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Extension of the rank function

by

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1. Statement of theorem to be proved

1.1. This note is devoted chiefly to proving the following theorem (the terminology is explained in § 2 below):

THEOREM 1. *Suppose that* $\mathcal{R}$ *is a regular associative ring with rank function* R *and* n *is any integer* $\geqslant 1$. *Then*:

(1.1) *There exists a unique rank function* R_n *on* $\mathcal{R}_n$ *with the property*: $R_n(E(e)) = nR(e)$ *whenever* e *is an idempotent in* $\mathcal{R}$ *and* $E(e)$ *is the matrix in* $\mathcal{R}_n$ *which has* e *for all diagonal entries and* 0 *for all other entries.*

(1.2) $R_n(A) = R_n(B)$ *whenever* $A \in \mathcal{R}_n$ *and* B *is obtained from* A *by interchanging two columns (rows), or by adding to any column (row) any right linear (left linear) combination of the other columns (rows).*

(1.3) $\mathcal{R}_n$ *is complete with respect to the metric of* R_n *if* $\mathcal{R}$ *is complete with respect to the metric of* R.

(1.4) *There exists a unique dimension function* D_n *on the lattice of all finitely generated right submodules of* $\mathcal{R}^n$ *with the property*: $D_n(x\mathcal{R}) = R(e)$ *whenever* x (*in* $\mathcal{R}^n$) *has idempotent* e *for one component and* 0 *for all other components.*

(1.5) *If* $A \in \mathcal{R}_n$ *and the columns of* A *are denoted by* $A_1, \ldots, A_n$, *then* $R_n(A) = D_n(A_1\mathcal{R}+\ldots+A_n\mathcal{R})$; *if* $A_1\mathcal{R}, \ldots, A_n\mathcal{R}$ *are independent right submodules, then* $R_n(A) = \sum_{i=1}^{n} D_n(A_i\mathcal{R})$.

2. Introduction

2.1. If $\mathcal{R}$ is a ring (not required to possess a unit) we write $\mathcal{R}_n$ to denote the ring of all $n\times n$ matrices with entries in $\mathcal{R}$ and we write $\mathcal{R}^n$ to denote the right $\mathcal{R}$-module of all vectors $x = (x^i)_{1\leqslant i\leqslant n}$ with components x^i in $\mathcal{R}$. A vector $x \in \mathcal{R}^n$ is said to be *controlled* at the i-th place by the idempotent e if $xe = x$, $x^i = e$, and $x^j = 0$ for $j > i$. If $A \in \mathcal{R}_n$, then A^t denotes the transpose of A.

2.2. An associative ring $\mathcal{R}$ is called *regular* if for each a in $\mathcal{R}$, $a\beta a = a$ for some β in $\mathcal{R}$. The following is a slight generalization of a theorem of J. von Neumann (see [4], Theorem 2.13; [1], § 3.4):

If $\mathcal{R}$ is regular then:

(2.1) $\mathcal{R}_n$ is regular.

(2.2) The principal right ideals of $\mathcal{R}_n$ form a sublattice, denoted by $P_r(\mathcal{R}_n)$, of the complete lattice of all right ideals of $\mathcal{R}_n$ ordered by inclusion; $P_r(\mathcal{R}_n)$ is a relatively complemented modular lattice with least element $0 = (0)$.

(2.3) The finitely generated right submodules of $\mathcal{R}^n$ form a sublattice, denoted by $F_r(\mathcal{R}^n)$, of the complete lattice of all right submodules of $\mathcal{R}^n$ ordered by inclusion; $F_r(\mathcal{R}^n)$ is a relatively complemented modular lattice with least element $0 = (0)$.

(2.4) If ϱ is the mapping defined for each principal right ideal I of $\mathcal{R}_n$ by the rule

$$\varrho(I) = \text{set of columns of elements in } I,$$

then ϱ is an order isomorphism of $P_r(\mathcal{R}_n)$ onto $F_r(\mathcal{R}^n)$.

