Proof. If $p \in L^\infty$ is positive definite and P is the corresponding linear functional, then there exists a finite positive Baire measure μ_p on I' ([2], p. 97) such that

$$P(f) = \int_I f(x)p(x)\,dx = \int_{I'} \hat{f}(a)\,d\mu_p(a)$$
$$= \int_{I'}\int_I f(x)\langle x, a\rangle\,dx\,d\mu_p(a) = \int_I f(x)\Big[\int_{I'} \langle x, a\rangle\,d\mu_p(a)\Big]\,dx.$$

Hence

$$p(x) = \int_{I'} \langle x, a\rangle\,d\mu_p(a)$$

almost everywhere.

Conversely, if

$$p(x) = \int_{I'} \langle x, a\rangle\,d\mu_p(a)$$

for some finite positive Baire measure μ_p on I', then $p \in L^\infty$, $p(x) = \overline{p(x)}$ and

$$\int_I (f*f^*)(x)p(x)\,dx = \int_I\int_{I'} (f*f^*)(x)\langle x, a\rangle\,d\mu_p(a)\,dx$$
$$= \int_{I'}\int_I (f*f^*)(x)\langle x, a\rangle\,dx\,d\mu_p(a) = \int_{I'} |\hat{f}(a)|^2\,d\mu_p(a) \geqslant 0.$$

Hence p is positive definite.

Corollary. *A function $p \in L^\infty$ is positive definite if and only if p is positive, monotone non-increasing and left-continuous.*

Proof. The above theorem shows that positive definite functions are positive, monotone decreasing and left-continuous. On the other hand, such a function determines, in the usual way, a finite positive Baire measure such that

$$p(x) = \mu_p[x, b] = \int_{I'} \langle x, a\rangle\,d\mu_p(a).$$

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Commutators of singular integrals

by

E. B. FABES and N. M. RIVIÈRE (Chicago, Ill.)

Introduction. Calderón and Zygmund considered in [2] singular integral operators, K, of type C_β^∞, $\beta > 1$, and proved results involving commutators of singular integral operators and the operator, Λ. It is the purpose of this paper to prove similar results for $K \in C_\beta^\infty$, $0 < \beta \leqslant 1$, and for the operator Λ^a, $a < \beta$, defined so that $\widehat{\Lambda^a f} = |x|^a \hat{f}$, where $\hat{f}$ denotes the Fourier transform of f.

Notation. $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$ will denote points of E^n. $C_0^\infty(E^n)$ denotes the class of functions $f \in C^\infty(E^n)$ with compact support.

"a.e." designates the phrase "almost everywhere with respect to Lebesgue measure".

$$x \circ y = \sum_{i=1}^n x_i y_i; \qquad \Sigma = \{x \in E^n : |x| = 1\}.$$

$$\|f\|_p = \Big(\int_{E^n} |f(x)|^p\,dx\Big)^{1/p}, \quad \hat{f}(x) = \int_{E^n} f(y)\,e^{2\pi i x \circ y}\,dy.$$

$\gamma = (\gamma_1, \ldots, \gamma_n)$ will denote a point in E^n with each γ_i representing a non-negative integer.

Finally,

$$(\partial/\partial x)^\gamma f(x) = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}}, \frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}} f(x).$$

Assume $f(x) \in C^\infty(E^n)$ and that every derivative, $(\partial/\partial x)^\gamma f$, satisfies $(\partial/\partial x)^\gamma f = O(|x|^{-n})$. For such f we define

$$(S_{j,\varepsilon}^a f)(x) = \int_{|y|>\varepsilon} f(x-y)\frac{y_j}{|y|^{n+1+a}}\,dy, \qquad 0 < a < 1.$$

Remark 1. *$\lim_{\varepsilon\to 0} (S_{j,\varepsilon}^a f)(x)$ exists point-wise for every x, and in $L^p(E^n)$, for every p $(1 < p < \infty)$.*

Proof. We have

$$(S_{j,\varepsilon}^{a} f)(x) = \int_{|y|>\varepsilon} f(x-y) \frac{y_j}{|y|^{n+1+a}} dy.$$

Since

$$\int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1+a}} dy = 0$$

and since $|f(x-y)-f(x)| \leqslant C|y|$, it is clear that for each x, $\lim_{\varepsilon\to 0} (S_{j,\varepsilon}^{a} f)(x)$ exists point-wise.

