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Singular integrals with mixed homogeneity

by

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A. P. Calderón and A. Zygmund have widely studied singular integrals whose kernels are functions defined on E^n and homogeneous of degree $-n$. Lately, Jones [4] considered singular integrals whose kernels satisfied a "homogeneity" property of the form $K(\lambda x, \lambda^m t) = \lambda^{-n-m} K(x, t)$, where $x \in E^n$, $t \in (0, \infty)$, m a positive integer (for more details see the appendix).

The purpose of this paper is to consider a general class of kernels $K(x)$, homogeneous in the sense that there are positive numbers $a_1, \ldots, a_n$ such that $K(\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n) = \lambda^{-\Sigma a_j} K(x)$.

In the first part of this work the continuity of these operators acting on $L^p(E^n)$ is considered. In the second part, these same considerations are applied to operators on $L^p(E^n)$ arising from kernels $K(x, y)$ satisfying $K(x, \lambda^{a_1} y_1, \ldots, \lambda^{a_n} y_n) = \lambda^{-\Sigma a_j} K(x, y)$. Finally, in the appendix it is shown that these kernels include those studied by B. F. Jones.

§1. SINGULAR INTEGRALS

I. A change of variables of polar type. Let $x = (x_1, \ldots, x_n) \in E^n$ and $a_1, \ldots, a_n$ real numbers, $a_j \geqslant 1$. Consider

$$F(x, \varrho) = \sum_{j=1}^{n} \frac{x_j^2}{\varrho^{2a_j}};$$

for a fixed x, $F(x, \varrho)$ is a decreasing function of ϱ ($\varrho > 0$) and therefore call the *unique solution* of $F(x, \varrho) = 1$ by $\varrho(x)$. The point

$$\left(\frac{x_1}{\varrho^{a_1}(x)}, \ldots, \frac{x_n}{\varrho^{a_n}(x)}\right) \in \Sigma_n \qquad (\Sigma_n = \{x \in E^n, |x| = 1\})$$

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and x can be written

$$x_1 = \varrho^{a_1}\cos\varphi_1 \dots \cos\varphi_{n-2}\cos\varphi_{n-1},$$
$$x_2 = \varrho^{a_2}\cos\varphi_1 \dots \sin\varphi_{n-1},$$
$$\dots\dots\dots\dots\dots$$
$$x_n = \varrho^{a_n}\sin\varphi_1$$

and $dx = \varrho^{(\Sigma a_j)-1}J(\varphi_1, \dots, \varphi_{n-1})\,d\varrho\,d\sigma$ where $d\sigma$ is the element of area of Σ_n and $1 \leqslant J(\varphi_1, \dots, \varphi_{n-1}) \leqslant M$, $J(\varphi_1, \dots, \varphi_{n-1}) \in C^\infty\big((0, 2\pi)^{n-2} \times (0, \pi)\big)$.

REMARK 1. *$\varrho(x)$ is a metric.*

Proof. Observe that $\varrho(x) \leqslant 1$ is equivalent to $|x| \leqslant 1$ and $\varrho(\lambda^{a_1}x_1, \dots, \lambda^{a_n}x_n) = \lambda\varrho(x)$ $(\lambda > 0)$. Call $\lambda_1 = \varrho(x)$, $\lambda_2 = \varrho(y)$, $\lambda = \lambda_1 + \lambda_2$.

From the preceding remarks it will be enough to see that

$$\left(\frac{x_1+y_1}{\lambda^{a_1}}, \dots, \frac{x_n+y_n}{\lambda^{a_n}}\right) = \left(\left(\frac{\lambda_1}{\lambda}\right)^{a_1} x_1^*, \dots, \left(\frac{\lambda_1}{\lambda}\right)^{a_n} x_n^*\right) + \left(\left(\frac{\lambda_2}{\lambda}\right)^{a_1} y_1^*, \dots, \left(\frac{\lambda_2}{\lambda}\right)^{a_n} y_n^*\right)$$
$$= \frac{\lambda_1}{\lambda}\left(\left(\frac{\lambda_1}{\lambda}\right)^{a_1-1} x_1^*, \dots, \left(\frac{\lambda_1}{\lambda}\right)^{a_n-1} x_n^*\right) + $$
$$+ \frac{\lambda_2}{\lambda}\left(\left(\frac{\lambda_2}{\lambda}\right)^{a_1-1} y_1^*, \dots, \left(\frac{\lambda_2}{\lambda}\right)^{a_n-1} y_n^*\right) \in S_n,$$

where $(x_1^*, \dots, x_n^*) \in \Sigma_n$, $(y_2^*, \dots, y_n^*) \in \Sigma_n$. Since $0 \leqslant \lambda_i/\lambda \leqslant 1$ and using the convexity of $S_n = \{x;\ |x| \leqslant 1\}$, the result follows.

II. Singular integrals and class preservation. Let $x \in E^n$, $K(x)$ a complex function defined in $E^n - (0)$, a_i real numbers, such that $1 \leqslant a_i$, $1 \leqslant i \leqslant n$. (It can always be assumed that $1 = a_1 \leqslant a_2 \leqslant \dots \leqslant a_n$.) We will assume

1) $K(\lambda^{a_1}x_1, \dots, \lambda^{a_n}x_n) = \lambda^{-(\Sigma a_i)}K(x_1, \dots, x_n)$, $\lambda > 0$.

If

$$\lambda = \begin{pmatrix} \lambda^{a_1} & 0 & \dots & 0 \\ 0 & \lambda^{a_2} & \dots & 0 \\ 0 & \dots & \dots & \lambda^{a_n} \end{pmatrix}$$

this property can be written as $K(\lambda x) = |\det(\lambda)|^{-1}K(x)$.

2) $\int_{\Sigma_n} |K(x)|\,d\sigma < \infty$ (we will assume for simplicity: $\int_{\Sigma_n} |K(x)|\,d\sigma \leqslant 1$)

$$\int_{\Sigma_n} K(x)J(\varphi_1, \dots, \varphi_{n-1})\,d\sigma = 0,$$

where $J(\varphi_1, \dots, \varphi_{n-1})$ is the function determined by the change of variables defined in Section I.

Let $f \in C_0^\infty(E^n)$ and set

$$K_\varepsilon(x) = \begin{cases} K(x); & \varrho(x) > \varepsilon, \\ 0 & \text{otherwise} \end{cases}$$

($\varrho(x)$ defined as in Section I).

Set

$$\tilde{f}_\varepsilon(x) = K_\varepsilon(f) = \int_{E^n} K_\varepsilon(x-y)f(y)\,dy$$

exists everywhere.

THEOREM 1. *If*

$$\int_{S_y} |K(x-y) - K(x)|\,dx \leqslant C \tag{1.1}$$

where $S_y = \{x,\ \varrho(x) \geqslant 4\varrho(y)\}$, C independent of y (say for simplicity $C > 1$), then for $f \in C_0^\infty(E^n)$

1) *$\|\tilde{f}_\varepsilon\|_p \leqslant A_pC\|f\|_p$ for $1 < p < \infty$ where A_p depends only on p, $a_1, \dots, a_n$, $\int_{\Sigma_n} |K(x)|\,d\sigma$.*

2) *There exists $\tilde{f} \in L^p(E^n)$ such that $\lim_{\varepsilon\to 0} \|\tilde{f} - \tilde{f}_\varepsilon\|_p = 0$.*

Part 1) of the theorem will be divided into two lemmas. We pass now to prove the theorem for $p = 2$.