(2.5) Suppose that $M \in F_r(\mathcal{R}^n)$. If $M = y_1\mathcal{R}+\ldots+y_n\mathcal{R}$ for some $y_1, \ldots, y_n$ in $\mathcal{R}^n$ such that for each i, y_i is controlled at the i-th place by some idempotent e_i in $\mathcal{R}$, then each $e_i\mathcal{R}$ is determined uniquely by M; indeed,

$$e_i\mathcal{R} = \{x^i \mid x \in M \text{ and } x^j = 0 \text{ for } j > i\}.$$

(2.6) If $M \in F_r(\mathcal{R}^n)$, then there exist $y_1, \ldots, y_n$ in $\mathcal{R}^n$ such that $M = y_1\mathcal{R}+\ldots+y_n\mathcal{R}$ and for each i, y_i is controlled at the i-th place by some idempotent e^i and, for all $j < i$, $e^j(y_i)^j = 0$.

2.3. Suppose that $\mathcal{R}$ is a regular ring. A real valued function R is called a *rank function* on $\mathcal{R}$ if it is defined for all a in $\mathcal{R}$ and

(2.7) $0 < R(a) < \infty$ for all $a \neq 0$.

(2.8) $R(a\beta) \leqslant R(a)$, $R(a\beta) \leqslant R(\beta)$ for all a, β in $\mathcal{R}$.

(2.9) $R(e+f) = R(e)+R(f)$ whenever e, f are idempotents in $\mathcal{R}$ which are orthogonal (a, β are called *orthogonal* if $a\beta = 0 = \beta a$).

It follows that $R(0) = 0, R(a) = R(\beta)$ whenever $a\mathcal{R} = \beta\mathcal{R}$ or $\mathcal{R}a = \mathcal{R}\beta$ (in particular $R(a) = R(-a)$),

$$R(e_1+\ldots+e_m) = \sum_{i=1}^{m} R(e_i)$$

whenever $e_1, \ldots, e_m$ are pairwise orthogonal idempotents, and the function $d(a, \beta) = R(a-\beta)$ is a metric on $\mathcal{R}$, to be called the *rank metric* (see [2], Lemma 3.2).

2.4. Suppose that L is a relatively complemented modular lattice with least element 0 so that von Neumann's theory of independence is valid for finite subsets of L (see [4], Chapter II, [1], § 2.3). A real valued function D is called a *dimension function* on L if it is defined for all a in L and

(2.10) $0 < D(a) < \infty$ for all $a \neq 0$.

(2.11) $D(a \cup b) = D(a)+D(b)$ whenever $a, b \in L$ with $a \cap b = 0$.

Since L is assumed to be relatively complemented and modular, the relations (2.10) and (2.11) imply:

$$D(0) = 0, \quad D(a \cup b)+D(a \cap b) = D(a)+D(b)$$

for all a, b in L, and

(2.12) $D(a_1 \cup \ldots \cup a_m) = \sum_{i=1}^{m} D(a_i)$ whenever $a_1, \ldots, a_m$ are independent.

3. Preliminary discussion

3.1. It is easily verified that (1.2) holds for *any* rank function on $\mathcal{R}_n$. In fact the conditions on A, B ensure that $A\mathcal{R}_n = B\mathcal{R}_n (\mathcal{R}_nA = \mathcal{R}_nB)$ (note that any finite subset $a_1, \ldots, a_m$ of $\mathcal{R}$ have a common right (left) unit: indeed, $\mathcal{R}a_1+\ldots+\mathcal{R}a_m$ is equal to $\mathcal{R}e (a_1\mathcal{R}+\ldots+a_m\mathcal{R} = f\mathcal{R})$ for some suitable idempotent $e(f)$ and hence $a_ie = a_i$ $(fa_i = a_i)$ for all i); hence A and B must have equal rank.

3.2. If $\mathcal{R}$ is *commutative* and R' is a rank function on $\mathcal{R}_n$, then the rule $R''(A) = R'(A^t)$ determines a rank function on $\mathcal{R}_n$. Thus, if $\mathcal{R}$ is commutative and a unique R_n exists as required in (1.1), then $R_n(A^t) = R_n(A)$; this equality may fail if $\mathcal{R}$ is not commutative.

3.3. We shall show now that every rank function R' on $\mathcal{R}_n$ is determined completely by the values $R'(E(e))$ where $E(e)$ are the special matrices which were defined in (1.1).

