslant 1$, at each point of a set E, then for every $\varepsilon > 0$ there is a closed subset H of E with $|E-H| < \varepsilon$ and a decomposition $f = f_1 + f_2$, where $f_1 \in C^k(Q^0)$, the differential of f_1 is the same as that of f in H; in particular $f_1 = f$, $f_2 = 0$ in H so that

$$\left(\frac{1}{|Q|}\int |f_2|^p d\xi\right)^{1/p} = o(h^k) \quad (h \to 0)$$

for $x \in H$ and cubes Q of edge h containing x. See [1], p. 189, *Corollary*.

We will consider two other kinds of differential. In what follows, all rectangles will have sides parallel to the coordinate axes.

Definition 2. *The function f has a $(k,p)'$ differential at x* if there is a polynomial $P(t)$ of degree k such that

$$\left(\frac{1}{|R|}\int_R |f(x+t)-P(t)|\;dt\right)^{1/p} = o(w), \quad w \to 0, \tag{1.2}$$

where R is an n-dimensional rectangle containing the origin and w is the maximum of the edges of R.

In the case $k = 0$ this implies the strong differentiability of the indefinite integral of f. By analogy we may call the $(k,p)'$ differential a *strong differential in L^p at x*. The polynomial $P(t)$ will be called the *differential of f at x*.

Definition 3. *The function f has a $(k,p)''$ differential at x* if it satisfies (1.2) for rectangles R with only two different edge lengths.

The question arises as to the relations among these definitions of differential. Clearly, for any point x,

$$(k,p)' \Rightarrow (k,p)'' \Rightarrow (k,p),$$

and it is easy to see that the implications are not reversible (except in the case $n = 2$, for then the notions $(k,p)'$ and $(k,p)''$ are identical). Also it is easy to see that (k,p) implies $(k,p-\varepsilon)$, etc. for $\varepsilon > 0$.

Let us assume then that f has a (k,p) differential at each point of a set E and ask whether it has a $(k,p)'$ or a $(k,p)''$ differential almost everywhere in E. If $k = 0$, $p = 1$, the answer is no: any integrable function has a $(0,1)$ differential almost everywhere, and Saks [2] has given an example of an integrable f which has a $(0,1)''$ differential almost nowhere. (He discusses the case $n = 2$ only but his argument is valid for general $n \geqslant 2$). If $k \geqslant 1$, however, the situation is surprisingly different and we will prove the following theorem:

THEOREM 1. *If $n = 2$ and $k \geqslant 1$, and if f has a (k,p) differential in a set E, $1 \leqslant p < \infty$, then f has a $(k,p)'$ differential almost everywhere in E.*

This theorem does not hold for $n > 2$ as the following result shows:

THEOREM 2. *Given $n \geqslant 3$, $k \geqslant 1$, $p \geqslant 1$, there exists a continuous function f in the n-dimensional unit cube and a set E of measure as close to 1 as we wish such that f has (k,p) differential everywhere in E but has a $(k,p)'$ differential nowhere in E.*

However for $n \geqslant 2$ we have the following substitutes for Theorem 1:

THEOREM 3. *If $k \geqslant 1$, $p \geqslant 1$, and f has a (k,p) differential in E, it has a $(k,p-\varepsilon)'$, $\varepsilon > 0$, differential almost everywhere in E.*

THEOREM 4. *If $k \geqslant 1$, $p \geqslant 1$, and f has a (k,p) differential in E, it has a $(k,p)''$ differential almost everywhere in E.*

Clearly Theorem 1 is a special case of Theorem 4.

In the remaining two theorems we treat the case $k = 0$ which has some exceptional properties. In this case we obtain substitutes for Theorems 3 and 4 by strengthening the hypotheses.

THEOREM 5. *If for every $x \in E$ the function f satisfies*

$$\frac{1}{|Q|}\int_Q |f(x+t)-f(x)|\,dt = o\left(\frac{1}{\log 1/h}\right) \tag{1.3}$$

where Q is a cube of edge h containing the origin, then almost everywhere in E the function f has a $(0,1)''$ differential, i.e.,

$$\frac{1}{|R|}\int_R |f(x+t)-f(x)|\,dt = o(1), \quad |R| \to 0, \tag{1.4}$$

where R is a rectangle containing the origin and with two different edge lengths.

THEOREM 6. *Let $\varphi(u) = u(\log^+ u)^{n-2}$ $(u \geqslant 0)$. If for every $x \in E$ the function f satisfies*

$$\frac{1}{|Q|}\int_Q \varphi\{|f(x+t)-f(x)|\}\,dt = o\left(\frac{1}{\log 1/h}\right) \tag{1.5}$$

then f has a $(0,1)'$ differential almost everywhere in E, i.e. (1.4) holds for arbitrary rectangles R containing the origin.

CHAPTER II

In this chapter we prove Theorem 4. As is explained in the proof of Theorem A in [3], it is sufficient to consider the case $f(x) = 0$ for $x \in E$, and hence $P(t) = 0$ at every point of density of E. If we replace $|f|^p$ by g and kp by a, Theorem 4 becomes equivalent to the following theorem:

THEOREM 4′. *Let $g(x)$ be defined, non-negative and integrable in the unit cube Q^0: $0 \leqslant x_j \leqslant 1, j = 1, 2, .., n$. Let a be a positive number and let Q and R denote respectively a cube with edge h and a rectangle with edges*

$$s_i = w,\ i = 1, 2, \ldots, j; \quad s_i = l,\ i = j+1, \ldots, n; \quad w \geqslant l.$$

If at each point x of a set $E \subset Q^0$ we have

$$\int_Q g(\xi)\,d\xi = o(h^{n+a}) \quad (h \to 0) \tag{2.1}$$

where $x \in Q$, then at almost every point $x \in E$

$$\int_R g(\xi)\,d\xi = o(w^a|R|) = o(w^{j+a}l^{n-j}) \quad (w \to 0) \tag{2.2}$$

where $x \in R$.

As in [3] (we do not repeat the argument here) we further reduce our theorem to the following form which is of independent interest:

THEOREM 4″. *There is a positive constant A depending only on the dimension n and a having the following property. Let $g(x) = g(x_1, \ldots, x_n)$ defined in a cube Q_0 be non-negative and integrable. Denote by U the set of points $x \in Q_0$ such that there is a cube $Q \supset x$ with*

$$\int_Q g(\xi)\,d\xi > h^{n+a} \tag{2.3}$$

and by V the set of points $x \in Q_0$ such that there is a rectangle $R \supset x$ with edges $s_i = w$ for $1 \leqslant i \leqslant j$ and $s_i = l$ for $j+1 \leqslant i \leqslant n$, with $w \geqslant l$ and

$$\int_R g(\xi)\,d\xi > w^a|R|. \tag{2.4}$$

Then

$$|V| \leqslant A\,|U|. \tag{2.5}$$

That Theorem 4″ implies Theorem 4′ is plausible if we note that U can be thought of as the set where (2.1) is not likely to hold, and V is the set where (2.2) is not likely to hold. Theorem 4″ asserts, then, that if (2.2) is not likely to hold neither is (2.1).

