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Cross-continuity vs. continuity

by

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A mapping T of the Hilbert space $\mathscr{H}$ (real or complex but not necessarily separable) into itself is said to be *δ-continuous* or *δ-cross-continuous* if there are constants C and $C^{\perp}$ such that for any couple of distinct points x_1 and x_2 in its domain

$$\|Tx_1-Tx_2\| \leqslant C\delta(\|x_1-x_2\|), \tag{1}$$

$$\|(Tx_1-Tx_2)^{\perp}\| \leqslant C^{\perp}\delta(\|x_1-x_2\|) \tag{2}$$

respectively, where $\delta(t)$ is a non-decreasing, non-negative, sub-additive function defined on the open positive half line ("triangular function"), and where

$$(Tx_1-Tx_2)^{\perp} = Tx_1-Tx_2-\frac{(Tx_1-Tx_2, x_1-x_2)}{\|x_1-x_2\|^2}(x_1-x_2).$$

For $\delta(t) = t^{\nu}$, $0 \leqslant \nu \leqslant 1$, δ-continuity coincides with the usual Hölder condition of exponent ν, also called *Lipschitz condition* if $\nu = 1$, and δ-cross-continuity yields the notion of cross-Hölder condition of exponent ν (or cross-Lipschitz condition if $\nu = 1$) introduced by one of the authors in a recent study of non-linear operators in Hilbert space [3], where the idea was immediately put to use without any further inquiry into its meaning. Clearly, since $(Tx_1-Tx_2)^{\perp}$ is the component of Tx_1-Tx_2 orthogonal to x_1-x_2, a Hölder condition implies a cross-Hölder condition of the same exponent. At the beginning the apparent absence of counter-examples led to the conjecture that perhaps the converse of this was also true and the new notion altogether superfluous. Counter-examples such as $x\log(1/\|x\|)$ — which is cross Lipschitzian but not Lipschitzian — arrived to later showed the conjecture false but were insufficient to establish for which ν's any ν-cross-Hölder mapping is ν-Hölder, or more generally still, for which δ's δ-cross-continuity and δ-continuity are equivalent, if ever. This broader question is our main concern in this article, to which we give the following somewhat unexpected answer (boundary behaviour being ignored): There is equivalence if and only if

$$\varlimsup_{t\to 0}\frac{1}{\delta(t)}\int_t^1 \delta(t/u)\,du < \infty.$$

According to this, in the special case $\delta(t) = t^\nu$, $0 \leqslant \nu \leqslant 1$, equivalence occurs for $\nu < 1$ and non-equivalence for $\nu = 1$. As a result of our discussion it will also appear that the distinction between both notions is a subtle one in the sense that δ-cross-continuity may be weaker than δ-continuity by a logarithmic factor at most.

If the dimension of the space is one, all cross-increments vanish and any mapping is δ-cross-continuous for any δ whatever. Thus $\dim \mathscr{H} \geqslant 2$ is a basic condition in all considerations that follow. As to the domains of definition of the mappings $T: \mathscr{H} \to \mathscr{H}$ considered here will be assumed to be dense subsets $\mathscr{D}(T)$ of open convex sets, that is, the $\mathscr{D}(T)$'s will be required to satisfy $\mathscr{D}(T) \subset \operatorname{int} \overline{\mathscr{D}}(T)$, with $\overline{\mathscr{D}}(T)$ convex. Behind this choice there is a double motivation: 1st if a continuity assertion is to be made about any point of $\mathscr{D}(T)$ such a point must be an accumulation point of $\mathscr{D}(T)$ and hence $\mathscr{D}(T)$ dense in itself; 2nd it is the possibility of moving about any point which gives strength to cross-continuity, strength that is maximized if $\mathscr{D}(T)$ is taken as dense in an open set. Easily constructed examples show that nothing of the nature we seek is to be expected of less substantial domains. Convexity is imposed only for convenience; its effect is to make distances within $\mathscr{D}$ coincide with distances in $\mathscr{H}$. For any $\mathscr{D}$ of the type described and any positive ϱ, $\mathscr{D}^{(\varrho)}$ will indicate the set of points in $\mathscr{D}$ at a distance $> \varrho$ from the boundary of $\overline{\mathscr{D}}$. The restriction of T to $\mathscr{D}^{(\varrho)}(T)$ will be denoted by $T^{(\varrho)}$.

For any T the functions of a real variable t,

$$\delta_T(t) = \sup_{\|x_1-x_2\|<t} \|Tx_1 - Tx_2\|, \tag{3}$$

$$\delta_T^{\perp}(t) = \sup_{\|x_1-x_2\|<t} \|(Tx_1 - Tx_2)^{\perp}\|, \tag{4}$$

are respectivelly called the *modulus of continuity* and the *cross-modulus of continuity* of T. As we shall see later they are both triangular functions. Their finiteness for some value of t expresses boundedness or cross-boundedness, whereas their approaching zero as $t \to 0$ says that the mapping is either uniformly continuous or uniformly cross-continuous as the case be. So, uniform continuity (cross-continuity) and δ-continuity (δ-cross-continuity) for some δ are the same thing, and therefore the comparison of the various degrees of uniform continuity or uniform cross-continuity can be effected by comparing the corresponding moduli of continuity:

DEFINITION. Given two triangular functions δ_1 and δ_2, δ_1 is said to be *weaker* than δ_2, in symbols $\delta_1 \precsim \delta_2$, if

$$\overline{\lim_{t\to 0}} \big(\delta_1(t)/\delta_2(t)\big) < +\infty.$$

If $\delta_1 \precsim \delta_2$ and $\delta_2 \precsim \delta_1$ hold simultaneously one says that δ_1 and δ_2 are *equivalent* and denotes the fact by $\delta_1 \sim \delta_2$.