Let

$$h(x) = K_{\varepsilon,R}(x) = \begin{cases} K(x), & \varepsilon \leqslant \varrho(x) \leqslant R, \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity B will always denote any constant depending only on $a_1, \dots, a_n$,

$$\hat{f}(x) = \int_{E_n} e^{i\pi x \circ y} f(y)\,dy$$

will denote the Fourier transform of $f(x)$ in the sense of $L^2(E^n)$ or $L^1(E^n)$ depending on whether $f \in L^2(E^n)$ or $f \in L^1(E^n)$. Here

$$x \circ y = \sum_{i=1}^n x_i y_i.$$

LEMMA 1. *$|\hat{h}(x)| \leqslant BC$ for $x \in \Sigma_n$.*

Proof. Observe that

$$\int_{0 \leqslant \varrho(x) \leqslant a} \varrho(x)\,|h(x)|\,dx \leqslant \int_0^a \Big\{\int_{\Sigma_n} |K(x)|\,J(\varphi)\,d\sigma\Big\}\,d\varrho \leqslant Ba \tag{1.2}$$

(using the change of variables introduced in Section I).

(1.3) $$\int\limits_{a\leqslant\varrho(x)\leqslant 4a}|K(x)|\,dx=\Big\{\int\limits_{a}^{4a}\int\limits_{\Sigma_n}\frac{1}{\varrho}|K(x)|\,J(\varphi)\,d\sigma\Big\}\,d\varrho\leqslant B.$$

Finally, using (1.1),

(1.4) $$\int\limits_{S_y}|h(x-y)-h(x)|\,dx$$

$$\leqslant 2\int\limits_{\varepsilon\leqslant\varrho(x)\leqslant 4\varepsilon}|K(x)|\,dx+\int\limits_{S_y}|K(x-y)-K(x)|\,dx+2\int\limits_{R\leqslant\varrho(x)\leqslant 4R}|K(x)|\,dx\leqslant BC.$$

Let $x\in\Sigma_n$

$$2\hat{h}(x)=\int\limits_{E^n}e^{i\pi(x\circ y)}[h(y)-h(y-x)]\,dy=\int\limits_{\varrho(y)\geqslant 4}e^{i\pi(x\circ y)}[h(y)-h(y-x]\,dy+$$

$$+\int\limits_{\varrho(y)<4}h(y)[e^{i\pi(x\circ y)}-1]\,dy-\int\limits_{\varrho(y)<4}h(y-x)[e^{i\pi(x\circ y)}+1]\,dy+\int\limits_{\varrho(y)<4}h(y-x)\,dy.$$

For the first term use (1.4) with $\varrho(x)=|x|=1$. Observe now that

$$|e^{i\pi(x\circ y)}-1|\leqslant B|y|\leqslant B\varrho(y)\quad\text{for}\quad\varrho(y)<4,$$

(Note $\varrho(y)\geqslant|y|$ for $|y|\leqslant 1$.)

$$|e^{i\pi(x\circ y)}+1|=|e^{i\pi(x\circ(y-x))}-1|\leqslant B|y-x|\leqslant B\varrho(y-x)$$

for $\varrho(y)\leqslant 4$.

Both estimates together with (1.3) give a bound for the second and third integrals. For the last one observe that $\{y;\varrho(x-y)<3\}\subset\{y;\varrho(y)<4\}$ and therefore it can be majorized by

$$\int\limits_{3\leqslant\varrho(y-x)\leqslant 5}|h(y-x)|\,dy\leqslant B,$$

and the lemma is established.

Observe that $K_{\varepsilon,R}(\lambda^{a_1}x_1,\ldots,\lambda^{a_n}x_n)=K_{\varepsilon/\lambda,R/\lambda}(x_1,\ldots,x_n)$ which implies $|\hat{K}_{\varepsilon,R}(x)|\leqslant BC$ for every $x\in E^n$.

Now if

$$\tilde{f}_{\varepsilon,R}(x)=\int\limits_{E^n}K_{\varepsilon,R}(x-y)f(y)\,dy\qquad(f\in L^2(E^n)),$$

using Parseval's identity

$$\|\tilde{f}_{\varepsilon,R}\|_2=\|\hat{K}_{\varepsilon,R}\hat{f}\|_2\leqslant BC\|\hat{f}\|_2=BC\|f\|_2.$$

Finally, since $f\in C_0^\infty(E^n)$, $\tilde{f}_{\varepsilon,R}(x)\to\tilde{f}_\varepsilon(x)$ everywhere as $R\to\infty$, and using Fattou's theorem,

(1.5) $$\|\tilde{f}_\varepsilon\|_2\leqslant BC\|f\|_2.$$

To prove the same result for every $L^p(E^n)$, $1<p<\infty$, we will prove first weak type (1, 1) and then interpolate.

The weak type result holds for convolution with more general functions than $K_\varepsilon(x)$. In fact, no homogeneity is needed for this result. More precisely:

Lemma 2. *Let $k(x)$ be a locally integrable function, satisfying condition (1.1), and for $f\in C_0^\infty(E^n)$ assume*

$$\tilde{f}(x)=\int\limits_{E^n}k(x-y)f(y)\,dy$$

satisfies

(1.5′) $$\|\tilde{f}\|_2\leqslant C\|f\|_2$$

(it is enough to assume (1.5′) for some $p>1$). Then

$$|\{x,\,|\tilde{f}(x)|\geqslant M\}|\leqslant\frac{BC}{M}\|f\|_1$$

where $|A|$ denotes the measure of the set A.

To prove this, we will need the following result:

Sublemma. *Let $f(x)\geqslant 0$, $f\in L^p(E^n)$, $1\leqslant p<\infty$. Then for every $M>0$ there exists a sequence of non-overlapping sets $\{I_k\}$ such that*

1) *$I_k=I_k^1\times I_k^2\times\ldots\times I_k^n$ (I_k^j one-dimensional intervals), $|I_k|=c_k^{a_j}$ (a_j as considered before), c_k depending only on k and determined by the sequence.*

2) *Consider any a such that $1<a^{a_j}\leqslant 2$ for every a_j and take any $r>0$ satisfying*

(1.6) $$1-\frac{1}{a^{a_j}}-\frac{1}{a^{ra_j}}\geqslant 0\quad\textit{for every } a_j.$$

Then

$$M\leqslant\frac{1}{|I_k|}\int\limits_{I_k}f(x)\,dx\leqslant 2^{rn}M.$$

3) *If $D_M=\bigcup_k I_k$, $f(x)\leqslant M$ in D'_M (the complement of D_M).*

Proof. Write $E^n=R_1\times R_2\times\ldots\times R_n$ (R_j real numbers, $R_j=(-\infty,\infty)$). Divide R_j into a mesh of intervals of measure $|I_0^j|=c_0^{a_j}$, project them into E^n and intersect the projections. This will produce in

E^n a mesh of the type described in (1). Let c_0 be large enough so that

$$\frac{1}{|I_0|}\int_{I_0} f(x)\,dx < M.$$

Proceed now by induction as follows: Set $c_k = c_{k-1}/a$. Every interval I_{k-1}^j can be divided into at most two intervals, the one on the left of measure $c_k^{a_j}$ if possible. Project and intersect in I_{k-1} forming at most 2^n sets, call them I_k, select from them those for which $|I_k| = c_k^{a_j}$ and

$$\frac{1}{|I_k|}\int_{I_k} f(x)\,dx \geqslant M,$$

on the rest proceed as before.