We need the following lemma the proof of which can be found in [3]:

LEMMA 1. *Let $g(x) = g(x_1, \ldots, x_n)$ be non-negative and integrable over a cube Q_0 of edge h_0 and suppose that*

$$h_0^{-(n+a)} \int_{Q_0} g(x)\,dx \leqslant 2^{n+a} \tag{2.6}$$

where $a > 0$. Then there is a sequence of non-overlapping cubes $Q_1, Q_2, \ldots$ contained in Q_0 with edges respectively $h_1, h_2, \ldots,$ such that

$$1 < h_k^{-(n+a)} \int_{Q_k} g(x)\,dx \leqslant 2^{n+a}, \qquad k = 1, 2, \ldots, \tag{2.7}$$

and $g(x) = 0$ almost everywhere in the complement of $\bigcup Q_k$.

We proceed with the proof of Theorem 4″. If (2.6) does not hold, then $U = Q_0 = V$ and we have (2.5) with $A = 1$. Thus we assume (2.6) and apply Lemma 1. Clearly, $\bigcup Q_k \subset U$. Let $\xi = (\xi_1, \ldots, \xi_n) = (\xi', \xi'')$, where $\xi' = (\xi_1, \ldots, \xi_j)$ and $\xi'' = (\xi_{j+1}, \ldots, \xi_n)$. Let $\hat{Q}_k$ be Q_k expanded three times about its center. Let $Q_k = Q_k' \times Q_k''$, where Q_k' and Q_k'' are the projections of Q_k onto the ξ' and ξ'' subspaces respectively. Let $\tilde{Q}_k = Q_k' \times J_k''$, where J_k'' is Q_k'' expanded γ times about its center; γ is the smallest odd integer for which

$$(\gamma - 1)^a \geqslant 2^{n+1+2a}3^n. \tag{2.8}$$

Let $V = V_1 \cup V_2$, where $V_2 = V \cap \bigcup \hat{Q}_k$ and $V_1 = V - V_2$. Clearly.

$$|V_2| \leqslant 3^n \sum |Q_k| \leqslant 3^n\,|U|, \tag{2.9}$$

and it remains to estimate $|V_1|$.

Let

$$V_1^{x''} = \{x' : (x', x'') \in V_1\}.$$

Since

$$|V_1| = \int_{Q_0''} |V_1^{x''}|\,dx'', \tag{2.10}$$

it suffices to estimate $|V_1^{x''}|$. Now $x' \in V_1^{x''}$ if and only if $(x', x'') \notin \bigcup \hat{Q}_k$ and there exists a rectangle $R = R' \times R''$ where R' is a j-dimensional cube of edge w and R'' is an $n-j$-dimensional cube of edge l with $w \geqslant l$, $x' \in R'$ and $x'' \in R''$, such that

$$\int_R g(\xi)\,d\xi > w^{j+a}l^{n-j}. \tag{2.11}$$

By an elementary Vitali theorem, there exists a finite disjoint collection $\mathscr{R} = \{R\}$ of such rectangles R for which

$$\beta \sum_{R \in \mathscr{R}} |R'| \geqslant |V_1^{x''}|, \tag{2.12}$$

where β is a constant depending only on the dimension j. For a fixed $R\in\mathcal{R}$,

$$w^{j+a}l^{n-j} < \int_R g(\xi)d\xi = \sum_k \int_{R\cap Q_k} g(\xi)d\xi = \sum\nolimits_1 \int_{R\cap Q_k} g(\xi)d\xi + \sum\nolimits_2 \int_{R\cap Q_k} g(\xi)d\xi, \tag{2.13}$$

where $\sum_1$ is taken over all Q_k such that $x''\in J_k''$ and $\sum_2$ over all Q_k such that $x''\notin J_k''$. Now $R\cap Q_k\neq 0$ and $x''\in J_k''$ imply $Q_k\subset\hat{R}$, where $\hat{R}$ is R expanded three times about its center and that

$$h_k \leqslant l\frac{2}{\gamma-1} \leqslant w\frac{2}{\gamma-1}.$$

Thus

$$\sum\nolimits_2 \int_{R\cap Q_k} g(\xi)d\xi \leqslant \sum\nolimits_2 2^{n+a}h_k^{n+a} \leqslant 2^{n+a}w^a\left(\frac{2}{\gamma-1}\right)^a \sum h_k^n \leqslant 2^{n+a}3^n|R|w^a\left(\frac{2}{\gamma-1}\right)^a \leqslant \tfrac{1}{2}w^{j+a}l^{n-j}.$$

Hence, in view of (2.13),

$$w^{j+a}l^{n-j} \leqslant 2\sum\nolimits_1 \int_{R\cap Q_k} g(\xi)d\xi. \tag{2.14}$$

Let

$$g_k(\xi'') = \begin{cases} \int_{Q_k'} g(\xi',\xi'')d\xi' & \text{if} \quad \xi''\in Q_k'', \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\int_{R\cap Q_k} g(\xi)d\xi \leqslant \int_{R''\cap Q_k''} d\xi'' \int_{Q_k'} g(\xi',\xi'')d\xi' = \int_{R''\cap Q_k''} g_k(\xi'')d\xi'',$$

and thus, by (2.14),

$$w^{j+a} \leqslant 2\sum_{Q_k\cap R\neq 0}{}_1 \frac{1}{l^{n-j}} \int_{R''\cap Q_k''} g_k(\xi'')d\xi'' \leqslant 2\sum_{Q_k\cap R\neq 0}{}_1 g_k^*(x'') = 2\sum_{Q_k\cap R\neq 0} g_k^*(x''), \tag{2.15}$$

where g_k^* is the $n-j$-dimensional Hardy-Littlewood maximal function in J_k'' with respect to cubes; that is,

$$g_k^*(x'') = \sup_{S\ni x''} \frac{1}{|S|}\int_S g_k(\xi'')d\xi'', \tag{2.16}$$

where S is a cube and $x''\in J_k''$. Thus

$$|R'| = w^j \leqslant \left[2\sum g_k^*(x'')\right]^{j/(j+a)} \leqslant 2\sum \left(g_k^*(x'')\right)^{j/(j+a)}, \tag{2.17}$$

where the summation is over all k such that $Q_k\cap R\neq 0$.

Now R is not contained in any $\hat{Q}_k$ since otherwise we would have $x\in V_2$. Hence $R\cap Q_k\neq 0$ implies $h_k\leqslant w$ and therefore R' must contain one of the 2^j vertices of Q_k'. Since the R' are disjoint, any k can appear in at most 2^j sums of (2.17). Thus

$$\sum_{R\in\mathcal{R}} |R'| \leqslant 2^{j+1}\sum_k \left(g_k^*(x'')\right)^{j/(j+a)}. \tag{2.18}$$

Hence, by (2.12),

$$|V_1^{x''}| \leqslant \beta 2^{j+1}\sum_k \left(g_k^*(x'')\right)^{j/(j+a)}. \tag{2.19}$$

We recall the fact that if f is a non-negative function defined in an m-dimensional cube Q of edge h, f^* is its maximal function, and $0<\delta<1$, then there exists a constant $A=A_{m,\delta}$ such that

$$\left(\int_Q \left(f^*(x)\right)^\delta dx\right)^{1/\delta} \leqslant A_{m,\delta}h^{m(1-\delta)/\delta}\int_Q f(x)dx.$$

Thus, by (2.10) and (2.19),

$$|V_1| \leqslant \beta\, 2^{j+1}\sum_k \int_{J_k''} \left(g_k^*(x'')\right)^{j/(j+a)}dx'' \leqslant A\sum_k h_k^{(n-j)(1-j/(j+a))}\left(\int_{Q_k''} g_k(x'')dx''\right)^{j/(j+a)} \leqslant A\sum_k h_k^{(n-j)(1-j/(j+a))}h_k^{(n+a)j/(j+a)} = A\sum h_k^n \leqslant A|U|, \tag{2.20}$$

where A stands for a generic constant depending on n and a.