The relation $\sim$ is in fact an "equivalence relation", and it is seen at once that it is the class of the triangular functions equivalent to δ what is attached to the family of δ-continuous or δ-cross-continuous mappings rather than the individual δ. A third modulus of continuity, the *parallel modulus of continuity* $\delta_T^{\parallel}(t)$, is defined by

$$\delta_T^{\parallel}(t) = \sup_{\|x_1-x_2\|<t} \|(Tx_1 - Tx_2)^{\parallel}\|, \tag{5}$$

where

$$(Tx_1 - Tx_2)^{\parallel} = \left(Tx_1 - Tx_2, \frac{x_1 - x_2}{\|x_1 - x_2\|}\right) \frac{x_1 - x_2}{\|x_1 - x_2\|},$$

is the component of $Tx_1 - Tx_2$ parallel to $x_1 - x_2$. Note the relation

$$\delta_T(t)^2 \leqslant \delta_T^{\parallel}(t)^2 + \delta_T^{\perp}(t)^2. \tag{6}$$

Our first step towards the announced goal is a basic inequality which serves to translate the geometric content of the hypotheses into analytical terms:

LEMMA 1. *For any three points x_1, x_2, and x_3 in the domain $\mathscr{D}(T)$ of a mapping T*

$$\left| \frac{(Tx_1 - Tx_3, x_1 - x_3)}{\|x_1 - x_3\|^2} - \frac{(Tx_2 - Tx_3, x_2 - x_3)}{\|x_2 - x_3\|^2} \right| \leqslant 24\sqrt{2}\, \frac{\delta_T^{\perp}(\|x_1 - x_3\| + \|x_2 - x_3\|)}{\min\{\|x_1 - x_3\|, \|x_2 - x_3\|, d(x_3, \operatorname{bdry} \overline{\mathscr{D}})\}} \tag{7}$$

where $d(x_3, \operatorname{bdry} \overline{\mathscr{D}})$ denotes the distance of x_3 to the boundary of $\overline{\mathscr{D}}$.

Proof. We start out from an identity valid for $x_1, x_2, x_3 \in \mathscr{D}$:

$$\begin{aligned} &[\|x_1 - x_3\|^2 \|x_2 - x_3\|^2 - |(x_1 - x_3, x_2 - x_3)|^2] \times \\ &\quad \times \left[\frac{(Tx_2 - Tx_3, x_2 - x_3)}{\|x_2 - x_3\|^2} - \frac{(Tx_1 - Tx_3, x_1 - x_3)}{\|x_1 - x_3\|^2} \right] \\ &= (x_1 - x_2, x_1 - x_3) \left((Tx_1 - Tx_3)^{\perp}, x_2 - x_3 - \frac{(x_2 - x_3, x_1 - x_3)}{\|x_1 - x_3\|^2} (x_1 - x_3) \right) + \\ &\quad + (x_1 - x_2, x_2 - x_3) \left((Tx_2 - Tx_3)^{\perp}, x_1 - x_3 - \frac{(x_1 - x_3, x_2 - x_3)}{\|x_2 - x_3\|^2} (x_2 - x_3) \right) + \\ &\quad + \big((Tx_1 - Tx_2)^{\perp}, (x_2 - x_3, x_1 - x_2)(x_1 - x_3) - (x_1 - x_3, x_1 - x_2)(x_2 - x_3) \big). \end{aligned} \tag{8}$$

To verify (8) observe first that the factors multiplying $(Tx_1 - Tx_3)^{\perp}$, $(Tx_2 - Tx_3)^{\perp}$, $(Tx_1 - Tx_2)^{\perp}$ scalarly are vectors respectively orthogonal to $x_1 - x_3$, $x_2 - x_3$, $x_1 - x_2$; then drop the symbols $\perp$ and check the resulting identity by expanding the scalar products according to the ordinary rules.

Simple calculations yield

$$(9)\qquad \left\| x_2 - x_3 - \frac{(x_2 - x_3, x_1 - x_3)}{\|x_1 - x_3\|^2}(x_1 - x_3)\right\| = \|x_2 - x_3\| \sqrt{1 - \left|\left(\frac{x_1 - x_3}{\|x_1 - x_3\|}, \frac{x_2 - x_3}{\|x_2 - x_3\|}\right)\right|^2},$$

$$(10)\qquad \left\| x_1 - x_3 - \frac{(x_1 - x_3, x_2 - x_3)}{\|x_2 - x_3\|^2}(x_2 - x_3)\right\| = \|x_1 - x_3\| \sqrt{1 - \left|\left(\frac{x_1 - x_3}{\|x_1 - x_3\|}, \frac{x_2 - x_3}{\|x_2 - x_3\|}\right)\right|^2},$$

$$(11)\qquad \|(x_2 - x_3, x_1 - x_2)(x_1 - x_3) - (x_1 - x_3, x_1 - x_2)(x_2 - x_3)\| = \|x_1 - x_2\|\,\|x_1 - x_3\|\,\|x_2 - x_3\| \sqrt{1 - \left|\left(\frac{x_1 - x_3}{\|x_1 - x_3\|}, \frac{x_2 - x_3}{\|x_2 - x_3\|}\right)\right|^2}.$$