Finally we will have selected a sequence $\{I_k\}$ of non-overlapping sets as described in (1).

For a selected I_k, let I_{k-l} the first I such that $I_k \subset I_{k-l}$ and I_{k-l} satisfies property (1), and using (1.6) $l \leqslant 2r$; hence

$$\frac{1}{|I_k|}\int_{I_k} f(x)\,dx \leqslant \frac{|I_{k-l}|}{|I_k|}\,\frac{1}{|I_{k-l}|}\int_{I_{k-l}} f(x)\,dx \leqslant 2^{rn}M$$

(since I_{k-l} was not selected) and property (2) is also satisfied by the selected sequence.

Call $D_M = \bigcup I_k$ (I_k selected sequence). If $x \notin D_M$, then there exists a sequence I_m of sets of type (1) such that $x \in I_m$, $|I_m| \to 0$ and

$$\frac{1}{|I_m|}\int_{I_m} f(y)\,dy < M.$$

But for this type of sets [1]

$$\frac{1}{|I_m|}\int_{I_m} f(y)\,dy \to f(x) \text{ a.e.}$$

and the sublemma is completed.

Proof of Lemma 2. Since $f(x) = f^+(x)+f^-(x)$ we can assume that $f(x) \geqslant 0$ and $|f(x)-f(y)| \leqslant c|x-y|$ with bounded support.

For $M > 0$, let $\{I_k\}$ be the sequence selected by the sublemma. Define

$$u(x) = \begin{cases} f(x) & \text{for } x \in D_M', \\ \dfrac{1}{|I_k|}\displaystyle\int_{I_k} f(x)\,dx & \text{for } x \in I_k. \end{cases}$$

(1) See Jensen, Marcinkiewicz, Zygmund, Fundamenta Mathematicae 25, p. 217-234, Theorem 6.

Set $v(x) = f(x)-u(x)$, then $v(x)$ satisfies:

(1.7) $$v(x) = 0 \text{ in } D_M', \qquad \int_{I_k} v(x)\,dx = 0;$$

(1.8) $$\int_{E^n} |v(x)|\,dx \leqslant 2\int_{E^n} f(x)\,dx.$$

Using properties (2) and (3) of the I_k's we have:

(1.9) $$\int_{E^n} u^2(x)\,dx = \int_{D_M'} f^2(x)\,dx + \sum_k \int_{I_k}\Big(\frac{1}{|I_k|}\int_{I_k} f(y)\,dy\Big)^2 dx$$
$$\leqslant \hat{M}\,\|f\|_1 + 2^{2rn}M^2|D_M| \leqslant BM\|f\|_1.$$

Since $u(x), v(x)$ have bounded support and satisfy a Lipschitz condition, $\tilde f(x) = \tilde u(x)+\tilde v(x)$ is defined everywhere,

(1.10) $$|\{x, |\tilde f(x)| \geqslant M\}| \leqslant \Big|\Big\{x, |\tilde u(x)| \geqslant \frac{M}{2}\Big\}\Big| + \Big|\Big\{x, |\tilde v(x)| \geqslant \frac{M}{2}\Big\}\Big|,$$

and

(1.11) $$\Big|\Big\{x, |\tilde u(x)| \geqslant \frac{M}{2}\Big\}\Big| \leqslant \frac{4}{M^2}\int_{E^n} |\tilde u|^2\,dx \leqslant \frac{4C}{M^2}\|u\|_2^2 \leqslant \frac{BC}{M}\|f\|_1$$

(using (1.5′) and (1.9)).

Call $S_\delta(x) = \{y, \varrho(y-x) \leqslant \delta\}$. Let β be the only positive solution of

$$\sum_{j=1}^n \beta^{-2a_j} = 1,$$

and x_k the symmetric center of I_k. Then $S_{\beta c_k}(x_k)$ covers precisely I_k (c_k defined in the sublemma). Set

$$S^*(x_k) = S_{5\beta c_k}(x_k) \quad \text{and} \quad D_M^* = \bigcup_k S_k^*.$$

Then

(1.12) $$\Big|\Big\{x, |\tilde v(x)| \geqslant \frac{M}{2}\Big\}\Big|$$
$$\leqslant \Big|\Big\{x, |\tilde v(x)| \geqslant \frac{M}{2}\Big\} \cap D_M^*\Big| + \Big|\Big\{x, |\tilde v(x)| \geqslant \frac{M}{2}\Big\} \cap (D_M^*)'\Big|$$

($(D_M^*)'$ complement of D_M^*), and

(1.13) $$\Big|\Big\{x, |\tilde v(x)| \geqslant \frac{M}{2}\Big\} \cap D_M^*\Big| \leqslant |D_M^*| \leqslant B\sum |I_k| = B|D_M| \leqslant \frac{B}{M}\|f\|_1.$$

By hypothesis

$$\int_{S_y} |k(x-x)-k(x)|\,dx \leqslant C. \tag{1.14}$$

Hence using (1.7)

$$\int_{(D_M^*)'} |\tilde{v}(x)|\,dx \leqslant \sum_k \int_{(D_M^*)'} \Big\{\Big|\int_{I_k} k(x-y)v(y)\,dy\Big|\Big\}\,dx$$
$$= \sum_k \int_{(D_M^*)'} \Big\{\Big|\int_{I_k} [k(x-y)-k(x-y_k)v(y)\,dy\Big|\Big\}\,dx$$
$$\leqslant \sum_k \int_{I_k} |v(y)|\Big\{\sum_{(D_M^*)'} |k\big((x-y_k)-(y-y_k)\big)-k(x-y_k)|\,dx\Big\}\,dy,$$

and since for $x\in(D_M^*)'$, $y\in I_k$, $\varrho(x-y_k)\geqslant 4\varrho(y-y_k)$, and applying (1.14)

$$\int_{(D_M^*)'} |\tilde{v}(x)|\,dx \leqslant B(C+1)\|h\|_1 \leqslant B(C+1)\|f\|_1.$$

Hence

$$\Big|\Big\{x;\,|\tilde{v}(x)|\geqslant\frac{M}{2}\Big\}\cap(D_M^*)'\Big| \leqslant \frac{2}{M}\int_{(D_M^*)'}|\tilde{v}(x)|\,dx \leqslant \frac{BC}{M}\|f\|_1. \tag{1.15}$$

Therefore collecting the results (1.11), (1.12), (1.13), (1.15) and applying them to (1.10),

$$|\{x;|\tilde{f}(x)|>M\}| \leqslant \frac{BC}{M}\|f\|_1. \tag{1.16}$$

Remark 2. *Let $K_\varepsilon(x)$ satisfying the conditions of Theorem 1. Then*

$$|\{x;|\tilde{f}_\varepsilon(x)|>M\}| \leqslant \frac{BC}{M}\|f\|_1.$$

Proof. $K_\varepsilon(x)$ is locally integrable and by using the same argument as for (1.3) and (1.4), (1.1) follows for K_ε. Finally using (1.5) it is of type (2,2).