To show L^p-convergence we note that for $\varepsilon < 1$, $(S_{j,\varepsilon}^{a} f)(x)$ is bounded uniformly in ε:

$$(S_{j,1}^{a} f)(x) - (S_{j,\varepsilon}^{a} f)(x) = \int_{\varepsilon<|y|<1} f(x-y) \frac{y_j}{|y|^{n+1+a}} dy$$

$$= \int_{\varepsilon<|y|<1} [f(x-y)-f(x)] \frac{y_j}{|y|^{n+1+a}} dy.$$

Since $\partial f/\partial x_i = O(|x|^{-n})$, for $|x| > 2$

$$(S_{j,1}^{a} f)(x) - (S_{j,\varepsilon}^{a} f)(x) \leqslant C|x|^{-n}.$$

Hence for all x, $(S_{j,1}^{a} f)(x) - (S_{j,\varepsilon}^{a} f)(x) \leqslant C(1+|x|^{-n})$. Since $(1+|x|^{-n}) \in L^p(E^n)$ for every $p, 1 < p < \infty$, remark 1 follows.

Let $B_\beta(E^n)$ = set of all functions $a(x)$ such that $a(x)$ is bounded and $|a(x)-a(y)| \leqslant C|x-y|^\beta$ $(0 < \beta \leqslant 1)$.

Remark 2. *Suppose $f \in C^\infty(E^n)$ and that for every γ, $(\partial/\partial x)^\gamma f(x) = O(|x|^{-n})$. If $a(x) \in B_\beta$, $\beta > a$, then $\lim_{\varepsilon\to 0} (S_{j,\varepsilon}^{a} af)(x)$ exists point-wise everywhere and also in L^p* for *every p* $(1 < p < \infty)$.

Proof. Since $af \in B_\beta \cap L^p(E^n)$, for every p $(1 < p < \infty)$, the point-wise limit is clear.

$$\int_{|y|>\varepsilon} a(x-y) f(x-y) \frac{y_j}{|y|^{n+1+a}} dy$$

$$= \int_{|y|>\varepsilon} [a(x-y)-a(x)] f(x-y) \frac{y_j}{|y|^{n+1+a}} dy + a(x) \int_{y>\varepsilon} f(x-y) \frac{y_j}{|y|^{n+1+a}} dy.$$

Using remark 1 and the fact that $a(x)$ is bounded, we see that the second term converges in $L^p(E^n)$ as $\varepsilon \to 0$ for every p $(1 < p < \infty)$.

Again for $\varepsilon < 1$, the first term is bounded in x, uniformly in $\varepsilon < 1$.

For $|x| > 2$,

$$\left| \int_{\varepsilon<|y|<1} [a(x-y)-a(x)] f(x-y) \frac{y_j}{|y|^{n+1+a}} dy \right| \leqslant \frac{C}{|x|^{n+1}}.$$

Hence the first term converges in $L^p(E^n)$ as $\varepsilon \to 0$.

Define $S_j^a f = \lim_{\varepsilon\to 0} S_{j,\varepsilon}^{a} f$.

Remark 3. *For $f \in C_0^\infty$,*

$$\widehat{S_j^a f} = C_{a,j} \frac{x_j}{|x|} |x|^a \hat{f},$$

$C_{a,j}$ absolute constant.