This together with (2.9) completes the proof of the theorem.

CHAPTER III

We now give the proof of Theorem 3. We may, of course, reduce the general case to that of differential 0. It is a curious fact that in this special case we have a stronger version of the theorem, namely, that the w in (1.2) may be taken to be the *smallest* edge of the rectangle R instead of the largest. To show this it is clearly enough to prove the following theorem:

THEOREM 3′. *Let $g(x) \geqslant 0$ be defined in the n-dimensional unit cube $Q^0: 0 \leqslant x_j \leqslant 1, j = 1, 2, \ldots, n$. Let a and ε be positive numbers and let Q and R denote respectively a cube with edge h and a rectangle with smallest edge w. Let $g \in L^{1+\varepsilon}(Q^0)$. If at each point x of a set $E \subset Q^0$ we have*

$$\int_Q g^{(1+\varepsilon)}(\xi)\, d\xi = o(h^{n+a(1+\varepsilon)}) \qquad (h \to 0) \tag{3.1}$$

for cubes $Q \supset x$, then at almost all points x of E:

$$\int_R g(\xi)\, d\xi = o(|R|\, w^a) \tag{3.2}$$

where $x \in R$.

We will prove Theorem 3′ by induction. For $n = 1$ there is nothing to prove. Thus we may assume it true for $n-1$. Let $\xi = (\xi_1, \ldots, \xi_n) = (\xi', \xi_n)$. Let

$$g^*(\xi) = \sup_{\xi_n \in J} \frac{1}{|J|} \int_J g(\xi', u_n)\, du_n. \tag{3.3}$$

Using the fact that if $\varphi(t)$ is defined and non-negative in an interval I, and if $1 < r < \infty$, then

$$\int_I (\varphi^*(t))^r dt \leqslant A_r \int_I \varphi^r(t)\, dt, \tag{3.4}$$

we see that (3.1) implies

$$\int_Q g^{*(1+\varepsilon)}(\xi)\, d\xi = o(h^{n+a(1+\varepsilon)}) \tag{3.5}$$

for $x \in E$, $Q \supset x$.

By Theorem A′, for almost every $x = (x_1, \ldots, x_n) = (x', x_n) \in E$ we have

$$\int_{Q'} g^{*(1+\varepsilon)}(\xi', x_n)\, d\xi' = o(h^{n-1+a(1+\varepsilon)}) \tag{3.6}$$

for $(n-1)$-dimensional cubes Q' containing x'. Let x_n be fixed such that (3.6) holds almost everywhere in $E^{x_n} = \{x' : (x', x_n) \in E\}$. By the induction hypothesis, if R' is a rectangle containing x',

$$\int_{R'} g^*(\xi', x_n)\, d\xi' = o(w^a |R'|) \tag{3.7}$$

holds almost everywhere in E^{x_n}. Let R be a rectangle containing (x', x_n) of smallest edge w, which we may assume parallel to the x_1 axis, and let $R = R' \times J$. Then

$$\int_R g(\xi)\, d\xi = \int_{R'} d\xi' \int_J g(\xi', \xi_n)\, d\xi_n \leqslant |J| \int_{R'} g^*(\xi', x_n)\, d\xi'$$
$$= o(|J|\, w^a |R'|) = o(w^a |R|).$$

Thus (3.2) holds almost everywhere in E.

CHAPTER IV

In this chapter we prove Theorem 5.

Let M be a large number; write $f = f_1 + f_2$, where $f_1 = f$ if $|f| \leqslant M$ and $f_1 = 0$ otherwise. If M is large enough, then $f_2(x) = 0$ and $|f_1(x)| \leqslant M/2$ for $x \in H \subset E$, with $|E - H|$ arbitrarily small. In view of the inequality $|f_2(x+t)| \leqslant 2|f(x+t) - f(x)|$ for $x \in H$, f_2 satisfies in H an inequality analogous to (1.3). Since f_1 is bounded, it has a differential $(0,1)'$ almost everywhere. Thus it is enough to consider f_2, that is the case when the function is zero in the set.

Theorem 5 is easily seen to be implied by the following theorem:

THEOREM 5′. *There is a constant A depending only on the dimension n and having the following property. Let $g(x) = g(x_1, \ldots, x_n)$ be non-negative and integrable in a cube Q_0. Denote by U the set of points $x \in Q_0$ such that there is a cube $Q \supset x$ with*

$$\int_Q g(\xi)\, d\xi > \frac{h^n}{\log 1/h}, \tag{4.1}$$

and by V the set of points $x \in Q_0$ such that there is a rectangle $R \supset x$ of edges $s_i = w$ for $1 \leqslant i \leqslant j$; $s_i = l$ for $j+1 \leqslant i \leqslant n$; $w \geqslant l$, with

$$\int_R g(\xi)\, d\xi > |R|. \tag{4.2}$$

Then

$$|V| \leqslant A\, |U|. \tag{4.3}$$

The proof of Theorem 5′ follows faithfully that of Theorem 4″ until we reach the analogue of (2.17) which now takes the form

$$|R'| = w^j \leqslant \sum g_k^*(x''). \tag{4.4}$$

Unlike, however, in the case of the proof of Theorem 4″ where we considered $g_k^{*j/(j+a)}$ we cannot now operate with the functions g_k^*, because they need not be integrable. We note however that if some of terms on the right of (4.4) exceed 1 we can replace them by 1 without invalidating the inequality since, in any case, $w \leq 1$. This shows that instead of the maximal function f^* of Hardy and Littlewood we may use its modification

$$f_*(x) = \min\{1, f^*(x)\} \tag{4.5}$$

and replace the g_k^* in (4.4) by g_{k*}.

Before we proceed with the proof we state two lemmas.

LEMMA 1. *Let $g(x) = g(x_1, \ldots, x_n)$ be non-negative and integrable over a cube Q_0 with edge $h_0 < \frac{1}{2}$ and suppose that*

$$\int_{Q_0} g(x)\,dx \leq 2^{n+1} \frac{h_0^n}{\log 1/h_0}. \tag{4.6}$$

Then there is a sequence of non-overlapping cubes $Q_1, Q_2, \ldots$ contained in Q_0 with edges respectively $h_1, h_2, \ldots$ such that

$$\frac{h_k^n}{\log 1/h_k} < \int_{Q_k} g(x)\,dx \leq 2^{n+1} \frac{h_k^n}{\log 1/h_k}$$

and $g(x) = 0$ almost everywhere in the complement of $\bigcup Q_k$.