If A is in $\mathcal{R}_n$ with columns $x_1, \ldots, x_n$, then for suitable y_i with properties as described in (2.6) we have $x_1\mathcal{R}+\ldots+x_n\mathcal{R} = y_1\mathcal{R}+\ldots+y_n\mathcal{R}$. Let E_i be the matrix with y_i as i-th column and all other columns 0. Easy calculations show that each E_i is idempotent in $\mathcal{R}_n$, $E_iE_j = 0$ if $i \neq j$, and $(E_1+\ldots+E_n)\mathcal{R}_n = A\mathcal{R}_n$. Hence,

$$R'(A) = \sum_{i=1}^{n} R'(E_i).$$

Thus the values of the $R'(E_i)$ determine $R'(A)$.

Next, for any idempotent $e \in \mathcal{R}$, let $E_i(e)$ be the matrix which has e for (i, i)-th entry and 0 for all other entries. Since y_i is controlled at the i-th place by the idempotent y_i^i, and $y_iy_i^i = y_i$, it follows from 3.1 that

$R'(E_i) = R'\big(E_i(y_i^i)\big)$. Thus R' is completely determined by the values of $R'\big(E_i(e)\big)$, where e varies over all idempotents in $\mathcal{R}$ and $i = 1, \ldots, n$.

Next, from 3.1 it follows that $R'\big(E_i(e)\big) = R'\big(E_j(e)\big)$ for all i, j. Since each $E_i(e)$ is idempotent in $\mathcal{R}_n$ and $E_i(e)E_j(e) = 0$ for $i \neq j$, it follows that $R'\big(E_i(e)\big) = R'\big(E(e)\big)/n$. Thus R' is completely determined by the values of $R'\big(E(e)\big)$.

This implies that if the rank function R_n in (1.1) exists at all it is unique and

$$R_n(A) = \sum_{i=1}^{n} R_n(E_i) = \sum_{i=1}^{n} R_n\big(E_i(y_i^i)\big) = \sum_{i=1}^{n} R(y_i^i).$$

3.4. It is now easy to verify that if the dimension function D_n in (1.4) exists at all it is determined uniquely by R.

Indeed, if $M \in F_r(\mathcal{R}^n)$, then for suitable y_i as described in (2.6) we have $M = y_1\mathcal{R} + \ldots + y_n\mathcal{R}$. Since the right modules $y_1\mathcal{R}, y_2\mathcal{R}, \ldots, y_n\mathcal{R}$ are independent, it follows that

$$D_n(M) = \sum_{i=1}^{n} D_n(y_i\mathcal{R}).$$

Next, let e be a common left unit for all y_i^j $(i, j = 1, \ldots, n)$, let z_i denote the vector with e as i-th component and 0 for all other components, and let w_i denote the vector with $e - y_i^i e$ as i-th component and all other components 0. Then

$$z_1\mathcal{R} + \ldots + z_{i-1}\mathcal{R} + w_i\mathcal{R} + y_i\mathcal{R} = z_1\mathcal{R} + \ldots + z_i\mathcal{R}$$

where each side is the sum of independent right modules. It follows that $D_n(y_i\mathcal{R}) = R(e) - R(e - y_i^i e) = R(y_i^i)$, and hence

$$D_n(M) = \sum_{i=1}^{n} R(y_i^i).$$

Thus $D_n(M)$ is determined uniquely by R.

3.5. From now on we shall take R_n and D_n to be defined as follows:

(3.1) If $M \in F_r(\mathcal{R}_n)$, then M has a representation $\sum y_i\mathcal{R}$ such that each y_i controlled at the i-th place; $D_n(M)$ is defined to be $\sum_{i=1}^{n} R(y_i^i)$.

(3.2) If $A \in \mathcal{R}_n$, then $A\mathcal{R}_n = E\mathcal{R}_n$ where E has columns $y_1, \ldots, y_n$ such that each y_i is controlled at the i-th place; $R_n(A)$ is defined to be $\sum_{i=1}^{n} R(y_i^i)$.