Proof of Theorem 1. The proof of 1) for $1<p\leqslant 2$ is an immediate corollary of (1.5) and Remark 1 by the use of Marcinkiewicz interpolation Theorem (see [7]). For $p>2$, take p' such that $1/p+1/p'=1$. Clearly $1<p'\leqslant 2$.

$$\|\tilde{f}_\varepsilon\|_p = \sup_{\substack{\|g\|_{p'}\leqslant 1\\ g\in C_0^\infty(E^n)}} \int_{E^n}\tilde{f}_\varepsilon(x)g(x)\,dx = \sup_{\substack{\|g\|_{p'}\leqslant 1\\ g\in C_0^\infty(E^n)}} \int_{E^n} f(x)\tilde{g}_\varepsilon(x)\,dx$$
$$\leqslant \|f\|_p\|\tilde{g}_\varepsilon\|_{p'} \leqslant A_{p'}B(C+1)\|f\|_p,$$

$A_{p'}$ depending only on p, and Part 1) is established.

2) Let $\varepsilon\leqslant\eta\leqslant 1$, $f\in C_0^\infty(E^n)$.

$$\|\tilde{f}_\varepsilon(x)-\tilde{f}_\eta(x)\|_p = \Big\|\int_{\varepsilon\leqslant\varrho(y)\leqslant\eta} K(y)f(x-y)\,dy\Big\|_p$$
$$= \Big\|\int_{\varepsilon\leqslant\varrho(y)\leqslant\eta} K(y)[f(x-y)-f(x)]\,dy\Big\|_p$$
$$\leqslant \Big\|\int_{\varepsilon\leqslant\varrho(x)\leqslant\eta} |x|\,|K(x)|\Big(\sum_j\Big|\frac{\partial}{\partial x_j}f(y+\theta x)\Big|\Big)\,dx\Big\|_p \leqslant B\eta\Big\|\sum_j\Big|\frac{\partial}{\partial y_j}f(y)\Big|\Big\|_p,$$

using Minkowski's inequality and (1.2) ($|x|\leqslant\varrho(x)$). Hence $\{\tilde{f}_\varepsilon\}$ forms a Cauchy sequence in $L^p(E^n)$ which proves part 2).

This result allows us to extend the operations $\tilde{f}_\varepsilon,\tilde{f}$ to the whole of $L^p(E^n)$ by continuity, since

$$\|\tilde{f}\|_p = \lim_{\varepsilon\to 0}\|\tilde{f}_\varepsilon\|_p \leqslant A_pB(C+1)\|f\|_p,$$

and we will denote the extension by

$$\tilde{f}_\varepsilon = K_\varepsilon(f) = \int_{\varrho(y)\geqslant\varepsilon} K(y)f(x-y)\,dy,$$

understood as a limit in L^p, and

$$\tilde{f} = K(f) = \lim_{\varepsilon\to 0}\int K_\varepsilon(y)f(x-y)\,dy \text{ in } L^p.$$

Remark 3. *If $K(x)$ is bounded in Σ_n, $\int_{\varrho(y)\geqslant\varepsilon} K(y)f(x-y)\,dy$ exists and coincides with $\tilde{f}_\varepsilon(x)$.*

Remark 4. *Assume $K(x)\in C^1(E^n-\{0\})$ and let*

$$C^* = \sup_{x\in\Sigma_n}\Big\{\sup_{j=1,\dots,n}\Big|\frac{\partial}{\partial x_j}K(x)\Big|\Big\}. \tag{1.17}$$

Then

$$\int_{S_y}|K(x-y)-K(x)|\,dx \leqslant BC^*.$$

Proof. We have

$$\int_{\varrho(x)\geqslant 4\varrho(y)} |K(x-y)-K(x)|\,dx = \int_{\Sigma_n} J(\varphi)\,d\sigma\Big\{\int_{4\varrho(y)}^{\infty} \varrho^{-1}(x)\,|K(x'-y_\varrho)-K(x')|\,d\varrho$$

where $x' = \big(x_1/\varrho^{a_1}(x), \ldots, x_n/\varrho^{a_n}(x)\big)\in\Sigma_n$ and

$$y_\varrho = \big(y_1/\varrho^{a_1}(x), \ldots, y_n/\varrho^{a_n}(x)\big) = \left(\left(\frac{\varrho(y)}{\varrho(x)}\right)^{a_1}\bar{y}_1, \ldots, \left(\frac{\varrho(y)}{\varrho(x)}\right)^{a_n}\bar{y}_n\right).$$

Since $\varrho(y)/\varrho(x)\leqslant 1/4$, $|y_\varrho|\leqslant 1/4$. Hence $|x'-y_\varrho|\leqslant 3/4$, and applying formula (1.17)

$$\int_{\varrho(x)\geqslant 4\varrho(y)} |K(x-y)-K(x)|\,dx \leqslant BC^*\int_{\Sigma_n} d\sigma\int_{4\varrho(y)} \varrho^{-1}(x)\left(\sum_{j=1}^{n}\left(\frac{\varrho(y)}{\varrho(x)}\right)^{a_j}\right)d\varrho$$

$$\leqslant BC^*\int_{4\varrho(y)}^{\infty}\frac{\varrho(y)}{\varrho(x)}\,d\varrho \leqslant BC^*$$

since $a_j\geqslant 1$.

III. $L^p(E^n)$ multipliers. Let $a=(a_1,\ldots,a_n)$, $a_j\geqslant 1$ as before. For $\beta=(\beta_1,\ldots,\beta_n)$ define

$$(a,\beta)=\sum_{j=1}^{n} a_j\beta_j \quad\text{and}\quad |\beta|=\sum_{j=1}^{n}\beta_j.$$

For $x=(x_1,\ldots,x_n)$ define

$$\lambda^a x = (\lambda^{a_1}x_1,\ldots,\lambda^{a_n}x_n) \quad\text{and}\quad (x)^\beta = x_1^{\beta_1}x_2^{\beta_2}\ldots x_n^{\beta_n}.$$

If $h(x)\in L^\infty(E^n)$ and $\varphi\in C_0^\infty(E^n)$, the operation $T(\varphi)=F^{-1}\big(hF(\varphi)\big)$ (where F is the Fourier transform operator) exists everywhere and moreover is of type (2, 2), i.e., $\|T\|_2\leqslant M\|\varphi\|_2$ where M is the essc. sup. of $h(x)$, by a simple application of Parseval's identity.

The function $h(x)$ is said to be a *multiplier* when

$$\|T\varphi\|_p\leqslant A_p\|\varphi\|_p \quad\text{for every } p,\quad 1<p<\infty.$$

Theorem. *Let $h(x)\in L^\infty(E^n)$, and assume $h(x)$ is N times continuously differentiable where $N>|a|/2$; moreover, assume that*

$$\text{(1.18)}\qquad \int_{R/2\leqslant\varrho(x)\leqslant 2R} |R^{(a,\beta)}(\partial/\partial x)^\beta h(x)|^2\frac{dx}{R^{|a|}}\leqslant C,\qquad |h(x)|\leqslant C\ \text{a.e.},$$

where C is independent of R, say $C\geqslant 1$, and $\varrho(x)$ defined by a as in Section I.