Taking absolute values in (8) one obtains by Schwartz' inequality and by virtue of (9), (10) and (11),

$$(12)\qquad \sqrt{1 - \left|\left(\frac{x_1 - x_3}{\|x_1 - x_3\|}, \frac{x_2 - x_3}{\|x_2 - x_3\|}\right)\right|^2}\left|\frac{(Tx_2 - Tx_3, x_2 - x_3)}{\|x_2 - x_3\|^2} - \frac{(Tx_1 - Tx_3, x_1 - x_3)}{\|x_1 - x_3\|^2}\right| \leqslant \frac{\|x_1 - x_2\|}{\|x_1 - x_3\|\,\|x_2 - x_3\|}\left[\|(Tx_1 - Tx_3)^{\perp}\| + \|(Tx_2 - Tx_3)^{\perp}\| + \|(Tx_1 - Tx_2)^{\perp}\|\right].$$

We are halfway towards (7); our next step is to get rid of the inconvenient square root factor. For this purpose we take an auxiliary point x_0 constructed as follows: Let

$$\theta = \arg(x_1 - x_3, x_2 - x_3),$$

$$m(x_1, x_2, x_3) = \min\{\|x_1 - x_3\|, \|x_2 - x_3\|, d(x_3, \operatorname{bdry} \mathcal{D})\},$$

and set

$$(13)\qquad x_0 = \begin{cases} x_3 + m(x_1, x_2, x_3)\dfrac{\dfrac{x_1 - x_3}{\|x_1 - x_3\|} - e^{i\theta}\dfrac{x_2 - x_3}{\|x_2 - x_3\|}}{\left\|\dfrac{x_1 - x_3}{\|x_1 - x_3\|} - e^{i\theta}\dfrac{x_2 - x_3}{\|x_2 - x_3\|}\right\|}, & \text{if } x_1 - x_3 \text{ and } x_2 - x_3 \text{ are not parallel,} \\ x_3 + m(x_1, x_2, x_3)u, & \text{with } u \text{ unitary and orthogonal to } x_1 - x_3 \text{ and } x_2 - x_3, \text{ if they are parallel.} \end{cases}$$

Assume first that $x_0 \in \mathcal{D}(T)$, and apply (12) twice with x_1 and x_2 successively replaced by x_0, and add the results. One obtains

$$(14)\qquad \sqrt{1 - \left|\left(\frac{x_2 - x_3}{\|x_2 - x_3\|}, \frac{x_0 - x_3}{\|x_0 - x_3\|}\right)\right|^2}\left|\frac{(Tx_2 - Tx_3, x_2 - x_3)}{\|x_2 - x_3\|^2} - \frac{(Tx_0 - Tx_3, x_0 - x_3)}{\|x_0 - x_3\|^2}\right| + \sqrt{1 - \left|\left(\frac{x_0 - x_3}{\|x_0 - x_3\|}, \frac{x_1 - x_3}{\|x_1 - x_3\|}\right)\right|^2}\left|\frac{(Tx_0 - Tx_3, x_0 - x_3)}{\|x_0 - x_3\|^2} - \frac{(Tx_1 - Tx_3, x_1 - x_3)}{\|x_1 - x_3\|^2}\right|$$
$$\leqslant \frac{\|x_2 - x_0\|}{\|x_2 - x_3\|\,\|x_0 - x_3\|}\left[\|(Tx_2 - Tx_3)^{\perp}\| + \|(Tx_0 - Tx_3)^{\perp}\| + \|(Tx_2 - Tx_0)^{\perp}\|\right] + \frac{\|x_1 - x_0\|}{\|x_1 - x_3\|\,\|x_0 - x_3\|}\left[\|(Tx_0 - Tx_3)^{\perp}\| + \|(Tx_1 - Tx_3)^{\perp}\| + \|(Tx_1 - Tx_0)^{\perp}\|\right].$$

Moreover,

$$(15)\qquad \sqrt{1 - \left|\left(\frac{x_2 - x_3}{\|x_2 - x_3\|}, \frac{x_0 - x_3}{\|x_0 - x_3\|}\right)\right|^2} = \sqrt{1 - \left|\left(\frac{x_1 - x_3}{\|x_1 - x_3\|}, \frac{x_0 - x_3}{\|x_0 - x_3\|}\right)\right|^2} = \sqrt{\frac{1}{2}\left(1 + \left|\left(\frac{x_1 - x_3}{\|x_1 - x_3\|}, \frac{x_2 - x_3}{\|x_2 - x_3\|}\right)\right|\right)} \geqslant \frac{1}{\sqrt{2}},$$

and in consequence

$$(16)\qquad \left|\frac{(Tx_2 - Tx_3, x_2 - x_3)}{\|x_2 - x_3\|^2} - \frac{(Tx_1 - Tx_3, x_1 - x_3)}{\|x_1 - x_3\|^2}\right| \leqslant \frac{\sqrt{2}\,\|x_2 - x_0\|}{\|x_2 - x_3\|\,\|x_0 - x_3\|}\left[\|(Tx_2 - Tx_3)^{\perp}\| + \|(Tx_0 - Tx_3)^{\perp}\| + \|(Tx_2 - Tx_0)^{\perp}\|\right] + \frac{\sqrt{2}\,\|x_1 - x_0\|}{\|x_1 - x_3\|\,\|x_0 - x_3\|}\left[\|(Tx_1 - Tx_3)^{\perp}\| + \|(Tx_0 - Tx_3)^{\perp}\| + \|(Tx_1 - Tx_0)^{\perp}\|\right].$$