We note that (3.1) and (3.2) give unique values for $D_n(M)$ and, $R_n(A)$, since in (3.1), (3.2) the $y_i^i\mathcal{R}$ are uniquely determined by M and A respectively. We note also that if the dimension and rank functions required in (1.4) and (1.1) respectively exist as required, they are determined by (3.1) and (3.2), as follows from 3.3 and 3.4.

Our problem has now been reduced to proving that D_n defined in (3.1) is a dimension function on $F_r(\mathcal{R}_n)$ and R_n defined in (3.2) is a rank function on $\mathcal{R}_n$, and that (1.3) holds.

We shall prove Theorem 1 by induction on n. Thus we shall suppose that m is an integer $\geqslant 1$ and that Theorem 1 holds for all $n < m$, and we need only establish that Theorem 1 holds for the case $n = m$.

3.6. Suppose that $M_1 = y_{11}\mathcal{R} + \ldots + y_{1m}\mathcal{R}$ and $M_2 = y_{21}\mathcal{R} + \ldots + y_{2m}\mathcal{R}$ with each y_{1i} and y_{2i} controlled at the i-th place. If $M_1 \subset M_2$ it follows from (2.5) that for each i, $(y_{1i})^i\mathcal{R} \subset (y_{2i})^i\mathcal{R}$. This shows that $M_1 \subset M_2 \to D_m(M_1) \leqslant D_m(M_2)$.

Next, if $A, B \in \mathcal{R}_m$, then the columns of AB are right linear combinations of the columns of A. The preceding paragraph now implies that $R_m(AB) \leqslant R_m(A)$.

Thus to prove Theorem 1, we need only verify that $R_m(AB) \leqslant R_m(B)$, that (2.9) holds for R_m on $\mathcal{R}_m$, that (2.11) holds for D_m on $F_r(\mathcal{R}^m)$, and prove the completeness theorem (1.3) for $\mathcal{R}_m$.

3.7. The isomorphism ϱ of (2.4) clearly has the property that $R_m(I) = D_m(M)$ whenever $\varrho(I) = M$. Suppose that (2.10) does hold in $F_r(\mathcal{R}^m)$ and that E_1 and E_2 are orthogonal idempotents in $\mathcal{R}_m$. Then $E_1\mathcal{R}_m \frown E_2\mathcal{R}_m = 0$ and $E_1\mathcal{R}_m + E_2\mathcal{R}_m = (E_1 + E_2)\mathcal{R}_m$. Hence we have $\varrho(E_1\mathcal{R}_m) \frown \varrho(E_2\mathcal{R}_m) = 0$ and

$$\begin{aligned} R_m(E_1 + E_2) &= D_m\Big(\varrho\big((E_1 + E_2)\mathcal{R}_m\big)\Big) \\ &= D_m\big(\varrho(E_1\mathcal{R}_m) + \varrho(E_2\mathcal{R}_m)\big) = D_m\big(\varrho(E_1\mathcal{R}_m)\big) + D_m\big(\varrho(E_2\mathcal{R}_m)\big) \\ &= R_m(E_1) + R_m(E_2). \end{aligned}$$

Thus the validity of (2.11) for D_m in $F_r(\mathcal{R}^m)$ will imply that of (2.8) for R_m in $\mathcal{R}_m$.

Moreover, since we assume that Theorem 1 holds for all $n < m$ and since each M in $F_r(\mathcal{R}^m)$ has a representation as described in (2.6), the validity of (2.11) for D_m in $F_r(\mathcal{R}^m)$ will clearly follow from the following lemma:

LEMMA. *Suppose that $M = y_1\mathcal{R} + \ldots + y_m\mathcal{R}$ with each y_i controlled in the i-th place. Suppose that x is a vector in $\mathcal{R}^m$ controlled by some idempotent f in the m-th place. Suppose that $x\mathcal{R} \frown M = 0$. Then $M + x\mathcal{R}$ has a representation $y_1'\mathcal{R} + \ldots + y_m'\mathcal{R}$ with each y_i' controlled at the i-th place and*

$$\sum_{i=1}^{m} R(y_i^i) + R(f) = \sum_{i=1}^{m} R(y_i'^i).$$

Thus we need to prove: the above Lemma, $R_m(AB) \leqslant R_m(B)$ and proposition (1.3).