Proof. Define

$$S_{j,\varepsilon}^{a}(x) = \begin{cases} x_j/|x|^{n+1+a} & \text{if} \quad |x| > \varepsilon, \\ 0 & \text{if} \quad |x| \leqslant \varepsilon, \end{cases}$$

$$\widehat{S_j^a f} = \lim_{\substack{\varepsilon\to 0 \\ \text{in } L^2}} \widehat{S_{j,\varepsilon}^{a} * f} = \lim_{\substack{\varepsilon\to 0 \\ \text{in } L^2}} \widehat{S_{j,\varepsilon}^{a}} \hat{f}.$$

Now,

$$\widehat{S_{j,\varepsilon}^{a}}(x) = \int_{\Sigma} \frac{y_j}{|y|} \left[\int_{\varepsilon}^{\infty} \frac{e^{2\pi i r\varrho(x' \circ y')}}{\varrho^{1+a}} d\varrho \right] d\sigma,$$

$$r = |x|, \qquad \varrho = |y|, \qquad x' = \frac{x}{|x|}, \qquad y' = \frac{y}{|y|}.$$

Assume $x \neq 0$ and set $s = \varrho r$. Hence,

$$\widehat{S_{j,\varepsilon}^{a}}(x) = r^a \int_{\Sigma} \frac{y_j}{|y|} \left[\int_{\varepsilon r}^{\infty} \frac{e^{2\pi i s(x' \circ y')}}{s^{1+a}} ds \right] d\sigma.$$

Since

$$\int_{\Sigma} \frac{y_j}{|y|} d\sigma = 0,$$

we have

$$|\widehat{S_{j,\varepsilon}^{a}}(x)| = r^a \left| \int_{\Sigma} \frac{y_j}{|y|} \left[\int_{\varepsilon r}^{\infty} \frac{e^{2\pi i s(x' \circ y')}-1}{s^{1+a}} ds \right] d\sigma \right| \leqslant C r^a.$$

Hence $|\widehat{S_{j,\varepsilon}^{a}}(x)| \leqslant C|x|^a$, C independent of ε.

It is also clear that for each x, as $\varepsilon \to 0$, $\widehat{S_{j,\varepsilon}^{a}}(x)$ tends pointwise to a litmi, which we denote by $\widehat{S_j^a}(x)$, which is homogeneous of degree a.

We assert that

$$\widehat{S_j^\alpha}(x) = C_{\alpha,j}\frac{x_j}{|x|}|x|^\alpha.$$

Indeed, let $Q(x)$ be a solid harmonic of degree $k>0$ and consider

$$\int_{E^n}\frac{x_j}{|x|^{n+1+\alpha}}\bar{Q}(x)e^{-\pi|x|^2} = \lim_{\varepsilon\to 0}\int_{E^n}\widehat{S_{j,\varepsilon}^\alpha}(x)\overline{\widehat{Q(x)e^{-\pi|x|^2}}}.$$

We now use the fact that

$$\widehat{Q(x)e^{-\pi|x|^2}} = (-i)^k Q(x)e^{-\pi|x|^2} \quad \text{(see [1])}.$$

Hence

$$\int_{E^n}\frac{x_j}{|x|^{n+1+\alpha}}\bar{Q}(x)e^{-\pi|x|^2}dx = C\int_{E^n}\widehat{S_j^\alpha}(x)\bar{Q}(x)e^{-\pi|x|^2}dx$$

$$= C\int_{\Sigma}\widehat{S_j^\alpha}(x')\bar{Q}(x')\,d\sigma.$$

But

$$\int_{E^n}\frac{x_j}{|x|^{n+1+\alpha}}\bar{Q}(x)e^{-\pi|x|^2}dx = C\int_{\Sigma}\frac{x_j}{|x|}\bar{Q}(x')\,d\sigma.$$

Therefore,

$$\int_{\Sigma}\frac{x_j}{|x|}\bar{Q}(x')\,d\sigma = C\int_{\Sigma}\widehat{S_j^\alpha}(x')\bar{Q}(x')\,d\sigma,$$

and from this our assertion follows.