Then

$$\|T(\varphi)\|_p = \big\|F^{-1}\big(hF(\varphi)\big)\big\|_p\leqslant A_pC\|\varphi\|_p,\qquad \varphi\in C_0^\infty(E^n),$$

where A_p depends only on a and p.

Proof. Let $\chi(t)\in C_0^\infty(-\infty,\infty)$ be positive in $\frac{1}{2}<|t|<2$ and zero otherwise. Set

$$\Phi(t)=\frac{\chi(t)}{\sum_{-\infty}^{\infty}\chi(2^{-j}t)}.$$

Clearly,

$$\sum_{j=-\infty}^{\infty}\Phi(2^{-j}t)=1,$$

and $\Phi(t)\in C_0^\infty(-\infty,\infty)$ with support in the interval $\frac{1}{2}\leqslant|t|\leqslant 2$. Let $\psi(x)=\Phi\big(\varrho(x)\big)\in C_\infty^0(E^n)$ and

$$\sum_{j=-\infty}^{\infty}\psi(2^{-ja}x)=1$$

($2^{-ja}x=(2^{-j})^a x$ in the sense previously defined). Write $h_j(x)=h(x)\psi(2^{-ja}x)$; then for $|\eta|\leqslant N$

$$\text{(1.19)}\qquad 2^{j(a,\eta)}(D_\eta h_j)(x)=\sum_{\beta+\gamma=\eta}2^{j(a,\beta)}(D_\beta f)(x)(D_\gamma\psi)(2^{-ja}x)$$

and since $(D_\gamma\psi)(2^{-ja}x)$ is absolutely bdd. and has support in $2^{j-1}\leqslant\varrho(x)\leqslant 2^{j+1}$, applying (1.18) to the right hand side of (1.19) with $R=2^j$, we have

$$\text{(1.20)}\qquad \int_{E^n}|2^{j(a,\eta)}D_\eta h_j(x)|^2\frac{dx}{2^{j|a|}}\leqslant BC.$$

Let $g_j(x)=\hat{h}_j(x)$. Applying Parseval's identity to (1.20) and adding over all η, $|\eta|\leqslant N$,

$$\text{(1.21)}\qquad \sum_{|\eta|\leqslant N}\int_{E^n}|2^{j(a,\eta)}(ix)^\eta|^2|g_j(x)|^2dx\leqslant BC\,2^{j|a|}.$$

Now,

$$\text{(1.22)}\qquad \sum_{|\eta|\leqslant N}|2^{j(a,\eta)}(ix)^\eta|^2\geqslant B(1+|2^{ja}x|^2)^N\geqslant B\big(1+\varrho^2(2^{ja}x)\big)^N.$$

The second part of this inequality follows immediately when $|2^{ja}x|\geqslant 1$ since then $|2^{ja}x|\geqslant\varrho(2^{ja}x)$ and becomes trivial when $|2^{ja}x|\leqslant 1$ (which is equivalent to $\varrho(2^{ja}x)\leqslant 1$).

Applying now (1.22) to (1.21),

$$\int_{E^n} (1+2^{2j}\varrho^2(x))^N |g_j(x)|^2 dx \leqslant BC2^{j|\alpha|}. \tag{1.23}$$

Using now Hölder's inequality and (1.23),

$$\int_{E^n} |g_j(x)|\,dx \leqslant (BC)^{1/2}2^{j|\alpha|/2}\Big(\int_{E^n} (1+2^{2j}\varrho^2(x))^{-N}dx\Big)^{1/2} \leqslant (BC)^{1/2} \tag{1.24}$$

provided $N > |\alpha|/2$.

Moreover

$$\int_{\varrho(x)\geqslant a} |g_j(x)|\,dx \leqslant (BC)^{1/2}2^{j|\alpha|/2}\Big(\int_{\varrho(x)\geqslant a} (1+2^{2j}\varrho^2(x))^{-N}dx\Big)^{1/2}$$

$$\leqslant (BC)^{1/2}(2^j a)^r \quad \text{where} \quad r = \frac{|\alpha|}{2} - N < 0.$$

In particular, the preceding inequality shows that

$$\int_{\varrho(x)\geqslant 4\varrho(y)} |g_j(x-y)-g_j(x)|\,dx \leqslant (BC)^{1/2}\big(2^j\varrho(y)\big)^r \tag{1.25}$$

which is a good estimate for that integral when $2^j\varrho(y)\geqslant 1$. When $2^j\varrho(y) = \varrho(2^{j\alpha}y)\leqslant 1$, we will proceed as follows:

Set $f_j(x) = (e^{-iy\circ x}-1)h_j(x)$. Hence

$$2^{j(\alpha,\eta)}D_\eta f_j(x) = \sum_{\substack{|\gamma|\geqslant 1\\ \beta+\gamma=\eta}} (-i2^{j\alpha}y)^\gamma e^{iy\circ x}2^{j(\alpha,\beta)}(D_\beta h_j)(x) +$$

$$+[e^{-iy\circ x}-1]2^{j(\alpha,\eta)}(D_\eta h_j)(x),$$

and therefore

$$\int_{E^n} |2^{j(\alpha,\eta)}D_\eta f_j(x)|^2 \frac{dx}{2^{j|\alpha|}} \leqslant BC2^{2j}\varrho^2(y). \tag{1.26}$$

Using that $|2^{j\alpha}y|^{|\gamma|}\leqslant|2^{j\alpha}y|\leqslant 2^j\varrho(y)$ when $|\gamma|\geqslant 1$ and since in the support of $D_\eta h_j$ $2^{j-1}\leqslant\varrho(x)\leqslant 2^{j+1}$, we have

$$|e^{-ix\circ y}-1| \leqslant B|x\circ y| = B|2^{j\alpha}x'y| = B|x'\circ 2^{j\alpha}y| \leqslant B|x'|\,|2^{j\alpha}y| \leqslant B2^j\varrho(y).$$

(Note $\varrho(x')\leqslant 2$, and inequality (1.20).)

Observing now that $\hat{f}_j(x) = g_j(x-y)-g_j(x)$ and proceeding with f_j as we did with h_j, we replace in inequalities (1.21)-(1.24) BC by $BC2^{2j}\varrho^2(y)$, and we finally obtain

$$\int_{E^n} |g_j(x-y)-g_j(x)|\,dx \leqslant (BC)^{1/2}2^j\varrho(y).$$

This inequality together with (1.27) shows that

$$\int_{\varrho(x)\geqslant 4\varrho(y)} |g_j(x-y)-g_j(x)|\,dx \leqslant BC\min\{2^j\varrho(y), (2^j\varrho(y))^r\}. \tag{1.27}$$

Let

$$G_M(x) = \sum_{j=-M}^{M} g_j(x).$$

(1.26) shows that

$$\int_{\varrho(x)\geqslant 4\varrho(y)} |G_M(x-y)-G_M(x)|\,dx \leqslant BC$$

independently of M and $\varrho(y)$.