This is an analogue of Lemma 1 of Chapter II in which the factor h^α is replaced by $(\log 1/h)^{-1}$, the proof is completely parallel and need not be repeated here.

LEMMA 2. *Let $f(x)$ be non-negative and integrable in an n-dimensional cube Q of edge $h < \frac{1}{2}$. Write*

$$f^*(x) = \sup_{S \ni x} \frac{1}{|S|} \int_S f(\xi)\,d\xi$$

where S is an n-dimensional cube, and

$$f_*(x) = \min\{1, f^*(x)\}.$$

Let $\alpha \geq 0$ be any number such that

$$h^{n+\alpha} \leq \int_Q f(x)\,dx.$$

Then

$$\int_Q f_*(x)\,dx \leq A\alpha \log\left(\frac{1}{h}\right) \int_Q f(x)\,dx + A \int_Q f(x)\,dx, \tag{4.7}$$

where A is a constant depending only on n.

Proof. Let

$$E_y = \{x : f^*(x) > y\}, \quad y > 0.$$

It is well known that there is a constant B depending only on the dimension n such that

$$|E_y| \leq By^{-1} \int_Q f(x)\,dx. \tag{4.8}$$

We split Q into three sets

$$F_1 = \{x : f^*(x) > 1\}; \quad F_2 = \{x : 1 \geq f^*(x) \geq h^\alpha\}; \quad F_3 = \{x : f^*(x) < h^\alpha\}.$$

We may assume that $\alpha > 0$ since if $\alpha = 0$ the lemma is trivially true with $A = 1$. Clearly,

$$\int_{F_1} f_*(x)\,dx = |F_1| = |E_1| \leq B \int_Q f(x)\,dx \tag{4.9}$$

by (4.8). For $x \in F_3$ we have $f_*(x) = f^*(x)$ and

$$\int_{F_3} f_*(x)\,dx = \int_{F_3} f^*(x)\,dx \leq h^\alpha |Q| \leq \int_Q f(x)\,dx. \tag{4.10}$$

Finally, if $x \in F_2$, then again $f_*(x) = f^*(x)$ and writing $|E_y| = e(y)$ we have

$$\begin{aligned} \int_{F_2} f^*(x)\,dx &= -\int_{h^\alpha}^{1} y\,de(y) = -ye(y)\Big|_{h^\alpha}^{1} + \int_{h^\alpha}^{1} e(y)\,dy \\ &\leq e(h^\alpha)h^\alpha + B\int_Q f(x)\,dx \int_{h^\alpha}^{1} y^{-1}\,dy \\ &\leq B\int_Q f(x)\,dx + B\log(1/h)\int_Q f(x)\,dx. \end{aligned} \tag{4.11}$$

Adding (4.9), (4.10) and (4.11) we obtain the lemma.

We can now complete the proof of Theorem 5′. We may assume that (4.6) holds. Apply Lemma 1. Let Q_k be the Q_k of Lemma 1, and let $\tilde{Q}_k, Q_k', Q_k'', J_k'', \hat{Q}_k$ have the same meaning as in Chapter II, except that now we take γ to be the smallest odd integer for which

$$\log\left(\frac{\gamma-1}{2}\right) \leq 2^{n+1}3^n.$$

Following the same steps as in Chapter II we show that if $x = (x', x'')$ is in V and in the complement of $\bigcup \hat{Q}_k$, then there exists an $R \ni x$, $R = R(w, \ldots, w, l, \ldots, l)$ such that (see (4.4))

$$w^j \leq 2\sum g_{k*}(x''),$$

and arguing as in the proof of Theorem 4'' we arrive at the inequality

(4.12) $$|V| \leqslant A \sum \int_{J_k''} g_{k*}(x'')dx''.$$

Let us now recall that

(4.13) $$\int_{J_k''} g_k(x'')\,dx'' = \int_{Q_k} g(x)\,dx > \frac{h_k^n}{\log(1/h_k)} \geqslant h_k^{n+1}.$$

Denote the side of J_k'' by t_k; thus $t_k = \gamma h_k$. Define β by the equation $\gamma^{-n-1} = 2^{-\beta}$. Then, by (4.13),

$$\int_{J_k''} g_k(x'')\,dx'' \geqslant t_k^{n-j+\beta+j+1} = t_k^{n-j+a_0}$$

where $a_0 = \beta+j+1$. Hence by Lemma 2 applied to the g_k and the $(n-j)$-dimensional space J_k'' we have

$$\int_{J_k''} g_{k*}(x'')\,dx'' \leqslant A a_0 \log(1/t_k) \int_{J_k''} g_k(x'')\,dx''$$

$$\leqslant A \log(1/h_k) \int_{Q_k} g(x)\,dx \leqslant A h_k^n.$$

Hence, by (4.12),

$$|V| \leqslant A \sum h_k^n = A \sum |Q_k| \leqslant A\,|U|.$$

This gives (4.3) and completes the proof of the theorem.

CHAPTER V

In this chapter we prove Theorem 6. Arguing as in the first paragraph of Chapter IV we reduce Theorem 6 to the case in which $f(x)$ is zero for x in E. That is,

THEOREM 6'. *Let $g \geqslant 0$ be defined in the n-dimensional unit cube Q^0. Let E be the set of all x such that*

(5.1) $$\int_Q g(\xi)\big(\log^+ g(\xi)\big)^{n-2} d\xi = o\left(\frac{h^n}{\log 1/h}\right)$$

where $Q \supset x$ is a cube of edge h. We also suppose

(5.2) $$g(x) = 0 \quad \textit{if} \quad x \in E.$$

Then for almost every $x \in E$,

(5.3) $$\int_R g(\xi)\,d\xi = o(|R|)$$

where R is a rectangle containing x.

We need several lemmas. We may clearly assume that $n \geqslant 3$.

LEMMA 1. *If g and E satisfy the hypotheses of Theorem 6', then given any $\varepsilon > 0$ there exists a subset H of E with $|E-H| < \varepsilon$ and a decomposition $g = g_1 + g_2$ with $g_1 \in L(\log^+ L)^{n-1}$, $g_1(x) = 0$ for x in E, and g_2 satisfies the hypotheses of Theorem 6'; moreover*

(5.4) $$g_2(\xi) > 1 \quad \textit{or} \quad g_2(\xi) = 0$$

and

(5.5) $$\int_Q g_2(\xi)\big(\log^+ g_2(\xi)\big)^{n-3} d\xi = o\,\frac{h^n}{(\log 1/h)^2} \qquad (h \to 0)$$

for cubes Q containing x, where $x \in H$.

Let $H \subset E$ be the set where

(5.6) $$\int_Q g(\xi)\big(\log^+ g(\xi)\big)^{n-2} d\xi \leqslant C\frac{h^n}{\log 1/h}$$

for all cubes Q of edge h containing x. If the constant C is large enough, $|E-H|$ is arbitrarily small. We may also suppose H closed. Now the complement of H or rather its interior can be expressed as a union of non-overlapping cubes S_k of edge l_k such that

(5.7) $$\frac{d_k}{2(\sqrt{n}+1)} \leqslant l_k \leqslant d_k$$

where d_k is the distance of S_k from H. Let $g_1(x) = g(x)$ if $x \in S_k$ and $g(x) \leqslant 1/l_k$; $g_1(x) = 0$ otherwise. Let $g(x) = g_1(x) + g_2(x)$. By (5.6) and (5.7) we see that

(5.8) $$\int_{S_k} g(\xi)\big(\log^+ g(\xi)\big)^{n-2} d\xi \leqslant \int_Q g(\xi)\big(\log^+ g(\xi)\big)^{n-2} d\xi$$
$$\leqslant C\frac{h^n}{\log 1/h} \leqslant B\frac{l_k^n}{\log 1/l_k}$$

where Q is the smallest cube containing S_k and a point of H.