Now, since $\|x_1 - x_0\| \leqslant \|x_1 - x_3\| + \|x_3 - x_0\|$, $\|x_2 - x_0\| \leqslant \|x_2 - x_3\| + \|x_3 - x_0\|$ and $\|x_3 - x_0\| = m(x_1, x_2, x_3)$, each term in square brackets on the right does no exceed $\delta_T^{\perp}(\|x_1-x_3\|+\|x_2-x_3\|)$, and (7) follows from (16) upon observing that

$$\frac{\|x_2-x_0\|}{\|x_2-x_3\|}+\frac{\|x_1-x_0\|}{\|x_1-x_3\|}\leqslant 2+\frac{\|x_3-x_0\|}{\|x_2-x_3\|}+\frac{\|x_3-x_0\|}{\|x_1-x_3\|}\leqslant 4.$$

If x_0 does not belong to $\mathscr{D}(T)$, the same conclusion can be reached by using in its place a sequence $x_0^{(n)}\in\mathscr{D}(T)$ with $\|x_0^{(n)}-x_3\|\leqslant m(x_1,x_2,x_3)$ converging to x_0, whose existence is assured by the nature of $\mathscr{D}(T)$ and the meaning of $m(x_1,x_2,x_3)$.

Lemma 2. *If $\delta_T^{\perp}(t_0)<+\infty$ for some $t_0>0$, T is bounded over any bounded set at a positive distance from the boundary of $\overline{\mathscr{D}}$.*

Proof. Let $\mathscr{B}$ be such a set and x_3 any of its points. Then it follows from formula (7) that

$$\|(Tx_1-Tx_3)^{\|}\|=\left|\frac{(Tx_1-Tx_3,\,x_1-x_3)}{\|x_1-x_3\|}\right|$$

remains bounded as x_1 runs over the intersection of $\mathscr{B}$ with a ball of radius $t_0/2$ about x_3, and as $\|Tx_1-Tx_3\|\leqslant\|(Tx_1-Tx_3)^{\|}\|+\|(Tx_1-Tx_3)^{\perp}\|$ so does $\|Tx_1-Tx_3\|$. Otherwise said, any point of $\mathscr{B}$ is the center of a ball of positive fixed radius over which T is bounded, and so the desired result is a consequence of the boundedness of $\mathscr{B}$.

Lemma 3. *The moduli of continuity δ_T and $\delta_T^{\perp}$ of a mapping $T:\mathscr{H}\to\mathscr{H}$ defined over a dense subset of a convex open set, are triangular functions. This also holds for $\delta_T^{\|}$ if T is locally bounded or $\mathscr{D}(T)$ convex and open.*

Proof. It is clear that the only property in question is subadditivity. That it holds for δ_T is almost immediate, for if $x_1,x_2\in\mathscr{D}(T)$, $\|x_1-x_2\|<\varrho_1+\varrho_2$ $(\varrho_1,\varrho_2>0)$, there is always an $x_0\in\mathscr{D}(T)$ with $\|x_1-x_0\|<\varrho_1$, $\|x_2-x_0\|<\varrho_2$, and hence $\|Tx_1-Tx_2\|\leqslant\|Tx_1-Tx_0\|+\|Tx_0-Tx_2\|\leqslant\delta_T(\varrho_1)+\delta_T(\varrho_2)$, which by the arbitrariness of x_1 and x_2 implies $\delta_T(\varrho_1+\varrho_2)\leqslant\delta_T(\varrho_1)+\delta_T(\varrho_2)$. Less simple is the situation for the other two moduli; let us prove $\delta_T^{\perp}(\varrho_1+\varrho_2)\leqslant\delta_T^{\perp}(\varrho_1)+\delta_T^{\perp}(\varrho_2)$. It is sufficient to consider the case when $\delta_T^{\perp}(\varrho_1)<+\infty$, $\delta_T^{\perp}(\varrho_2)<+\infty$, for otherwise there is nothing to prove. For any x_1 and x_2 in $\mathscr{D}(T)$ with $\|x_1-x_2\|<\varrho_1+\varrho_2$ let x_0 be the point of the segment $tx_1+(1-t)x_2$, $0\leqslant t\leqslant 1$, dividing it into two parts of length proportional to ϱ_1 and ϱ_2, and take a sequence $x_0^{(n)}\in\mathscr{D}(T)$ with $\|x_1-x_0^{(n)}\|<\varrho_1$, $\|x_2-x_0^{(n)}\|<\varrho_2$ converging to x_0. Since

$$\frac{x_1-x_2}{\|x_1-x_2\|}=\frac{x_1-x_0}{\|x_1-x_0\|}=\frac{x_0-x_2}{\|x_0-x_2\|},$$

we may write

$$(Tx_1-Tx_2)^{\perp}=(Tx_1-Tx_0^{(n)})^{\perp}+(Tx_0^{(n)}-Tx_2)^{\perp}+$$
$$+\left\{\left[\left(Tx_1-Tx_0^{(n)},\frac{x_1-x_0^{(n)}}{\|x_1-x_0^{(n)}\|}\right)\frac{x_1-x_0^{(n)}}{\|x_1-x_0^{(n)}\|}-\left(Tx_1-Tx_0^{(n)},\frac{x_1-x_0}{\|x_1-x_0\|}\right)\frac{x_1-x_0}{\|x_1-x_0\|}\right]+\right.$$
$$\left.+\left[\left(Tx_0^{(n)}-Tx_2,\frac{x_0^{(n)}-x_2}{\|x_0^{(n)}-x_2\|}\right)\frac{x_0^{(n)}-x_2}{\|x_0^{(n)}-x_2\|}-\left(Tx_0^{(n)}-Tx_2,\frac{x_0-x_2}{\|x_0-x_2\|}\right)\frac{x_0-x_2}{\|x_0-x_2\|}\right]\right\}.$$