4. A useful dimension relation

We shall now prove that the function D_n defined in (3.1) satisfies the relation.

(4.1) For each $x = (x^1, \ldots, x^n)$ in $\mathscr{R}^n$, $D_n(x\mathscr{R}) = R(g)$ if $\mathscr{R}g = \sum_{i=1}^{n} \mathscr{R}x^i$.

For the case $n = 1$, relation (4.1) follows directly from the definition of D_n. We now proceed by induction on n.

Let β be chosen in $\mathscr{R}$ so that $x^n\beta x^n = x^n$, and set $y = x - x\beta x^n$. Then $x\mathscr{R} = y\mathscr{R} + x\beta x^n\mathscr{R} = y\mathscr{R} + x\beta x^n\beta\mathscr{R}$ and $y^n = 0$, $(x\beta x^n\beta)^n = x^n\beta$. It follows from the definition of D_n and the inductive assumption that

$$D_n(x\mathscr{R}) = D_n(y\mathscr{R}) + R(x^n\beta) = R(h) + R(x^n)$$

if h is chosen to be an idempotent in $\mathscr{R}$ such that

$$\mathscr{R}h = \sum_{i=1}^{n-1} \mathscr{R}y^i = \sum_{i=1}^{n-1} \mathscr{R}(x^i - x^i\beta x^n)$$

and we need only prove that $R(h) + R(x^n) = R(g)$.

Clearly $\mathscr{R}h + \mathscr{R}x^n = \mathscr{R}g$ so it is sufficient to prove that $\mathscr{R}h \frown \mathscr{R}x^n = (0)$. Suppose now that $ah = \gamma x^n$. Then for suitable δ_i,

$$ah = ah\beta x^n = a\Big(\sum_{i=1}^{n-1} \delta_i y^i\Big)\beta x^n = a\Big(\sum_{i=1}^{n-1} \delta_i(x^i - x^i\beta x^n)\Big)\beta x^n = 0.$$

This shows that $\mathscr{R}h \frown \mathscr{R}x^n = (0)$ and hence that

$$D_n(x\mathscr{R}) = Rg \quad \text{if} \quad \mathscr{R}g = \sum_{i=1}^{n} \mathscr{R}x^i.$$

5. Proof of the Lemma of 3.7

5.1. We consider the Lemma first for the special case $f\mathscr{R} \frown y_m^m\mathscr{R} = (0)$. For this case we let g be an idempotent such that $g\mathscr{R} = f\mathscr{R} + y_m^m\mathscr{R}$. By a decomposition theorem of von Neumann there exist orthogonal idempotents h, k such that $\mathscr{R}f = \mathscr{R}h$, $\mathscr{R}y_m^m = \mathscr{R}k$, and $g = h + k$ (see [4], Lemma 3.2; [2], (2.12)). Thus, without changing $x\mathscr{R}$ or $y_m\mathscr{R}$ we can replace x by xh, f by h, y_m by $y_m k$ and y_m^m by k. After these replacements have been made the Lemma will be satisfied by the choice $y_i' = y_i$ for $i < m$ and $y_m' = x + y_m$, since with the new x and y_m:

$$(x + y_m)\mathscr{R} = x\mathscr{R} + y_m\mathscr{R},$$

$$x + y_m \text{ is controlled at the } m\text{-th place by } f + y_m^m,$$

and

$$R(y_m^m) + R(f) = R(f + y_m^m).$$

This establishes the Lemma for the case $\mathscr{R}f \frown \mathscr{R}y_m^m = 0$.

5.2. Next we consider the Lemma for the special case that $f\mathscr{R} \subset y_m^m\mathscr{R}$. For this case we have: $x\mathscr{R} + y_m\mathscr{R} = x'\mathscr{R} + y_m\mathscr{R}$ where $x' = x - y_m f$.