Let

$$H_M(x) = \sum_{j=-M}^{M} h_j(x).$$

Then $|H_M(x)|\leqslant BC$ and $\hat{H}_M = G_M$. Therefore $G_M(x)$ satisfies the conditions of Lemma 2 in the previous section. Then for $\varphi(x)\in C_0^\infty(E^n)$

$$T_M(\varphi) = \int_{E^n} G_M(x-y)\varphi(y)\,dy = F^{-1}\big(H_M F(\varphi)\big)$$

satisfies $\|T_M(\varphi)\|_p\leqslant A_pC\|\varphi\|_p$, A_p depending on p and α only. Now H_M are uniformly bdd. and converge everywhere to $h(x)$. Therefore,

$$H_M(x)F(\varphi)(x)\to h(x)F(\varphi)(x)$$

in $L^1(E^n)$ by Lebesgue dominating theorem, and hence

$$F^{-1}\big(H_MF(\varphi)\big)(x)\to F^{-1}\big(hF(\varphi)\big)(x) \text{ a.e.}$$

Using now Fattou's lemma

$$\|T(\varphi)\|_p = \big\|F^{-1}\big(hF(\varphi)\big)\big\|_p \leqslant A_p\|\varphi\|_p$$

and the theorem is proved.

This result has been proved by Hörmander in [4] when $\alpha_1=\alpha_2=\ldots=\alpha_n=1$.

§ 2. VARIABLE KERNELS

In this section we consider kernels $K(x,y)$ with the following properties ($x\in E^n$, $y\in E^n$):

1) For every x fixed $K(x,y)$ is a singular kernel in y as treated in § 1 with homogeneities $1\leqslant\alpha_1\leqslant\alpha_2\ldots\leqslant\alpha_n$.

2) For every x fixed $K(x,y)\in C^\infty(E^n-\{0\})$ and

$$\sup_{y\in\Sigma_n}\left|\left(\frac{\partial}{\partial y}\right)^\beta K(x,y)\right|\leqslant C_\beta$$

independent of x.

Define

$$K_\varepsilon(x,y)=\begin{cases}K(x,y) & \text{for}\quad \varrho(y)\geqslant\varepsilon,\\ 0 & \text{otherwise}.\end{cases}$$

For $f\in C_0^\infty(E^n)$ set

$$\tilde f_\varepsilon(x)=K_\varepsilon(f)=\int_{E^n}K_\varepsilon(x,x-y)f(y)\,dy.$$

$\tilde f_\varepsilon(x)$ is well defined everywhere.

THEOREM 1. *For* $f\in C_0^\infty(E^n)$, $1<p<\infty$, *we have*

1) $\|\tilde f_\varepsilon\|_p\leqslant A_p\|f\|_p$, A_p *independent of* ε *and* f.

2) *There exists* $\tilde f$ *such that* $\lim_{\varepsilon\to 0}\|\tilde f_\varepsilon-\tilde f\|_p=0$.

Before proceeding with the proof of the theorem, we will state certain properties of n-dimensional spherical harmonics which will be needed in the proof.

Let $Y_l(x)$ be an n-dimensional spherical harmonic of degree l; then

$$(2.1)\qquad x\in\Sigma_n;\qquad \left|\left(\frac{\partial}{\partial x}\right)^\beta Y_l(x)\right|\leqslant Bl^{\left(\frac{n-2}{2}+|\beta|\right)}.$$

Let

$$\{Y_{k,l}(x)\},\qquad 1\leqslant k\leqslant\binom{l+n-1}{n-1}-\binom{l+n-3}{n-1}$$

be an orthonormal base for the space of all spherical harmonics of degree l. Then $\{Y_{k,l}(x)\}$ for all k,l is a complete orthonormal system of functions over Σ_n.

If $f\in C^\infty(\Sigma_n)$ and if $f(x)\sim\sum_{k,l}a_{k,l}Y_{k,l}(x)$ $(x\in\Sigma_n)$ is the Fourier series development of $f(x)$ with respect to $\{Y_{k,l}\}$ where

$$a_{k,l}=\int_{\Sigma_n}f(x)Y_{k,l}(x)\,d\sigma,$$

then, for every $r>1$,

$$(2.2)\qquad |a_{k,l}|\leqslant A_r l^{-2r}\sup_{\substack{|\beta|=2r\\ x\in\Sigma_n}}\left|\left(\frac{\partial}{\partial x}\right)^\beta f(x)\right|,$$

A_r depending only on r and n.

For the proof of this result the reader can see [3].

Proof of Theorem 1. Since $J(\varphi_1,\ldots,\varphi_{n-1})\in C^\infty(\Sigma_n)$, for $y\in\Sigma_n$ and calling $\varphi=(\varphi_1,\ldots,\varphi_{n-1})$, we consider

$$K(x,y)J(\varphi)=\sum_{k,l}a_{k,l}(x)Y_{k,l}(y).$$

Hence

$$K(x,y)=\sum_{k,l}a_{k,l}(x)Y_{k,l}(y)/J(\varphi).$$

Define

$$\varrho_\varepsilon(x)=\begin{cases}\varrho(x) & \text{when}\quad \varrho(x)>\varepsilon,\\ 0 & \text{otherwise}.\end{cases}$$

Given $y\in E^n$ write $y=\left(\varrho^{a_1}(y)\bar y_1,\ldots,\varrho^{a_n}(y)\bar y_n\right)$ where $\bar y=(\bar y_1,\ldots,\bar y_n)\in\Sigma_n$. Then the series

$$\sum_{k,l}a_{k,l}(x)\frac{y_{k,l}(\bar y)}{J(\varphi)\,\varrho_\varepsilon^{\Sigma a_j}(y)}$$

converges in $L^q(E^n)$, $1<q<\infty$, to $K_\varepsilon(x,y)$, using (2.1) for $\beta=0$ and (2.2) for r large enough. Therefore

$$\tilde f_\varepsilon(x)=\sum_{k,l}a_{k,l}(x)\int_{E_n}\frac{Y_{k,l}(\bar y)}{J(\varphi)\,\varrho_\varepsilon^{\Sigma a_j}(y)}f(x-y)\,dy\qquad\text{a.e.}$$

Observe that, on the other hand, $Y_{k,l}(\bar y)/J(\varphi)\varrho_\varepsilon^{\Sigma a_j}(y)$ is a singular integral kernel as treated in Section II of § 1; and therefore

$$\|\tilde f_\varepsilon\|_p\leqslant A_p\left(\sum_{k,l}\max|a_{k,l}(x)|\left\{1+\sum_j\sup\left|\frac{\partial}{\partial y_j}Y_{k,l}(\bar y)\right|\right\}\right)\|f\|_p.$$

Using now (2.1) and (2.2) with $\beta=1$, $r>\frac{3}{2}n-1$

$$\|\tilde f_\varepsilon\|_p\leqslant A_p\Big(\sum_{k,l}l^{-r}(1+l^{(n-2)/2+1})\Big)\|f\|_p$$

$$\leqslant A_p\Big(\sum_l l^{-r}(1+l^{(n-2)/2+1})\Big(\binom{l+n-1}{n-1}-\binom{l+n-3}{n-1}\Big)\Big)\|f\|_p$$

$$\leqslant A_p\Big(\sum_l l^{-r}(1+l^{n/2})l^{(n-1)/2}\Big)\|f\|_p\leqslant A_p\|f\|_p,$$

A_p independent of ε and f.

The proof of 2) is the same as in Theorem 1, Section II of § 1 applied term by term to the series.