Let R_j denote the j^{th} Riesz transform, i.e.

$$R_j(f)(x) = \lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}}\int_{|x-y|>\varepsilon}\frac{(x_j-y_j)}{|x-y|^{n+1}}f(y)\,dy,$$

Definition. For $f\in C_0^\infty(E^n)$ set

$$\Lambda^\alpha f = \sum_{j=1}^{n}\frac{1}{CC_{\alpha,j}}R_j S_j^\alpha(f),$$

where

$$\widehat{R_j f} = C\frac{x_j}{|x|}\hat{f}.$$

We have, using remark 3, that $\widehat{\Lambda^\alpha f} = |x|^\alpha\hat{f}$.

From now on $K(x,y)$ will denote a function which, for each x, is homogeneous of degree $-n$ in y and satisfies

$$\int_{\Sigma}K(x,y)\,d\sigma_y = 0:$$

Set

$$Kf(x) = \lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}}\int_{|x-y|>\varepsilon}K(x,x-y)f(y)\,dy.$$

In the next result, K is independent of the first variable.

LEMMA. *Let $f\in C_0^\infty(E^n)$, $a(x)\in B_\beta(E^n)$, $K(x)\in C^1(E^n-(0))$. Define*

$$H(f) = \int[a(x)-a(y)]K(x-y)f(y)\,dy.$$

Then for $0<\alpha<\beta\leqslant 1$,

$$\|H(\Lambda^\alpha f)\|_p \leqslant C\|f\|_p.$$

To prove the lemma we need the following

REMARK. *For $f\in C_0^\infty(E^n)$, $R_j(S_j^\alpha f) = S_j^\alpha(R_j f)$.*

This is immediate by use of Fourier transform knowing the fact that $R_j f\in C^\infty$ and any derivative of $R_j(f)$ behaves like $|x|^{-n}$ at infinity. Therefore, using remark 1, $S_j^\alpha(R_j f)\in L^p(E^n)$, $1<p<\infty$.

Proof of Lemma. We have

$$H(\Lambda^\alpha f) = \sum_{j=1}^{n}\frac{1}{CC_{\alpha,j}}HR_j(S_j^\alpha f) = \sum_{j=1}^{n}\frac{1}{CC_{\alpha,j}}HS_j^\alpha(R_j f).$$

Set $R_j f = g$. So

$$HS_j^\alpha(g)$$

$$= \int[a(x)-a(y)]K(x-y)\lim_{\substack{\delta\to 0\\ \text{in } L^p}}(S_{j,\delta}^\alpha g)(y)\,dy$$

$$= \lim_{\substack{\delta\to 0\\ \text{in } L^p}}\int K(x-y)[a(x)-a(y)](S_{j,\delta}^\alpha g)(y)\,dy$$

$$= \lim_{\substack{\delta\to 0\\ \text{in } L^p}}\lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}}\int_{|x-y|>\varepsilon}K(x-y)\left\{\int_{|y-z|>\delta}\frac{[a(x)-a(z)]}{|y-z|^{n+1+\alpha}}(y_j-z_j)g(z)\,dz\right\}dy+$$

$$+\lim_{\substack{\delta\to 0\\ \text{in } L^p}}\lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}}\int_{|x-y|>\varepsilon}K(x-y)\left\{\int_{|y-z|>\delta}\frac{[a(z)-a(y)]}{|y-z|^{n+1+\alpha}}(y_j-z_j)g(z)\,dz\right\}dy.$$

For the second term, set $h_\delta(y)$ equal to the term in brackets. The second term then becomes $\lim\limits_{\substack{\delta\to 0\\ \text{in } L^p}} K(h_\delta)$. Since $a(z)$ is bounded and $|a(z)-a(y)| \leqslant A|y-z|^\beta$, $\beta > \alpha$, it is clear that $h_\delta(y)$ is a Cauchy sequence in L^p and therefore converges in L^p to a function, $h(y)$, such that $\|h\|_p \leqslant C\|g\|_p$. Since K is a continuous operator,

$$\lim_{\substack{\delta\to 0\\ \text{in } L^p}} K(h_\delta) = K(h) \quad \text{and} \quad \|K(h)\|_p \leqslant C\|h\|_p \leqslant C\|g\|_p.$$