Now $x'\mathscr{R}, y_1\mathscr{R}, \ldots, y_{m-1}\mathscr{R}$ are independent right submodules of $\mathscr{R}^{m-1}$ if the m-th components (which are all 0) are ignored. Thus, to prove the Lemma for the present case it is sufficient to show that $D_{m-1}(x'\mathscr{R}) = R(f)$, or equivalently (by (4.1)) that

$$\sum_{i=1}^{m-1} \mathscr{R}(x')^i = \mathscr{R}f.$$

We have

$$\sum_{i=1}^{m-1} \mathscr{R}(x')^i = \sum_{i=1}^{m-1} \mathscr{R}(x^i - y_m^i f) \subset \mathscr{R}f.$$

Hence by von Neumann's decomposition theorem ([4], Lemma 3.2; [2], (2.12)) there exists an idempotent g such that

$$\mathscr{R}g = \sum_{i=1}^{m-1} \mathscr{R}(x')^i \quad \text{and} \quad gf = fg = g.$$

Then $(x')^i(f-g) = 0$ for $i = 1, \ldots, m-1$; hence $x'(f-g) = 0$ and $x(f-g) = y_m f(f-g)$; since $x\mathscr{R} \frown y_m\mathscr{R} = (0)$ it follows that $x(f-g) = 0$. Since $x^m = f$, it follows that $f(f-g) = 0$, hence $f - g = 0$. This proves that

$$\sum_{i=1}^{m-1} \mathscr{R}(x')^i = \mathscr{R}f$$

and completes the proof of the Lemma for the special case that $f\mathscr{R} \subset y_m^m\mathscr{R}$.

5.3. Now we consider the Lemma for the general case. We use von Neumann's decomposition theorem (already used in 5.2) to obtain an idempotent g such that $\mathscr{R}g = \mathscr{R}f \frown \mathscr{R}y_m^m$ and $fg = gf = g$.

By 5.2, the Lemma holds for M and $(xg)\mathscr{R}$ (in place of M and $x\mathscr{R}$). Let $M' = M + (xg)\mathscr{R}$, $x' = x(f-g)$. Then M', $x'\mathscr{R}$ are independent, x' is controlled at the m-th place by $f - g$ and

$$M' = z_1\mathscr{R} + \ldots + z_m\mathscr{R}_m$$

with $z_m = y_m$ and each z_i controlled at the i-th place. Hence $\mathscr{R}(f-g) \frown \mathscr{R}(z_m^m) = 0$. Now 5.1 applies and completes the proof of the Lemma.

6. Proof that $R_m(AB) \leqslant R_m(B)$

6.1. Choose X in $\mathcal{R}_m$ so that $BX = E$, say, has columns $E_1, \ldots, E_m$ such that each E_i is controlled at the i-th place and $EB = B$. Then $BXB = EB = B$ and as we have already proved: $R_m(AB) \geqslant R_m(ABX) \geqslant R_m(ABXB)$, $R_m(B) \geqslant R_m(BX) \geqslant R_m(BXB)$. It is therefore sufficient to prove that $R_m(AE) \leqslant R_m(E)$.

6.2. We now have:

$$R_m(AE) = D_m\Big(\sum_{i=1}^{m}(AE)_i\mathcal{R}\Big) \leqslant \sum_{i=1}^{m} D_m\big((AE)_i\mathcal{R}\big),$$

$$R_m(E) = \sum_{i=1}^{m} D_m(E_i\mathcal{R}).$$

Thus it is sufficient to prove that for each i

$$D_m\big((EA)_i\mathcal{R}\big) \leqslant D_m(E_i\mathcal{R}).$$

We have: $\mathcal{R}(AE)^j_i \subset \mathcal{R}E^i_i$ for all j hence

$$\sum_{j=1}^{m}\mathcal{R}(AE)^j_i \subset \mathcal{R}E^i_i.$$

By (4.1) it follows that $D_m\big((AE)_i\mathcal{R}\big) \leqslant R(E^i_i)$. Since $D_m(E_i\mathcal{R}) = R(E^i_i)$, the proof of the inequality $R_m(AB) \leqslant R_m(B)$ is complete.

7. Proof of completeness theorem (1.3)

7.1. We now suppose that $\mathcal{R}$ is complete with respect to the metric of the rank R and we wish to show that $\mathcal{R}_m$ is complete with respect to the rank R_m. Thus we suppose that $A_1, A_2, \ldots$ is an infinite sequence of elements in $\mathcal{R}_m$ such that $R_m(A_q - A_p) \to 0$ as $p, q \to \infty$ and we wish to show that for some A in $\mathcal{R}_m$, $R_m(A - A_p) \to 0$ as $p \to \infty$.