It follows from (5.8) that

$$\int_{S_k} g_1(\xi)\big(\log^+ g_1(\xi)\big)^{n-1} d\xi \leqslant \log(1/l_k) \int_{S_k} g(\log^+ g)^{n-2} d\xi \leqslant B l_k^n,$$

and hence $g_1 \in L(\log^+ L)^{n-1}$.

If $Q\cap S_k\neq 0$, $Q\cap H\neq 0$, then $h\sqrt{n}\geqslant l_k$, as easily seen from (5.7). If $\xi\in S_k$, then

$$g_2(\xi)\big(\log^+g_2(\xi)\big)^{n-3}\leqslant(\log 1/l_k)^{-1}g_2(\xi)\big(\log^+g_2(\xi)\big)^{n-2}.$$

Hence, if $x\in H$, $x\in Q$,

$$\int\limits_Q g_2(\log^+g_2)^{n-3}d\xi=\sum\int\limits_{S_k\cap Q}g_2(\log^+g_2)^{n-3}d\xi \tag{5.9}$$

$$\leqslant\sum(\log 1/l_k)^{-1}\int\limits_{S_k\cap Q}g_2(\log^+g_2)^{n-2}d\xi$$

$$\leqslant\big(\log 1/(h\sqrt{n})\big)^{-1}\int\limits_Q g_2(\log^+g_2)^{n-2}d\xi=o\left(\frac{h^n}{(\log 1/h)^2}\right).$$

Thus, since g_1 satisfies (5.3) (by a familiar theorem on the strong differentiability of integrals), we may replace g by g_2 and E by H and assume (5.5).

LEMMA 2. *If $f(t)\geqslant 0$ is defined for $0\leqslant t\leqslant h\leqslant\frac12$ and*

$$\int\limits_0^h f\,dt\geqslant h^a$$

for some $a\geqslant 1$, then

$$\int\limits_0^h f^*(t)\,dt\leqslant A(a-1)\log(1/h)\int\limits_0^h f\,dt+A\int\limits_0^h f\log^+f\,dt+A\int\limits_0^h f\,dt. \tag{5.10}$$

Proof. We may assume $f(t)$ non-increasing. Then

$$\int\limits_0^h f^*(t)\,dt=\int\limits_0^h\Big(t^{-1}\int\limits_0^t f(s)\,ds\Big)dt=\int\limits_0^h f(t)\log\Big(\frac{h}{t}\Big)\,dt.$$

Set

$$\int\limits_0^h f\,dt=v,\qquad U=\Big\{t: f(t)\geqslant\frac{v}{t\log^3(h/t)}\Big\}.$$

If $t\in U$, then

$$\frac{h}{t}\leqslant\frac{h}{v}f(t)\log^3\Big(\frac{h}{t}\Big)$$

and hence

$$\log(h/t)\leqslant 2\log^+f(t)+2\log^+h/v$$
$$\leqslant 2\log^+f(t)+2(a-1)\log 1/h.$$

It follows that

$$\int\limits_U f(t)\log(h/t)\,dt\leqslant 2(a-1)\log 1/h\int\limits_0^h f\,dt+2\int\limits_0^h f\log^+f\,dt.$$

If t is in the complement of U, then

$$f(t)\log(h/t)\leqslant\frac{v}{t\log^2(h/t)}$$

and hence

$$\int\limits_{U'}f(t)\log(h/t)\,dt\leqslant v\int\limits_0^h t^{-1}\log^{-2}(h/t)\,dt\leqslant A\int\limits_0^h f\,dt.$$

LEMMA 3. *If $f(t)$ is defined for $0\leqslant t\leqslant h<\frac12$ and takes values either >1 or 0, then, A denoting an absolute constant,*

$$\int\limits_0^h f^*(\log^+f^*)^m dt \tag{5.11}$$

$$\leqslant Am\int\limits_0^h f(\log^+f)^{m+1}dt+Aa\log 1/h\int\limits_0^h f(\log^+f)^m dt,$$

provided $m\geqslant 1$ and

$$\int\limits_0^h f(\log^+f)^m dt\geqslant h^a,\qquad a\geqslant 1.$$

Proof. It suffices to prove (5.11) for f non-increasing. In this case, as we will show later,

$$f^*(t)\big(\log^+f^*(t)\big)^m\leqslant\Big(f(t)\big(\log^+f(t)\big)^m\Big)^*. \tag{5.12}$$

This will imply that

$$\int\limits_0^h f^*(t)\big(\log^+f^*(t)\big)^m dt\leqslant\int\limits_0^h\big(f(\log^+f)^m\big)^*dt. \tag{5.13}$$

Applying now Lemma 2 to the function $f(t)\big(\log^+f(t)\big)^m$ and using (5.13) we obtain (5.11). Thus it is enough to prove (5.12).

Let a be such that $f(t)>1$ for $0\leqslant t<a$ and $f(t)=0$ for $t>a$. If $t\leqslant a$, then (5.12) is just Jensen's inequality for the convex function $\varphi(u)=u(\log u)^m$. Let now

$$v_1=\int\limits_0^a f(s)\,ds,\qquad v_2=\int\limits_0^a f(s)\big(\log^+f(s)\big)^m ds$$

and suppose that $t > a$. Clearly,

$$f^*(t) = t^{-1}\int_0^a f(s)\,ds = t^{-1}v_1, \qquad \big(f(t)(\log^+ f(t))^m\big)^* = t^{-1}v_2.$$

Hence

$$(5.14)\qquad f^*(t)\big(\log^+ f^*(t)\big)^m = t^{-1}v_1(\log^+ t^{-1}v_1)^m$$
$$\leqslant \frac{a}{t}\,\frac{v_1}{a}\Big(\log^+\frac{v_1}{a}\Big)^m = t^{-1}af^*(a)\big(\log^+ f^*(a)\big)^m.$$

Using (5.12) for $t = a$ we see that the right-hand side of (5.14) does not exceed

$$\frac{a}{t}\,\frac{v_2}{a} = \frac{v_2}{t} = \big(f(t)(\log^+ f(t))^m\big)^*.$$

This completes the proof of (5.12) and so also of Lemma 3.

Lemma 4. *Let $f(x)$ be non-negative and integrable over the n-dimensional unit cube Q^0. Let*

$$(5.15)\qquad \int_Q f(\xi)\,d\xi = o\left(\frac{h^n}{\log 1/h}\right), \qquad x \in Q,$$

for all $x \in E$. Then for almost every $x = (x', x_n) \in E$ we have

$$(5.16)\qquad \int_I f(\xi', x_n)\,d\xi' = o\left(\frac{h^{n-1}}{\log 1/h}\right), \qquad x' \in I,$$

where Q is an n-dimensional cube of edge h and I an $(n-1)$-dimensional cube of edge h.