Set

$$K_\varepsilon(x-y) = \begin{cases} K(x-y) & \text{if } |x-y| > \varepsilon,\\ 0 & \text{if } |x-y| \leqslant \varepsilon.\end{cases}$$

We can then write the first term as

$$\lim_{\substack{\delta\to 0\\ \text{in } L^p}} \int_{E^n} [a(x)-a(z)]g(z)\Big\{\lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}} \int_{E^n} K_\varepsilon(x-y) S^a_{j,\delta}(y-z)\,dy\Big\}dz$$
$$= \lim_{\substack{\delta\to 0\\ \text{in } L^p}} \int_{E^n} [a(x)-a(z)]K(S^a_{j,\delta})(x-z)g(z)\,dz.$$

Claim. (i) *There is a function, call it $K(S^a_j)(x)$, homogeneous of degree $-n-a$ such that*

(ii) $$\int_\Sigma |K(S^a_j)(x)|\,d\sigma < C\Big(\operatorname*{Max}_{|x|=1} |(\partial/\partial x_i)K| + \operatorname*{Max}_{|x|=1} |K|\Big),$$

(iii) $$\big\||x|^a[K(S^a_{j,\delta}) - (K(S^a_j))_\delta](x)\big\|_1 \leqslant C\Big(\operatorname*{Max}_{|x|=1} |(\partial/\partial x_i)K| + \operatorname*{Max}_{|x|=1}|K|\Big),$$

C independent of δ.

Proof. Suppose $x \neq 0$, $\delta < \delta' < |x|/2$,

$$K(S^a_{j,\delta})(x) - K(S^a_{j,\delta'})(x) = \lim_{\varepsilon\to 0}\int K_\varepsilon(x-y)[S^a_{j,\delta}(y) - S^a_{j,\delta'}(y)]\,dy$$
$$= \int_{\delta<|y|<\delta'} [K(x-y)-K(x)]S^a_j(y)\,dy.$$

$$|K(S^a_{j,\delta})(x) - K(S^a_{j,\delta'})(x)| \leqslant C \operatorname*{Max}_{\substack{i=1,\dots,n\\ x\in\Sigma}} |(\partial/\partial x_i)K(x)||x|^{-(n+1)} \int_{|y|\leqslant\delta} |y|\,|S^a_j(y)|\,dy$$
$$\leqslant \frac{M\delta}{|x|^{n+1}}.$$

$K(S^a_{j,\delta})$ forms a Cauchy sequence in the L^∞-norm outside any neighborhood of the origin. Hence there is $K(S^a_j)^*(x)$ such that

$$\|K(S^a_{j,\delta})(x) - K(S^a_j)^*(x)\|_\infty \to 0 \quad \text{in} \quad |x| > a > 0.$$

Since $K_\varepsilon(S^a_{j,\delta})(\lambda x) = \lambda^{-n-a}K_{\varepsilon/\lambda}(S^a_{j,\delta/\lambda})(x)$, we have for each $\lambda > 0$

(1) $$K(S^a_j)^*(\lambda x) = \lambda^{-n-a}K(S^a_j)^*(x) \quad \text{for a.e. } x.$$

For each $x \neq 0$, (1) holds for a.e. $\lambda \in (0,\infty)$.

Let B = set of points x such that it is not certain that (1) holds for a.e. λ. Since $|B| = 0$, there is a sphere, Σ_ϱ, of radius ϱ, such that $\Sigma_\varrho \cap B$ has measure 0 over Σ_ϱ.