7.2. Suppose that $B \in \mathcal{R}_m$ and that $B^i_j = a$. We shall show that $R_m(B) \geqslant R(a)$.

Let e be an idempotent with $\mathcal{R}e = \mathcal{R}a$. Then for suitable B', B'' in $\mathcal{R}_m$ we have: $(B'BB'')^i_j = e$ and all other $(B'BB'')^k_s = 0$. Hence

$$R_m(B) \geqslant R_m(B'B) \geqslant R_m(B'BB'') = R(e) = R(a).$$

7.3. Suppose that $B \in \mathcal{R}_m$ with columns $B_1, \ldots, B_m$. We shall show that

$$R_m(B) \leqslant \sum_{i,j=1}^{m} R(B^j_i).$$

We have, using (4.1):

$$R_m(B) = D_m(B_1\mathcal{R} + \ldots + B_m\mathcal{R}) \leqslant \sum_{i=1}^{m} D_m(B_i\mathcal{R}) \leqslant \sum_{i=1}^{m}\sum_{j=1}^{m} R(B^j_i).$$

7.4. Now in 7.1, for fixed i, j, we have because of 7.2: $R\big((A_q)^i_j - (A_p)^i_j\big) \to 0$ as $p, q \to \infty$. Since $\mathcal{R}$ is assumed to be complete, there exists a^i_j in $\mathcal{R}$ such that $R\big(a^i_j - (A_p)^i_j\big) \to 0$ as $p \to 0$. Define A by the relations $A^i_j = a^i_j$; it follows from 7.3 that $R_m(A - A_p) \to 0$ as $p \to \infty$. This proves (1.3) and completes the proof of Theorem 1.

8. Remarks

8.1. If $\mathcal{R}$ is a division ring, then $\mathcal{R}$ is regular and there is a unique (normalized) rank function R^0 on $\mathcal{R}$ with $R^0(1) = 1$; namely $R^0(a) = 0$ if $a = 0$, $R^0(a) = 1$ if $a \neq 0$. Then R^0_n coincides with the classical left row, right column rank on $\mathcal{R}_n$.

8.2. Theorem 1 continues to hold as stated if rank function, dimension function, metric are replaced by semi-rank, semi-dimension, semi-metric respectively; this means that the conditions $R(a) > 0$ for $a \neq 0$, $D(M) > 0$ for $M \neq 0$, $d(a, b) > 0$ for $a \neq b$ are replaced by $R(a) \geqslant 0$, $D(M) \geqslant 0$, $d(a, b) \geqslant 0$ respectively.

8.3. Theorem 1 continues to hold as stated if rank, dimension and metric have values in the positive semi-group G^+ of any totally ordered commutative group G provided that for each $a \in G^+$ and each $n > 1$ then exists a unique $b \in G$ with $a = b + \ldots + b$ (n addends).

8.4. An alternative proof of Theorem 1 can be obtained as follows: prove Theorem 1 first for the case $n = 2$, then by induction for $n = 2^m$ for all $m \geqslant 1$; then by restriction ($\mathcal{R}_m$ can be considered as the set of those $2^m \times 2^m$ matrices which have zero entries outside the upper-left $m \times m$ corner) for m.

9. Inductive limits

9.1. Let I be an ordered directed set (this means that any two elements i, j in I have an upper bound in I). Suppose that $\mathcal{R}_i$ is a ring for each i and that for each i, j with $i \leqslant j$, there is given a ring homomorphism $\varphi_{ji}: \mathcal{R}_i \to \mathcal{R}_j$ such that whenever $i \leqslant j \leqslant k$, we have

$$\varphi_{kj}\varphi_{ji} = \varphi_{ki}.$$

Then we define a relation by the rule: $(a, i) \equiv (\beta, j)$ shall mean that a is in $\mathcal{R}_i$, β is in $\mathcal{R}_j$ and for some γ in some $\mathcal{R}_k$ with $i \leqslant k, j \leqslant k$:

$\varphi_{ki}\alpha = \varphi_{kj}\beta$. The relation $\equiv$ is clearly an equivalence relation on the set $S = \{(\alpha, i) \mid i \in I, \alpha \in \mathscr{R}_i\}$.