THEOREM 2. *For $f \in C_0^\infty(E^n)$ let*

$$\tilde{f}_\varepsilon^*(x) = \int_{E^n} \overline{K}_\varepsilon(y, x-y) f(y)\, dy;$$

then for $1 < p < \infty$

1) $\|\tilde{f}_\varepsilon^*\|_p \leqslant A_p \|f\|_p$.
2) *There exists* $\tilde{f}^* \in L^p(E^n)$ *such that* $\lim_{\varepsilon\to 0} \|\tilde{f}_\varepsilon^* - \tilde{f}^*\|_p = 0$.

Proof. 1) follows exactly as in previous proof once it is observed that

$$\tilde{f}_\varepsilon^*(x) = \sum_{k,l} \int_{E^n} \frac{Y_{k,l}(\bar{y})}{J(\varphi)\,\varrho_\varepsilon^{\Sigma a_j}} \overline{a_{k,l}}(x-y) f(x-y)\, dy \quad \text{a.e.}$$

For 2). Given $\delta > 0$ select first N large enough so that for $\varepsilon \leqslant 1$

$$\Big\| \sum_{\substack{k,l\\ l>N}} \int_{E^n} \frac{Y_{k,l}(\bar{y})}{J(\varphi)\,\varrho_\varepsilon^{\Sigma a_j}} \overline{a_{k,l}}(x-y) f(x-y)\, dy \Big\|_p \leqslant \frac{\delta}{4}.$$

This is possible by using part 1).

For the terms where $l \leqslant N$ using part 2) of Theorem 1 of Section II in § 1, there exists an ε_0; $\varepsilon, \eta \leqslant \varepsilon_0$, such that

$$\Big\| \sum_{\substack{k,l\\ l\leqslant N}} \int_{\varepsilon \leqslant \varrho(y) \leqslant \eta} \frac{Y_{k,l}(\bar{y})}{J(\varphi)\,\varrho^{\Sigma a_j}} \overline{a_{k,l}}(x-y) f(x-y)\, dy \Big\|_p \leqslant \frac{\delta}{2}$$

which proves $\|\tilde{f}_\varepsilon^* - \tilde{f}_\eta^*\|_p \leqslant \delta$ for $\varepsilon, \eta \leqslant \varepsilon_0$, and therefore $\{\tilde{f}_\varepsilon^*\}$ is a Cauchy sequence.

Theorem 1 and Theorem 2 allow us to extend the operations $\tilde{f}_\varepsilon, \tilde{f}, \tilde{f}_\varepsilon^*, \tilde{f}^*$ to the whole of $L^p(E^n)$. These operations we will call

$$\tilde{f}_\varepsilon = K_\varepsilon(f) = \int_{E^n} K_\varepsilon(x, x-y) f(y)\, dy,$$

$$\tilde{f}_\varepsilon^* = K_\varepsilon^*(f) = \int_{E^n} \overline{K}_\varepsilon(y, x-y) f(y)\, dy,$$

$$\tilde{f} = K(f) = \text{P.V.} \int K(x, x-y) f(y)\, dy = \lim_{\varepsilon\to 0} K_\varepsilon f \quad \text{in } L^p,$$

$$\tilde{f}^* = K^*(f) = \text{P.V.} \int \overline{K}(y, x-y) f(y)\, dy = \lim_{\varepsilon\to 0} K_\varepsilon^* f \quad \text{in } L^p.$$

Observe that K_ε^* and K^* are the conjugate operations of K_ε and K.

§ 3. APPENDIX

Similar singular integrals to those presented here have been studied when $a_1 = a_2 = \ldots = a_n = 1$ by Mihlin [6], and by Calderon and Zygmund [1], [2], [3] who first developed the theory of variable kernels for indices a_i of the preceding form [2].

Lately, Jones [5] studied a similar problem where $a_1 = a_2 = \ldots = a_n = 1$, $a_{n+1} = a$, but using a different truncation. The conditions of Jones are:

Let $x \in E_{n-1}$, $t \in E_1 = \{\text{positive real numbers}\}$.

(a) $K(\lambda x, \lambda^a t) = \lambda^{-n-a} K(x, t)$, $\lambda > 0$, $a \geqslant 1$.

Call $\Omega(x) = K(x, 1)$, $K(x, t) = t^{-n/a-1} K(t^{-1/a}x, 1) = t^{-n/a-1}\Omega(t^{-1/a}x)$.

(3.1) (b) $\int_{E_n} (1+|x|)|\Omega(x)|\, dx \leqslant C$,

(3.2) $\int_{E_n} \Omega(x)\, dx = 0$;

(3.3) (c) $\int_{E_n} |\Omega(x-y) - \Omega(x)|\, dx \leqslant C|y|$,

(3.4) $\int_{E_n} |\Omega((1+\delta)x) - \Omega(x)|\, dx \leqslant C\delta$ for $\delta \leqslant 1$;

(d) $\int_{|x|>a} |\Omega(x)|\, dx \leqslant Ca^{-n}$.

Under these conditions the truncation used in [5] is

$$K_\varepsilon'(x, t) = \begin{cases} K(x, t) & \text{for } t \geqslant \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

We will show that conditions (a), (b), (c) imply conditions (1), (2) of Section II, § 1 and (1.1). Moreover, if K_ε is the truncation defined in Section II, § 1, $K_\varepsilon - K_\varepsilon' \in L^1(E^{n+1})$ and

(3.5) $\|K_\varepsilon - K_\varepsilon'\|_1 \leqslant C$, C independent of ε.

This result will immediately imply the continuity of the operator K_ε'. Observe that (3.1) implies for $a > 0$ that

(3.6) $$\int_a^{ka} dt \int_{E_n} |K(x, t)|\, dx = \int_a^{ka} \frac{1}{t}\Big\{\int_{E_n} |\Omega(x)|\, dx\Big\} dt \leqslant C \ln k$$

and

$$(3.7)\qquad \int_0^{a^\alpha} dt \int_{|x|\geqslant a} |K(x,t)|\,dx = \int_0^{a^\alpha} \frac{1}{t}\Big\{\int_{|x|\geqslant a/t^{1/\alpha}} |\Omega(x)|\,dx\Big\}\,dt$$

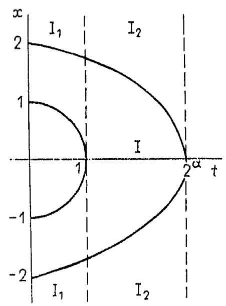
$$\leqslant \int_0^{a^\alpha} \frac{t^{1/\alpha}}{a}\,\frac{1}{t}\Big\{\int_{E_n} |x|\,|\Omega(x)|\,dx\Big\} \leqslant C.$$

With these two observations and condition (a) we will prove now (2) of II, § 1 and (3.5).

For that

$$\int_{1\leqslant\varrho(x,t)\leqslant 2} K(x,t)\,dx\,dt = \int_I K(x,t)\,dxdt + \int_{I_1} K(x,t)\,dxdt - \int_{I_2} K(x,t)\,dxdt,$$

where $I = \{(x,t);\ 1\leqslant t\leqslant 2\}$, $I_k = \{(x,t);\ 0\leqslant t\leqslant k,\ \varrho(x,t)\geqslant k\}$.