But as

$$\frac{x_1-x_0^{(n)}}{\|x_1-x_0^{(n)}\|}\to\frac{x_1-x_0}{\|x_1-x_0\|},\qquad\frac{x_0^{(n)}-x_2}{\|x_0^{(n)}-x_2\|}\to\frac{x_0-x_2}{\|x_0-x_2\|},$$

and since $Tx_0^{(n)}$ is a bounded sequence by Lemma 2, the term in braces of the identity above tends to zero as $n\to\infty$. Therefore,

$$\|(Tx_1-Tx_2)^{\perp}\|\leqslant\varlimsup_{n\to\infty}\|(Tx_1-Tx_0^{(n)})^{\perp}\|+\varlimsup_{n\to\infty}\|(Tx_0^{(n)}-Tx_2)^{\perp}\|$$
$$\leqslant\delta_T^{\perp}(\varrho_1)+\delta_T^{\perp}(\varrho_2),$$

whence $\delta_T^{\perp}(\varrho_1+\varrho_2)\leqslant\delta_T^{\perp}(\varrho_1)+\delta_T^{\perp}(\varrho_2)$, as we wished to prove. The proof of subadditivity for $\delta_T^{\|}$ is similar and is left to the reader's care.

Before proceeding with our study of the relations between continuity and cross-continuity we shall stop to gather some information about triangular functions (for a fuller discussion of this material we refer the reader to [2], Ch. VI).

Lemma 4. *Any non-negative function $\delta(t)$, $t>0$, for which $\delta(t)/t$ does not increase is sub-additive* ([2], p. 83). *In particular, any non-negative, non-decreasing concave function is a triangular function.*

Proof. If $\delta(t)/t$ is not increasing, and $t_1,t_2>0$,

$$\delta(t_1+t_2)=\frac{\delta(t_1+t_2)}{t_1+t_2}t_1+\frac{\delta(t_1+t_2)}{t_1+t_2}t_2\leqslant\frac{\delta(t_1)}{t_1}t_1+\frac{\delta(t_2)}{t_2}t_2=\delta(t_1)+\delta(t_2).$$

On the other hand, if $\delta(t)$ is non-negative, non-decreasing and concave, and if $0<\beta<\alpha$,

$$\frac{\delta(t)}{t}\leqslant\varlimsup_{\varepsilon\downarrow 0}\frac{\beta\delta(t)+(1-\beta)\,\delta\big(\varepsilon/(1-\beta)\big)}{\beta t}\leqslant\varlimsup_{\varepsilon\downarrow 0}\frac{\delta(\beta t+\varepsilon)}{\beta t}\leqslant\frac{\delta(\alpha t)}{\beta t}.$$

Letting $\beta\uparrow\alpha$ it follows that $\delta(t)/t$ does not increase and hence that δ is a triangular function.

Lemma 5. *For any triangular function $\delta(t)$:*

a. $\lim\limits_{t\to 0}\delta(t)=0$ *implies continuity for every value of t;*

b. $\lim_{t\to 0}\delta(t)/t$ *exists and is equal to* $\sup_t \delta(t)/t$;

c. *we have*

(17) $$\frac{\sum a_i\delta(t_i)}{\sum a_i} \leqslant 2\delta\left(\frac{\sum a_i t_i}{\sum a_i}\right), \qquad a_i \geqslant 0.$$

Proof. a is an obvious consequence of sub-additivity. As to b we have if $t_1 < t_2$,

(18) $$\delta(t_2)/t_2 = \delta\big((t_2/t_1)t_1\big)/t_2 = \delta\big(([t_2/t_1]+1)t_1\big)/t_2 \leqslant \frac{1}{t_2}\,([t_2/t_1]+1)\,\delta(t_1) \leqslant \frac{t_2+t_1}{t_2}\;\frac{\delta(t_1)}{t_1}.$$

So, letting $t_1 \to 0$ first,

(19) $$\delta(t_2)/t_2 \leqslant \varliminf_{t_1\to 0} \frac{(t_2+t_1)}{t_2}\;\frac{\delta(t_1)}{t_1} = \varliminf_{t_1\to 0}\frac{\delta(t)}{t},$$

and then $t_2 \to 0$,

$$\varlimsup_{t\to 0}\frac{\delta(t)}{t} \leqslant \varliminf_{t\to 0}\frac{\delta(t)}{t}.$$

Hence $\lim_{t\to 0}\delta(t)/t$ exists and by (19) coincides with $\sup_t \delta(t)/t$.

Now we prove c; it is enough to do it under the assumption $\sum a_i = 1$. We have

$$\sum_i a_i\,\delta(t_i) = \sum_i a_i\,\delta\left(\frac{t_i}{\sum_k a_k t_k}\sum_k a_k t_k\right) \leqslant \sum_i a_i\left(t_i\Big/\Big(\sum_k a_k t_k\Big)+1\right)\delta\Big(\sum_k a_k t_k\Big) = 2\delta\Big(\sum a_k t_k\Big).$$

This lemma has interesting interpretations. Part b says that the function t is comparable with any triangular function $\delta(t)$ and that if $\delta(t)$ is strictly weaker than t, $\delta(t) \equiv 0$. In particular, $\delta_T(t) \prec t$ strictly amounts to $T = \text{const}$, whereas if this holds for $\delta_{T}^{\perp}(t)$, then

$$Tx = Tx_0 + \frac{(Tx-Tx_0,\, x-x_0)}{\|x-x_0\|^2}\,(x-x_0) + (Tx-Tx_0)^{\perp} = Tx_0 + C(x-x_0),$$

because $(Tx-Tx_0)^{\perp}$ vanishes and $(Tx-Tx_0,\, x-x_0)/\|x-x_0\|^2 = \text{const}$ by (8). As to c, it may be read to mean that the non-negative, non-decreasing concave function

(20) $$\hat\delta(t) = \sup_{\sum a_i = 1,\, \sum a_i t_i \leqslant t} \sum a_i\delta(t_i)$$

satisfies

(21) $$\delta(t) \leqslant \hat\delta(t) \leqslant 2\delta(t),$$

and hence is equivalent to δ.