Define

$$K(S^a_j)(x) = \begin{cases} \left(\dfrac{\varrho}{|x|}\right)^{n+a} K(S^a_j)^*\left(\dfrac{x\varrho}{|x|}\right) & \text{if } \dfrac{x\varrho}{|x|} \notin B,\\ 0 & \text{otherwise}.\end{cases}$$

$K(S^a_j)$ differs from $K(S^a_j)^*$ in a set of measure 0 and is homogeneous of degree $-n-a$

$$C_a\int_\Sigma K(S^a_j)(x)\,d\sigma = \int_{1<|x|<2} K(S^a_j)(x)\,dx.$$

For $|x| > 1$,

$$\big|\big(K(S^a_j) - K(S^a_{j,1/2})\big)(x)\big| \leqslant \frac{M}{|x|^{n+1}}$$

and

$$\int_{1<|x|<2} |K(S^a_{j,1/2})|\,dx \leqslant C\big(\operatorname*{Sup}_{|x|=1}|K|\big) = CM_1.$$

Therefore

$$\int_\Sigma |K(S^a_j)(x)|\,d\sigma \leqslant C(M+M_1).$$

Set $\mu_\delta(x) = |x|^a\big[K(S^a_{j,\delta})(x) - (K(S^a_j))_\delta(x)\big]$.

Since $\mu_\delta(x) = \delta^{-n}\mu_1(x/\delta)$, it is sufficient to show

(iii) for $\mu_1(x)$

$$\int_{|x|<2}\mu_1(x)\,dx \leqslant \int_{|x|<2}|x|^a|K(S^a_{j,1})(x)|\,dx + \int_{1<|x|<2}|K(S^a_j)|\,|x|^a\,dx$$
$$\leqslant C(M_1+M),$$

$$\int_{|x|\geqslant 2}|\mu_1(x)|\,dx = \int_{|x|\geqslant 2}|x|^a\big|K(S^a_{j,1})(x) - \big(K(S^a_j)\big)(x)\big|\,dx$$
$$\leqslant M\int_{|x|\geqslant 2}\frac{|x|^a}{|x|^{n+1}}\,dx \leqslant BM.$$

Therefore

$$\lim_{\substack{\delta\to 0\\ \text{in } L^p}} \int [a(x)-a(z)]K(S^a_{j,\delta})(x-z)g(z)\,dz$$
$$= \lim_{\substack{\delta\to 0\\ \text{in } L^p}} \int [a(x)-a(z)][K(S^a_{j,\delta})-K(S^a_j)_\delta](x-z)g(z)\,dz+$$
$$+\lim_{\substack{\delta\to 0\\ \text{in } L^p}} \int [a(x)-a(z)]K(S^a_j)_\delta(x-z)g(z)\,dz.$$

Note. By an argument similar to that of remark 2 we infer that for g

$$\lim_{\delta\to 0}\int [a(x)-a(z)](KS^a_j)_\delta(x-z)g(z)\,dz$$

exists in the L^p-sense.

Hence

$$\|H(\Lambda^a f)\|_p \leqslant BA(1+M+M_1)\|g\|_p,$$

so that

$$\|H(\Lambda^a f)\|_p \leqslant C_p\|f\|_p,$$

where $C_p = B_p A(1+M+M_1)$, B_p depending on p, a, n only.

Definition. $K(x,y)\in C^\infty_\beta$, $\beta\geqslant 0$, if for each x, $K(x,y)\in C^\infty\big(E^n-(0)\big)$ as a function of y and each derivative, $(\partial/\partial y)^\gamma K(x,y)$, satisfies for $y'\in\Sigma$,

$$|(\partial/\partial y)^\gamma K(x_1,y')-(\partial/\partial y)^\gamma K(x_2,y')| \leqslant C|x_1-x_2|^\beta.$$

If $a(x)\in B_\beta(E^n)$, $0<\beta\leqslant 1$, then we define $\|a\|_\beta$ to be the sum of the supremum of $|a(x)|$ and the infimum of the numbers M such that

$$|a(x)-a(y)| \leqslant M|x-y|^\beta.$$

We set

$$Kf(x) = \lim_{\substack{\varepsilon\to 0\\ \text{in } L^p}} \int_{|x-y|>\varepsilon} K(x, x-y)f(y)\,dy.$$

We also let $K^*, K^\#, K_1K_2, K_1\circ K_2$, denote respectively the adjoint of K, the pseudo-adjoint of K, the product of K_1 and K_2, and the pseudo-product of K_1 and K_2 (see [2]).