The existence of the three preceding integrals is given by (3.6) and (3.7). This will in particular show that

$$\int_{1\leqslant\varrho(x,t)\leqslant 2} |K(x,t)|\,dxdt = \ln 2 \int_{\Sigma_n} |K(x,t)|J(\varphi)\,d\sigma < \infty.$$

Since

$$\int_{I_1} K(x,t)\,dxdt = \int_{I_2} K(x,t)\,dxdt$$

by a simple use of the homogeneity of $K(x,t)$; and

$$\int_I K(x,t)\,dx\,dt = \int_1^2 \frac{1}{t}\Big\{\int_{E_n} \Omega(x)\,dx\Big\}\,dt = 0,$$

$$\int_{1\leqslant\varrho(x,t)\leqslant 2} K(x,t)\,dxdt = \ln 2 \int_{\Sigma_n} K(x,t)J(\varphi)\,d\sigma = 0.$$

To prove (3.5),

$$\int_{E_{n+1}} |K_\varepsilon(x,t) - K'_\varepsilon(x,t)|\,dxdt = \int_{I_\varepsilon} |K(x,t)|\,dxdt = \int_{I_1} |K(x,t)|\,dxdt \leqslant C.$$

Finally using conditions (a) and (b) we will prove (1.1).

Call $A_1 = \{(x,t);\ |t|\geqslant 2^\alpha\varrho^\alpha(y,s)\}$, $A_2 = \{(x,t);\ |t|\leqslant 2^\alpha\varrho^\alpha(y,s),\ |x|\geqslant 2\varrho(y,s)\}$. Then

$$S_{(y,s)} = \{(x,t);\ \varrho(x,t)\geqslant 4\varrho(y,s)\} \subset A_1\cup A_2,$$

$$\int_{S_{(y,s)}} |K(x-y,t-s) - K(x,t)|\,dxdt$$

$$\leqslant \int_{A_1} |K(x-y,t-s)-K(x,t)|\,dxdt + \int_{A_2} |K(x-y,t-s)-K(x,t)|\,dxdt.$$

Now

$$\int_{A_2} |K(x-y,t-s)-K(x,t)|\,dxdt \leqslant \int_{A_2} |K(x-y,t-s)|\,dxdt + \int_{A_2} |K(x,t)|\,dxdt,$$

$$\int_{A_2} |K(x,t)|\,dxdt \leqslant C$$

using (3.7).

$$\int_{A_2} |K(x-y,t-s)|\,dxdt = \int_{(y,s)+A_2} |K(x,t)|\,dxdt \leqslant BC$$

using (3.6), (3.7) and

$$(y,s)+A_2 \subset \{(x,t),\ |t|\leqslant\varrho^\alpha(y,s);\ |x|\geqslant\varrho(y,s)\}$$
$$\subset \{(x,t),\ \varrho^\alpha(y,s)\leqslant|t|\leqslant 3^\alpha\varrho^\alpha(y,s)\}.$$

Also

$$\int_{A_1} |K(x-y,t-s)-K(x,t)|\,dxdt = \int_{A_1} \left|\frac{\Omega\big((t-s)^{1/\alpha}(x-y)\big)}{(t-s)^{n/\alpha+1}} - \frac{\Omega(t^{-1/\alpha}x)}{t^{n/\alpha+1}}\right| dxdt$$

$$\leqslant \int_{2^\alpha\varrho^\alpha(y,s)}^{\infty} \frac{1}{(t-s)^{n/\alpha+1}}\Big\{\int_{E_n} \big|\Omega\big((t-s)^{-1/\alpha}(x-y)\big) - \Omega\big((t-s)^{-1/\alpha}x\big)\big|\,dx\Big\}\,dt +$$

$$+ \int_{2^\alpha\varrho(y,s)}^{\infty} \frac{1}{(t-s)^{n/\alpha+1}}\Big\{\int_{E_n} \big|\Omega\big((t-s)^{-1/\alpha}x\big) - \Omega(t^{-1/\alpha}x)\big|\,dx\Big\}\,dt +$$

$$+ \int_{2^\alpha\varrho(y,s)}^{\infty} \left|\frac{1}{(t-s)^{n/\alpha+1}} - \frac{1}{t^{n/\alpha+1}}\right| \Big\{\int_{E_n} |\Omega(t^{-1/\alpha}x)|\,dx\Big\}\,dt = P+Q+R.$$

$$P = \int_{2^a\varrho^a(y,s)}^{\infty} \frac{1}{(t-s)} \Big\{ \int_{E_n} |\Omega(x-(t-s)^{-1/a}y) - \Omega(x)|\,dx \Big\}\,dt$$

$$\leqslant C|y| \int_{2^a\varrho^a(y,s)}^{\infty} \frac{1}{(t-s)^{1+1/a}}\,dt \leqslant C$$

using (3.3).

A similar inequality is obtained for Q using (3.4).

$$R = \int_{2^a\varrho^a(y,s)}^{\infty} t^{n/a} \left| \frac{1}{(t-s)^{n/a+1}} - \frac{1}{t^{n/a+1}} \right| \Big\{ \int_{E_n} |\Omega(x)|\,dx \Big\}\,dt$$

$$\leqslant C \int_{2^a\varrho^a(y,s)}^{\infty} t^{n/a} \frac{|s|}{|t+s|^{n/a+2}}\,dt \leqslant \left(\frac{1}{2}\right)^{n/a+2} C|s| \int_{2^a\varrho^a(y,s)}^{\infty} \frac{1}{t^2}\,dt \leqslant BC,$$

and the condition (1.1) is finally proved.

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On the convergence structure of Mikusiński operators

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1. Introduction. Let $\mathscr{C}$ denote the complex algebra, the elements of which are continuous complex-valued functions of a non-negative real variable with the operation of multiplication defined by finite convolution; the operations of addition and scalar multiplication defined in the usual manner. $\mathscr{C}$ has no zero divisors, hence the quotient field may be constructed. This field, which we will denote by $\mathscr{M}$, is called the *field of operators*, and is the foundation of the operational calculus developed by Mikusiński [4].

Mikusiński [4] (Part Two, Chapter I, p. 144) states a definition of convergence of sequences of operators. Urbanik [6] has shown that there is no topology satisfying the first axiom of countability in which convergence of sequences is convergence in the sense defined by Mikusiński. The definition of convergence as given by Mikusiński is generalized to nets and filters and is referred to as *M-convergence*. We show that M-convergence defines a Limitierung, τ_M, on the field of operators which is the direct limit of Limitierungen on subspaces of $\mathscr{M}$. It is shown that the Limitierung, τ_M, is not topological. Thus there is no topology on $\mathscr{M}$ for which convergence of nets and filters is precisely M-convergence.

Some properties of the limit space $(\mathscr{M}, \tau_M)$ are investigated and the notion of a linear limit space is defined. The topology defined by Norris in [5] is shown to be the direct limit of Limitierungen on certain subspaces of the field of operators

2. Preliminaries. If the complex algebra $\mathscr{C}$ is provided with the topology of compact convergence it is a routine matter to verify that $\mathscr{C}$ is a topological complex algebra. The collection

$$\mathfrak{B}(f) = \{B(a, \varepsilon, f): a \geqslant 0, \varepsilon > 0\},$$

where

$$B(a, \varepsilon, f) = \{g \in \mathscr{C}: \max_{0\leqslant t\leqslant a} |f(t)-g(t)| < \varepsilon\},$$

is a fundamental system of neighborhoods of the element $f \in \mathscr{C}$ with respect to the topology of compact convergence.