Lemma 6. *For any triangular* δ,

(22) $$\delta^*(t) = \begin{cases} t\delta(1) + \int_t^1 \delta(t/u)\,du, & 0 < t \leqslant 1,\\ \delta(1), & t > 1.\end{cases}$$

is a concave triangular function. Moreover,

(23) $$\delta(t) + t\big(\delta(1)-\delta(t)\big) \leqslant \delta^*(t) \leqslant t\delta(1) + 2(1-t)\,\delta\left(\frac{t}{1-t}\log\frac{1}{t}\right), \qquad 0 < t < 1;$$

(24) $$\delta\big(\min(t_1,t_2)\big)\left|\frac{1}{t_1}-\frac{1}{t_2}\right| \leqslant \left|\frac{\delta^*(t_1)}{t_1}-\frac{\delta^*(t_2)}{t_2}\right| \leqslant \delta\big(\max(t_1,t_2)\big)\left|\frac{1}{t_1}-\frac{1}{t_2}\right|, \qquad 0 < t_1, t_2 < 1.$$

Proof. $\delta^*(t)$ is obviously non-negative. Taking t/u as the integration variable we have, if $0 < t < 1$,

(25) $$\delta^*(t) = t\delta(1) + t\int_t^1 \delta(x)/x^2\,dx,$$

whence

(26) $$\frac{d\delta^*(t)}{dt} = \delta(1) + \int_t^1 \delta(x)/x^2\,dx - \frac{\delta(t)}{t} \geqslant \delta(1) + \delta(t)\left(\frac{1}{t}-1\right) - \frac{\delta(t)}{t} = \delta(1)-\delta(t) \geqslant 0,$$

showing that $\delta^*(t)$ does not decrease. Further, if $1 > t_2 > t_1 > 0$,

$$\frac{d\delta^*(t_2)}{dt_2} - \frac{d\delta^*(t_1)}{dt_1} = \frac{\delta(t_1)}{t_1} - \frac{\delta(t_2)}{t_2} - \int_{t_1}^{t_2}\frac{\delta(x)}{x^2}\,dx \leqslant \frac{\delta(t_1)}{t_1} - \frac{\delta(t_2)}{t_2} - \delta(t_1)\left(\frac{1}{t_1}-\frac{1}{t_2}\right) \leqslant \frac{\delta(t_1)-\delta(t_2)}{t_2} \leqslant 0.$$

Thus, $\delta(t)$ is concave and an appeal to Lemma 4 proves its triangularity. The first of inequalities (23) follows at once from the definition of $\delta^*(t)$ upon noticing that $\delta(t)$ is the minimum value of $\delta(t/u)$ in the interval $(t, 1)$. As to the second, we use (25), replace $\delta(x)$ by the concave function $\hat{\delta}(x)$ defined by (20) and apply Jensen's inequality ([1], p. 151) to get

$$\delta^*(t) \leqslant t\delta(1) + t\int_t^1 \hat{\delta}(x)/x^2\,dx$$

$$\leqslant t\delta(1) + t\Big(\int_t^1 dx/x^2\Big)\hat{\delta}\left(\frac{\int_t^1 dx/x}{\int_t^1 dx/x^2}\right) = t\delta(1) + (1-t)\hat{\delta}\left(\frac{t\log t^{-1}}{1-t}\right)$$

which since $\hat{\delta}(t) \leqslant 2\delta(t)$ yields (23).

It remains to check (24). Through the change of integration variable $x = u/t$ we write, if $t_1 < t_2$,

$$\frac{\delta^*(t_1)}{t_1} - \frac{\delta^*(t_2)}{t_2} = \int_{t_2^{-1}}^{t_1^{-1}} \delta(u^{-1})\,du$$

and deduce (24) by simply replacing $\delta(u^{-1})$ by its minimum and maximum values $\delta(t_1)$ and $\delta(t_2)$ respectively.

Remark. By requiring that $\delta^*(t)$ be stronger than $\delta(t)$ but weaker than $\delta(t\log 1/t)$, (24) indicates how small the marging of variability left to $\delta^*(t)$ is. This is to be kept in mind for the proper comprehension of what follows.

We are now conveniently equiped to reasume our discussion of the main topic.

Lemma 7. *For any $T: \mathscr{H} \to \mathscr{H}$ not of the form $Tx = y_0 + Cx$, defined over a dense subset of a convex open set*

$$(27)\qquad \delta^{\|}_{T^{(\varrho)}} \prec (\delta^{\perp}_T)^*, \qquad \varrho > 0,$$

where $T^{(\varrho)}$ is the restriction of T to $\mathscr{D}^{(\varrho)}(T) = \{x \in \mathscr{D},\ d(x, \operatorname{bdry}\overline{\mathscr{D}}) > \varrho\}$.