The proof of this lemma is analogous to that of Theorem A′ and we omit it here.

We now pass to the proof of Theorem 6′. For $n = 2$ it is just a special case of Theorem 5. Thus we assume it true for $n-1$ and prove it for n. By Lemma 1 we may assume that g satisfies (5.4) and (5.5). Let

$$g^*(\xi) = g(\xi', \xi_n) = \sup_J |J|^{-1}\int_J g(\xi', u_n)\,du_n$$

where J is an interval containing ξ_n. We wish to show that

$$(5.17)\qquad \int_Q g^*(\xi)\big(\log^+ g^*(\xi)\big)^{n-3}\,d\xi = o\left(\frac{h^n}{\log 1/h}\right)$$

for $x \in E$, $Q \ni x$. By Lemma 3 the left-hand side of (5.17) does not exceed

$$(5.18)\qquad A\int_{Q'}\Big(\alpha(\xi')\log(1/h)\int_{Q_n} g(\xi', \xi_n)\big(\log^+ g(\xi', \xi_n)\big)^{n-3}d\xi_n\Big)\,d\xi' +$$
$$+A\int_Q g(\xi)\big(\log^+ g(\xi)^{n-2}\big)\,d\xi,$$

where $\alpha(\xi')$ is the α of Lemma 3. The second term here is $o\{h^n/(\log 1/h)\}$ by the hypothesis of the theorem. To estimate the first term we split the $\xi' \in Q'$ into two classes, S_1 and S_2, such that in S_1

$$(5.19)\qquad \int_{Q_n} g(\xi', \xi_n)\big(\log^+ g(\xi', \xi_n)\big)^{n-3}d\xi_n \geqslant h^3,$$

and in S_2 the opposite inequality holds. For $\xi' \in S_2$ let $\beta(\xi')$ be defined by the equation

$$h^{\beta(\xi')} = \int_{Q_n} g(\xi', \xi_n)\big(\log^+ g(\xi', \xi_n)\big)^{n-3}d\xi_n.$$

Hence the first term of (5.18) does not exceed

$$(5.20)\qquad A\log(1/h)\int_Q g(\xi)\big(\log^+ g(\xi)\big)^{n-3}d\xi + A\int_{S_2}\log(1/h)\,\beta(\xi')\,h^{\beta(\xi')}\,d\xi'.$$

The first term of (5.20) is $o\big(h^n(\log 1/h)^{-1}\big)$ by (5.5). Using the fact that $\beta h^{\beta-2}\log 1/h \leqslant C$ for all $\beta \geqslant 3$ and $h \leqslant \frac{1}{2}$, we see that the second term in (5.20) does not exceed $Ah^{n+1} = o\big(h^n(\log 1/h)^{-1}\big)$. Thus (5.17) holds.

By Lemma 4, for almost all x_n, we have for almost all x' such that $(x', x_n) \in E$ the relation

$$(5.21)\qquad \int_I g^*(\xi', x_n)\big(\log^+ g^*(\xi', x_n)\big)^{n-3}d\xi' = o\big(h^{n-1}(\log 1/h)^{-1}\big),$$

where I is an n-dimensional cube containing x'. By the induction hypothesis this implies that for almost all such x'

$$\int_{R'} g^*(\xi', x_n)\,d\xi' = o(|R'|),$$

where R' is an $(n-1)$-dimensional rectangle containing x'. But if $R = R' \times J$ with $x_n \in J$, then

$$\int_R g(\xi)\,d\xi = \int_R g(\xi', \xi_n)\,d\xi'\,d\xi_n \leqslant |J|\int_{R'} g^*(\xi', x_n)\,d\xi'$$
$$= o(|R'|\,|J|) = o(|R|).$$

This completes the proof of the theorem.

CHAPTER VI

In this chapter we prove Theorem 2. We restrict ourselves to the special case $n = 3$, $k = 1$, $p = 1$ but the construction is general and can be routinely extended to the general case. We will construct a function which is bounded, in fact the characteristic function of a set but again the changes needed to make the function continuous are minor and can be supplied by the reader.

LEMMA 1. *Let $N > 0$ be an integer and let L be a lamina with thickness $h = N^{-1}(\log N)^{-1/2}$ contained in the unit cube Q^0 and parallel o the (x_1, x_2) plane. Let $\varepsilon > 0$. Then there exists a function f_L, the characteristic function of a subset of L such that*

(1) *If Q is any cube of edge $q \geqslant h$, we have*

$$\int_Q f_L \leqslant A\frac{q^3 h}{(\log N)^{1/2}}. \tag{6.1}$$

(2) *Except in a set of measure less than ε, for all x at a distance less than $1/2N(\log N)^{1/4}$ from L there exist arbitrarily small rectangles $R = R(w_1, w_2, w_3)$, $w_3 \geqslant w_2$, $w_3 \geqslant w_1$ containing x such that*

$$\frac{1}{w_3|R|}\int_R f_L \geqslant 1. \tag{6.2}$$

To prove this we need the following lemma (see [2]):

LEMMA 2. *Let S be a 2-dimensional square, $\varepsilon > 0$, N any positive integer. There exists $g(x_1, x_2)$, the characteristic function of a subset of S such that*

$$\int_S g \leqslant 2|S|/N\log N. \tag{1'}$$

(2′) *Except for $x = (x_1, x_2)$ in a set G of measure less than $\varepsilon|S|$, there exists a rectangle R' containing x such that*

$$|R'|^{-1}\int_{R'} g \geqslant 1/N.$$

Proof of Lemma 1. Let S_0 be the unit square $0 \leqslant x_1 \leqslant 1$, $0 \leqslant x_2 \leqslant 1$. Let L be of the form $S_0 \times I$, where $I \subset [0 \leqslant x_3 \leqslant 1]$ is an interval of length h. Cut S_0 into non-overlapping squares S of side h. Since h is going to be very small, the residual 'fringe' may be disregarded.

We define g on S_0 by defining it on each S to be the g of Lemma 2. We check properties (1) and (2):

$$\int_Q f_L \leqslant h \sum_{S\times I\cap Q\neq 0}\int_S g \leqslant h\frac{A}{N\log N}q^2 \tag{1}$$
$$= Aq^2h^2(\log N)^{-1/2} \leqslant Aq^3 h(\log N)^{-1/2}.$$

(2) Let $G_L = (\bigcup G)\times[0, 1]$, where the G are those of (2′) of Lemma 2. Let x be in the complement of G_L and let it have distance less than $1/2N(\log N)^{1/4}$ from L. Let J_w be an interval of length $w = 1/N(\log N)^{1/4}$ containing both I and x_3. Let R' be a rectangle of Lemma 2 corresponding to (x_1, x_2) and let $R = R'\times J_w$. Then

$$\frac{1}{w|R|}\int_R f_L = \frac{1}{w^2}\frac{h}{|R'|}\int_{R'} g \geqslant \frac{h}{w^2 N} = 1. \tag{6.3}$$