Theorem. *Let $K_1(x,y), K_2(x,y)\in C^\infty_\beta(E^n)$. Then for $0<a<\beta$,*

a) $K_1\Lambda^a-\Lambda^a K_1$, b) $(K_1^*-K_1^\#)\Lambda^a$ and c) $(K_1\circ K_2-K_1K_2)\Lambda^a$ are bounded operators from $L^p(E^n)$ to $L^p(E^n)$.

Proof. Suppose $f\in C^\infty_0(E^n)$ and let $[Y_{k,m}]$ denote the family of spherical harmonics, which are complete and orthonormal over Σ (see [1]). Let $K_1 = \Sigma_{k,l}a_{k,l}Y_{k,l}$.

a) $$(K_1\Lambda^a-\Lambda^a K_1)f = \Sigma_{k,l}\Big(\sum_{j=1}^n a_{k,l}Y_{k,l}R_jS^a_j - R_jS^a_j a_{k,l}Y_{k,l}\Big)f$$
$$= \Sigma_{k,l}\sum_{j=1}^n (a_{k,l}R_jS^a_j - R_jS^a_j a_{k,l})Y_{k,l}f.$$

Using previous lemma and following same argument of Calderón and Zygmund in [2], and of Calderón in [1], we have

$$\|(K_1\Lambda^a-\Lambda^a K_1)f\|_p \leqslant C_p\Sigma_{k,l}\|a_{k,l}\|_\beta\|Y_{k,l}f\|_p \leqslant C_p\|f\|_p.$$

b) $K_1^* = \Sigma Y_{k,l}\bar a_{k,l}$, $K_1^\# = \Sigma\bar a_{k,l}Y_{k,l}$,

$$(K_1^\#-K_1^*)\Lambda^a f = \Sigma_{k,l}[\bar a_{k,l}Y_{k,l}-Y_{k,l}\bar a_{k,l}]\Lambda^a f.$$

Therefore

$$\|(K_1^\#-K_1^*)\Lambda^a f\|_p \leqslant C\Big(\Sigma\|a_{k,l}\|_\beta\big(1+\sup_{|x|=1}|Y_{k,l}|+\sup_{\substack{|x|=1\\ i=1,\ldots,n}}|(\partial/\partial x_i)Y_{k,l}|\big)\Big)\|f\|_p$$
$$\leqslant C_pf_p.$$

c) Suppose $K_1 = \Sigma a_{k,l}Y_{k,l}$,

$$K_2 = \Sigma_{\lambda,\mu}b_{\lambda,\mu}Y_{\lambda,\mu},$$
$$K_1\circ K_2 = \Sigma_{l,m,\mu,\lambda}a_{l,m}b_{\lambda,\mu}Y_{l,m}Y_{\lambda,\mu},$$
$$K_1K_2 = \Sigma_{l,m,\mu,\lambda}a_{l,m}Y_{l,m}b_{\lambda,\mu}Y_{\lambda,\mu},$$
$$(K_1K_2-K_1\circ K_2)\Lambda^a f = \Sigma a_{l,m}[Y_{l,m}b_{\lambda,\mu}-b_{\lambda,\mu}Y_{l,m}]\Lambda^a Y_{\lambda,\mu}(f).$$

Therefore

$$\|(K_1K_2-K_1\circ K_2)\Lambda^a f\|_p$$
$$\leqslant C\Sigma\|a_{l,m}\|_\beta\|b_{\lambda,\mu}\|_\beta\big(1+\sup_{|x|=1}|Y_{l,m}|+\sup_{\substack{|x|=1\\ i=1,\ldots,n}}|(\partial/\partial x_i)Y_{l,m}|\big)\|Y_{\lambda,\mu}f\|_p \leqslant C_p\|f\|_p.$$

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