Proof. If $x_1, x_2, x_3 \in \mathscr{D}^{(\varrho)}$ and $\|x_1 - x_3\| < \varrho$, $\|x_2 - x_3\| < \varrho$, (7) yields

$$\left|\frac{(Tx_1 - Tx_3, x_1 - x_3)}{\|x_1 - x_3\|^2}\right| \leqslant \left|\frac{(Tx_2 - Tx_3, x_2 - x_3)}{\|x_2 - x_3\|^2}\right| + 48\sqrt{2}\,\frac{\delta^{\perp}_T(\max\{\|x_1 - x_3\|, \|x_2 - x_3\|\})}{\min\{\|x_1 - x_3\|, \|x_2 - x_3\|\}}.$$

Assuming $\|x_1 - x_3\| < st \leqslant \|x_2 - x_3\| < t \leqslant \varrho$, we have by definition of $\delta^{\|}_{T^{(\varrho)}}$,

$$\left|\frac{(Tx_1 - Tx_3, x_1 - x_3)}{\|x_1 - x_3\|}\right| \leqslant s\delta^{\|}_{T^{(\varrho)}}(t) + 48\sqrt{2}\,\delta^{\perp}_T(t),$$

and taking the sup on the left,

$$(28)\qquad \delta^{\|}_{T^{(\varrho)}}(st) \leqslant s\delta^{\|}_{T^{(\varrho)}}(t) + 48\sqrt{2}\,\delta^{\perp}_T(t), \qquad 0 < s \leqslant 1,\ 0 < t \leqslant \varrho.$$

Consider now the identity

$$\delta^{\|}_{T^{(\varrho)}}(s^n t) = s^n\delta^{\|}_{T^{(\varrho)}}(t) + [\delta^{\|}_{T^{(\varrho)}}(s^n t) - s\delta^{\|}_{T^{(\varrho)}}(s^{n-1}t)] + s[\delta^{\|}_{T^{(\varrho)}}(s^{n-1}t) - s\delta^{\|}_{T^{(\varrho)}}(s^{n-2}t)] + \ldots + s^{n-1}[\delta^{\|}_{T^{(\varrho)}}(st) - s\delta^{\|}_{T^{(\varrho)}}(t)],$$

and apply (28) to each term on the right. This gives

$$\delta^{\|}_{T^{(\varrho)}}(s^n t) = s^n\delta^{\|}_{T^{(\varrho)}}(t) + 48\sqrt{2}\{\delta^{\perp}_T(s^{n-1}t) + s\delta^{\perp}_T(s^{n-2}t) + \ldots + s^{n-1}\delta^{\perp}_T(t)\}$$

$$s^n\delta^{\|}_{T^{(\varrho)}}(t) + 48\sqrt{2}s^{n-1}\sum_{i=0}^{n}\frac{\delta^{\perp}_T(s^i t)}{s^i} \leqslant s^n\delta^{\|}_{T^{(\varrho)}}(t) + 48\sqrt{2}s^{n-1}\sum_{i=0}^{n}\frac{\hat{\delta}^{\perp}_T(s^i t)}{s^i},$$

where $\hat{\delta}^{\perp}_T$ is the concave triangular function defined by (20) corresponding to $\delta^{\perp}_T$. Since $\hat{\delta}^{\perp}_T(st)/s$ is a decreasing function of s, the sum on the right can be estimated by an integral in the usual way:

$$\delta^{\|}_{T^{(\varrho)}}(s^n t) \leqslant s^n\delta^{\|}_{T^{(\varrho)}}(t) + 48\sqrt{2}s^{n-1}\int_0^n \hat{\delta}^{\perp}_T(s^x t)/s^x\,dx,$$

which since $\hat{\delta}^{\perp}_T \leqslant 2\delta^{\perp}_T$, and through the substitution $u = s^{n-x}$ yields

$$\delta^{\|}_{T^{(\varrho)}}(s^n t) \leqslant s^n\delta^{\|}_{T^{(\varrho)}}(t) + \frac{48\sqrt{2}}{s\log 1/s}\int_{s^n}^1 \hat{\delta}^{\perp}_T(ts^n/u)\,du$$

$$\leqslant s^n\delta^{\|}_{T^{(\varrho)}}(t) + \frac{96\sqrt{2}}{s\log 1/s}\int_{s^n}^1 \delta^{\perp}_T(ts^n/u)\,du.$$

This in turn, setting $t = \varrho$ and replacing s^n by s becomes

$$(29)\qquad \delta^{\|}_{T^{(\varrho)}}(\varrho s) \leqslant \delta^{\|}_{T^{(\varrho)}}(\varrho)s + \frac{96\sqrt{2}}{s^{1/n}\log s^{-1/n}}\int_s^1 \delta^{\perp}_T(\varrho s/u)\,du, \qquad 0 < s < 1.$$

Now take a fixed s_0 with $0 < s_0 < 1$, and for any s in the interval $0 < s < s_0$ let n be the positive integer such that $s^{1/n} \leqslant s_0 < s^{1/(n+1)}$, that is, set

$$n \doteq \left[\log\frac{1}{s}\Big/\log\frac{1}{s_0}\right].$$

With this choice $s^{1/n} > s^{1+1/n} \geqslant s_0^2$ and $s^{1/n}\log 1/s^{1/n} > s_0^2 \log 1/s_0^2$, which inserted in (29) leads to

$$(30)\qquad \delta^{\|}_{T(\varrho)}(\varrho s) \leqslant \delta^{\|}_{T(\varrho)}(\varrho)s + \frac{96\sqrt{2}}{s_0^2\log 1/s_0^2}\int_s^1 \delta_T^{\perp}(\varrho s/u)\,du$$

$$\leqslant \delta^{\|}_{T(\varrho)}(\varrho)s + \frac{96\sqrt{2}}{s_0^2\log 1/s_0^2}\Big[\delta_T^{\perp}(\varrho)s + \int_s^1 \delta_T^{\perp}(\varrho s/u)\,du\Big], \qquad 0 < s < s_0.$$