Now we construct the function f of Theorem 2. Let j_0 be an integer to be chosen later. Let $N_j = 4^{2^j}$, $j = j_0, j_0+1, \ldots$;

$$\varepsilon_j = 2^{-j}/N_j(\log N_j)^{1/4}, \qquad h_j = 1/N_j(\log N_j)^{1/2}.$$

We begin by constructing parallel laminae of distance $N_{j_0}^{-1}(\log N_{j_0})^{-1/}$ apart and thickness h_{j_0}. In each lamina we construct the function f_L of Lemma 1. Between these laminae we construct parallel laminae of thickness h_{j_0+1} at distance $N_{j_0+1}^{-1}(\log N_{j_0+1})^{-1/4}$ apart and define the corresponding function f_L in each of these laminae. We proceed for $j = j_0+2, \ldots$ Let

$$f = \sum f_L,$$

where the sum is extended over all the laminae we have constructed. Let $E = Q_0 - \bigcup \hat{L} - \bigcup G_L$, where $\hat{L}$ is L expanded three times. At the j-th stage we add no more than $N_j(\log N_j)^{1/4}$ new laminae, so that

$$\sum|\hat{L}| \leqslant 3\sum_{j_0}^{\infty}\frac{N_j(\log N_j)^{1/4}}{N_j(\log N_j)^{1/2}} = \frac{3}{(\log 4)^{1/4}}\sum_{j_0}^{\infty}2^{-j/4}. \tag{6.4}$$

Clearly,

$$\sum|G_L| \leqslant \sum_{j_0}^{\infty}2^{-j}. \tag{6.5}$$

If j_0 is large, the sum of the right-hand sides of (6.4) and (6.5) will be small. Thus the measure of E can be as close to 1 as we like. We will now show that, in E, f has a (1, 1) differential but not a (1, 1)′ differential.

Let $x \in E$, Q any cube containing x. If $Q \cap L \neq 0$, then $q \geqslant h$ because x is in the complement of $\hat{L}$. Thus

$$\frac{1}{q^4}\int_Q f = \frac{1}{q^4}\sum_{Q\cap L\neq 0}\int_Q f_L \leqslant \frac{A}{q}\sum \frac{h}{(\log N)^{1/2}}$$

$$\leqslant A\frac{1}{q}\max_{L\cap Q\neq 0}\frac{1}{(\log N)^{1/2}}\sum_{L\cap Q\neq 0} h \leqslant A \max_{Q\cap L\neq 0}\frac{1}{(\log N)^{1/2}} \to 0 \qquad (q\to 0).$$

For every $x \in Q^0$ there exists for every j a lamina at the j-th stage whose distance from x does not exceed $1/2N_j(\log N_j)^{1/4}$. Let $x \in E$. Fix j and choose such an L. Let R be the rectangle of Lemma 2 corresponding to this L. Then

$$\frac{1}{w_3|R|}\int_R f \geqslant \frac{1}{w_3|R|}\int_R f_L \geqslant 1.$$

Clearly $w_3 \to 0$ as $j\to\infty$.

This completes the proof of Theorem 2 as stated. By placing the aminae closer together we could also obtain

$$\limsup \frac{1}{w_3|R|}\int_R f = +\infty$$

for $x \in E$, $R \supset x$.

CHAPTER VII

In this chapter we include some additional observations.

1. In the preceding chapters we have shown that under suitable conditions the function f has at almost every point of a set E either a $(k, p)'$ or $(k, p)''$ differential. It is natural to inquire about intermediate results where the conclusion would be that f has almost everywhere in E a k-th differential in L^p with respect to rectangles having s different sides, $2 \leqslant s \leqslant n$. The results can of course be novel only for $n \geqslant 3$. The following theorem contains Theorem 5 and 6 as special cases.

THEOREM 7. *Let $f \in L(\log^+ L)^{s-2}(Q^0)$ and suppose that at each point x of a set $E \subset Q^0$ we have*

$$(7.1)\qquad \int_Q |f(x+t)-f(x)|\left(\log^+|f(x+t)-f(x)|\right)^{s-2}dt = o\left(\frac{h^n}{\log 1/h}\right) \qquad (h\to 0)$$

where Q are cubes of sides h containing the point 0. Then, for almost every $x \in E$,

$$(7.2)\qquad \int_R |f(x+t)-f(x)|\,dt = o(|R|)$$

where R is a rectangle containing 0 and having s different edge lengths.

Theorem 8 below is, like Theorem 3, an n-dimensional substitute for Theorem 1, if $s = n$.

THEOREM 8. *Let $f \in L^p(\log^+ L)^{s-2}$, $1 \leqslant p < \infty$, $2 \leqslant s \leqslant n$. Suppose that $k \geqslant 1$ and that, for each $x \in E$,*

$$(7.3)\qquad \left(\frac{1}{|Q|}\int_Q |f(x+t)-P_x(t)|^p\left(\log^+|f(x+t)-P_x(t)|\right)^{s-2}dt\right)^{1/p} = o(h^k),$$

$$(7.4)\qquad \left(\frac{1}{|Q|}\int_Q |f(x+t)-P_x(t)|^p dt\right)^{1/p} = o\left(\frac{h^k}{(\log 1/h)^{(s-2)/p}}\right)$$

where Q are cubes of side h containing the point 0.

Then, for almost every $x \in E$,

$$(7.5)\qquad \left(\frac{1}{|R|}\int_R |f(x+t)-P_x(t)|^p dt\right)^{1/p} = o(w^k) \qquad (w\to 0),$$

where R is a rectangle containing 0 with s different edges the longest of which is w.

Though the proofs of Theorems 7 and 8 follow the general lines of other proofs of this paper, they are a little more involved; in particular, they require a slight refinement of the decomposition lemma of Calderón and Zygmund stated in Chapter I giving the moduli of continuity of the derivatives of order k of the function f_1. We omit the proofs here.

2. The following remark is obvious and does not require any further comment: when in conclusions of theorems of this paper we speak of *equal edges* of R, we might actually assume that these edges are *essentially equal*, in the sense that the ratio of any two of them remains bounded.

3. Theorems 1, 3-8 can be strengthened in the following manner: the o's in the hypotheses can be replaced by O without affecting the conclusions of the theorems. Actually, in the case of Theorems 1, 3, 4 there is no novelty in this generalization, for it is well known (see [1]) that if we replace the o in (1.1) by O (the terms of degree k of $P(t)$ are then arbitrary) and if the new relation, with $P(t) = P_x(t)$, holds at each point of a set E, then f has a (k, p) differential almost everywhere in E. The situation is different when we have powers of $\log 1/h$ in the hypotheses. We state and prove the strengthening of Theorem 5 only; for Theorem 6 modifications of the proof are similar.

THEOREM 5_A. *If for every $x \in E$ the function f satisfies*

$$(7.6)\qquad \frac{1}{|Q|}\int_Q |f(x+t)-f(x)|\,dt = O\left(\frac{1}{\log 1/h}\right)$$

where Q is a cube containing the origin, then almost everywhere in E:

$$(7.7)\qquad \frac{1}{|R|}\int_R |f(x+t)-f(x)|\,dt = o(1),$$

where R is a rectangle containing the origin, with two different edge lengths and diameter tending to 0.