The equivalence classes of S form a ring called the *inductive limit* and denoted by $\mathscr{R} = \varinjlim(\mathscr{R}_i, \varphi_{ji}) = \varinjlim \mathscr{R}_i$, with respect to the following operations:

(9.1) If u, v are the equivalence classes of (α, i), (β, j) respectively, then for any k with $i \leqslant k$, $j \leqslant k$ the sum $u+v$ is defined to be the equivalence class of $(\varphi_{ki}\alpha + \varphi_{kj}\beta, k)$ and the product uv is defined to be the equivalence class of $(\varphi_{ki}\alpha\varphi_{kj}\beta, k)$.

It is easily verified that if each $\mathscr{R}_i$ is regular, then $\varinjlim \mathscr{R}_i$ is also regular; if each φ_{ji} is injective, then the mapping $\alpha \to$ (equivalence class of (α, i)) determines an injective ring embedding of $\mathscr{R}_i$ in $\varinjlim \mathscr{R}_i$; if each $\mathscr{R}_i$ is a regular rank ring and each mapping φ_{ji} preserves the rank, then the function

$$R(\text{equivalence class of } (\alpha, i)) = \text{rank of } \alpha \text{ in } \mathscr{R}_i$$

is a rank function on $\varinjlim \mathscr{R}_i$.

9.2. Let N denote set of integers $\{1, 2, 3, \ldots\}$ and write $m|n$ to mean: $m, n \in N$ and $n = mp$ for some $p \in N$.

Suppose that $\mathscr{R}$ is an associative ring and let $\mathscr{R}_n$ denote the matrix ring. For $m, n \in N$ with $m|n$ we define an injective ring isomorphism $\varphi_{n,m}: \mathscr{R}_m \to \mathscr{R}_n$ as follows: if $A \in \mathscr{R}_m$, then $\varphi_{n,m}(A)$ shall be the $n \times n$ matrix with A's down the diagonal and zeros elsewhere; more precisely,

$$(\varphi_{n,m}(A))^{rm+i}_{rm+j} = A^i_j \quad \text{for} \quad r = 0, 1, \ldots, \left(\frac{n}{m} - 1\right), 1 \leqslant i, j \leqslant m$$

and

$$\varphi_{n,m}(A) \text{ has all other entries } 0.$$

Now suppose that $I \subset N$ and that any pair m, n in I have a common multiple in I. Then the inductive limit

$$\mathscr{R}_I = \varinjlim(\mathscr{R}_m, \varphi_{n,m})_{n,m \in I}$$

is defined as a special case of 9.1.

9.3. Suppose next that $\mathscr{R}, N, I$ are as in 9.2 and also that $\mathscr{R}$ is a regular ring with normalized rank function R. Theorem 1 now implies that for each $\mathscr{R}_n$ the function R_n/n is a normalized rank on $\mathscr{R}_n$, to be denoted also without fear of ambiguity by R; with this choice of rank, each mapping $\varphi_{n,m}$ preserves the rank. Hence $\mathscr{R}_I$ is again a regular rank ring and its rank will be denoted again by R.

It is easily seen that if I is infinite and $\mathscr{R} \neq (0)$, then $\mathscr{R}_I$ is not complete (even if $\mathscr{R}$ is complete). But by [2], (1.4), (1.5) and (1.6), the completion of $\mathscr{R}_I$ in the rank metric, denoted $\hat{\mathscr{R}}_I$, is again a regular ring with a rank (again denoted by R) which is an extension of that of the rank on $\mathscr{R}_I$; the ring $\hat{\mathscr{R}}_I$ is complete with respect to its rank metric.

The study of the dependence of $\mathscr{R}_I$ and $\hat{\mathscr{R}}_I$ on $\mathscr{R}$ and I was initiated by J. von Neumann [5], [3] for the case that $\mathscr{R}$ is a division ring. We shall continue this study in subsequent notes.

References

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