We assume f defined in a cube Q_0 of edge $h_0 \leqslant \frac{1}{2}$. As in the proof of Theorem 5 we split f into a sum of two functions, $f = f_1+f_2$, where f_1 is bounded, and so satisfies the conclusions of the theorem, and f_2 is zero in a set $G \subset E$ with $|E-G|$ small. Replacing $|f_2|$ by g, we have

$$(7.8)\qquad \int_Q g(t)\,dt = O\left(h^n\Big/\log\frac{1}{h}\right)$$

for $x \in G$, $Q \supset x$. Dividing g by a suitable constant and neglecting a subset of G of arbitrarily small measure we may suppose (after renaming G) that

$$(7.9)\qquad \int_Q g(t)\,dt \leqslant \left(h^n\Big/\log\frac{1}{h}\right)$$

for all $x \in G$ and all $Q \supset x$. In particular,

$$(7.10)\qquad \int_{Q_0} g(t)\,dt \leqslant 2^{n+1}h_0^n\Big/\log\frac{1}{h_0}.$$

Apply Lemma 4 of Chapter V to g and Q_0. We obtain cubes Q_k for which

$$(7.11)\qquad h_k^n\Big/\log\frac{1}{h_k} < \int_{Q_k} g\,dt \leqslant 2^{n+1}h_k\Big/\log\frac{1}{h_k}$$

and $g = 0$ in the complement of $\bigcup Q_k$.

Clearly G is contained in the complement of $\bigcup Q_k$. Let x be a point of density of the complement of $\bigcup Q_k$, and let $Q \supset x$ be a cube of edge h. If h is small, then $Q \cap Q_k \neq 0$ implies that $\hat{Q}_k \subset \hat{Q}$, where $\hat{Q}$ is Q expanded three times. Thus

$$(7.12)\qquad \int_Q g\,dt \leqslant \sum_{Q_k\cap Q\neq 0}\int_{Q_k} g\,dt \leqslant \sum_{Q_k\cap Q\neq 0} 2^{n+1}h_k^n\Big/\log\frac{1}{h_k} \leqslant \frac{2^{n+1}}{\log(1/h)}\sum_{Q_k\cap Q\neq 0} h_k^n.$$

But

$$(7.13)\qquad \sum_{Q_k\cap Q\neq 0} h_k^n = \sum_{Q_k\cap Q\neq 0} |Q_k| = o(|\hat{Q}|) = o(h^n).$$

From (7.12) and (7.13) we obtain that

$$\int_Q g\,dt = o\left(h\Big/\log\frac{1}{h}\right).$$

Thus the O in (7.8) can be replaced by o and Theorem 5_A follows from Theorem 5.

4. Theorem 9 below is an analogue of Theorem A for *strong* differentials in L^p; it is not particularly deep but perhaps deserves a proof. We consider a function $f(x) = f(x_1, \ldots, x_n) \in L^p(Q_0)$, $1 \leqslant p < \infty$, where Q_0 is the unit cube: $0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$. We write $x' = (x_1, \ldots, x_m)$, $x'' = (x_{m+1}, \ldots, x_n)$ and for any rectangle (in particular, cube) R we denote by R' and R'' the projections of R onto the subspaces of x' and x'', so that $R = R' \times R''$.

THEOREM 9. *If $f(x)$ has at each point $x \in E \subset Q_0$ a $(k, p)'$ differential, then at almost all points $x \in E$ it also has a $(k, p)'$ differential with respect to the variable x'.*

Using the decomposition theorem stated in Chapter I, we may reduce the general case to that of differential 0. Write $|f|^p = g$. Omitting from E a set of arbitrarily small measure we may assume that given any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that for any rectangle R containing a point $x \in E$ and of diameter $\leqslant \delta$ we have

$$(7.14)\qquad \int_R g\,dx \leqslant \varepsilon w^{kp}|R|$$

where w is the maximum edge length of R.

Let R_1' be any fixed rational subrectangle of Q_0' (i.e., a rectangle whose vertices have rational co-ordinates), and let

$$h(x'') = h_{R_1'}(x'') = \int_{R_1'} g(\xi', x'')\,d\xi'.$$

The family of functions $h(x'')$ is denumerable. Each function $h(x'')$ is integrable over Q_0'' and (a) its indefinite integral has a regular derivative (i. e., derivative with respect to cubes $Q'' \subset Q_0''$) equal to $h(x'')$ for almost all points $x'' \in Q_0''$. Moreover, (b) for almost all $x'' \in Q_0''$ the function $f(x', x'')$, as a function of x', is integrable over Q_0'.

Consider now any point $x = (x', x'') \in E$ for which we have both (a), no matter what rational rectangle R_1' we take, and (b). In (7.14) we consider only rectangles $R \supset x$ of the form $R_1' \times Q''$, so that $|R| = |R_1'|\,|Q''|$,

$R_1' \supset x'$, $Q'' \supset x''$. The requirements (a) and (b) eliminate only a subset of E of measure 0. Dividing both sides of (7.14) by $|Q''|$ and making the passage to the limit $|Q''| \to 0$, which is justified by (a), we see that

$$\int_{R_1'} g(\xi', x'')\,d\xi' \leqslant \varepsilon w^{kp} |R_1'|,$$

where now w is the largest edgelength of R_1'. The last inequality has been established for rational rectangles R_1' containing x', but, in view of (b), it holds, by continuity, for all rectangles R' containing x' and of diameter $\leqslant \delta$. This completes the proof of Theorem 9.

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On some properties of a class of singular integrals

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Introduction. Our purpose is to extend some known properties of the singular integrals of Calderón and Zygmund to a more general class of operators introduced in [5]. These singular integrals are convolution operators by quasihomogeneous kernels having mean value zero on certain differentiable manifold surrounding the origin (in the case of parabolic kernels, see [6]).

The aim of this paper is twofold. Firstly, we study the pointwise convergence of the quasi-homogeneous singular integrals and the behaviour of their maximal operators. Similar questions have been considered in our joint paper with E. B. Fabes (cf. [9]) for the different kind of parabolic singular integrals introduced by Jones in [4]. The same argument of [9], that is essentially a suitable modification of the method used by Calderón and Zygmund in [1], could be repeated for this general case, changing the computations to adequate them to the truncation of the kernels used here. Nevertheless it may be of interest to reconsider the question since an adaptation of the general method of "subordination of operators" given by Cotlar in [3], that can be used for the singular integrals of Calderón and Zygmund, enables us to get also a complementary result for the case $p = 1$ not considered in [9] and the pointwise convergence even for integrable functions.

Secondly, we consider the classes $T_u^p(x_0)$ studied by Calderón and Zygmund in [2], conveniently generalized, and prove that they are preserved under quasi-homogeneous singular integral operators.

In § 1 we give the definition of quasi-homogeneous functions and kernels and state some results about the singular integrals given by convolution with those kernels.

In § 2 we study the maximal operators of these integrals and obtain as a consequence that the quasi-homogeneous singular integrals converge in the pointwise sense for functions in L^p, $p \geqslant 1$.

In § 3 we give a generalization of the classes $T_u^p(x_0)$ to the case where a different number of derivations may be taken in each variable and prove some basic properties of these classes.