Since Cs is weaker than any non-vanishing triangular function, (30) implies that as a function of s, $\delta^{\|}_{T(\varrho)}(\varrho s)$ is weaker than the triangular function constructed by the *-operation (cf. (22)) out of $\delta_T^{\perp}(\varrho s)$. As these are respectively equivalent to $\delta^{\|}_{T(\varrho)}$ and $(\delta_T^{\perp})^*$ the Lemma is proved.

The remark that (23), (27) and the relation $\delta_{T(\varrho)}(t) \leqslant \delta^{\|}_{T(\varrho)}(t) + \delta_T^{\perp}(t)$ transfer continuity from $\delta_T^{\perp}$ to $\delta_{T(\varrho)}$ leads to the important corollary:

THEOREM 1. *Uniform cross-continuity over a dense set $\mathscr{D}$ of an open convex set implies uniform continuity over $\mathscr{D}^{(\varrho)} = \{x \in \mathscr{D},\, d(x, \operatorname{bdry}\overline{\mathscr{D}}) > \varrho\}$, for any $\varrho > 0$.*

Next we shall exhibit a special class of mappings closely associated with triangular functions among which we shall find the examples and counterexamples that our theory requires.

LEMMA 8. *For any triangular function δ the modulus of continuity and the cross-modulus of continuity of the mapping*

$$(31)\qquad Tx = \frac{x}{\|x\|}\,\delta^*(\|x\|) \qquad (\delta^* \text{ defined by } (22)),$$

over any open ball about the origin are equivalent to δ^* and δ respectively.

Proof. It is sufficient to carry the demonstration for the unit ball only. By definition

$$\delta_T(t) = \sup_{\|x_1-x_2\|<t} \|Tx_1 - Tx_2\|$$
$$= \sup_{\|x_1-x_2\|<t} \{\delta^*(\|x_1\|)^2 + \delta^*(\|x_2\|)^2 - 2\cos\theta\,\delta^*(\|x_1\|)\,\delta^*(\|x_2\|)\}^{1/2},$$

where we have set $\cos\theta = \operatorname{Re}(x_1/\|x_1\|, x_2/\|x_2\|)$.

Through simple manipulations one may write

$$\delta_T(t) = \sup_{\|x_1-x_2\|<t} \{(\delta^*(\|x_2\|) - \delta^*(\|x_1\|))^2 + 4\sin^2\theta/2\,\delta^*(\|x_1\|)\,\delta^*(\|x_2\|)\}^{1/2}$$
$$\leqslant \sup_{\|x_1-x_2\|<t} \{\delta^*(\big|\|x_2\| - \|x_1\|\big|) + 2|\sin\theta/2|\sqrt{\delta^*(\|x_1\|)\delta^*(\|x_2\|)}\}$$
$$\leqslant \delta^*(t) + \sup_{\substack{(\|x_2\|-\|x_1\|)^2+\\ +4\sin^2\theta/2\|x_1\|\|x_2\|<t^2}} \{2|\sin\theta/2|\sqrt{\delta^*(\|x_1\|)\delta^*(\|x_2\|)}\}.$$

For fixed $\|x_1\|$ and $\|x_2\|$ the maximum value of $|\sin\theta/2|$ is either 1, if $\|x_1\| + \|x_2\| \leqslant t$, or $\{t_2 - (\|x_1\| - \|x_2\|)^2/4\|x_1\|\,\|x_2\|\}^{1/2}$ otherwise. Hence

$$(32)\qquad \delta_T(t) \leqslant \delta^*(t) + \sup_{\|x_1\|+\|x_2\|\leqslant t} 2\sqrt{\delta^*(\|x_1\|)\,\delta^*(\|x_2\|)} +$$
$$+ \sup_{\substack{\|\|x_2\|-\|x_1\|\|\\ \leqslant t \leqslant \|x_2\|+\|x_1\|}} \sqrt{\frac{t^2 - (\|x_2\| - \|x_1\|)^2}{\|x_1\|\,\|x_2\|}\,\delta^*(\|x_1\|)\,\delta^*(\|x_2\|)}$$
$$\leqslant 3\delta^*(t) + \sup_{\substack{\|\|x_2\|-\|x_1\|\|\\ \leqslant t \leqslant \|x_2\|+\|x_1\|}} \sqrt{\frac{t^2 - (\|x_2\| - \|x_1\|)^2}{\|x_1\|\,\|x_2\|}\,\delta^*(\|x_1\|)\,\delta^*(\|x_2\|)}.$$

To fix ideas we shall assume $\|x_1\| \leqslant \|x_2\|$ and shall treat the cases $\|x_1\| \leqslant t/2$ and $\|x_1\| > t/2$ separately. In the first case, $\|x_2\| - \|x_1\| \leqslant t \leqslant \|x_1\| + \|x_2\|$ imply $t/2 \leqslant \|x_2\| \leqslant 3t/2$ and $t - \|x_2\| \leqslant \|x_1\|$, which give

$$\frac{t^2 - (\|x_2\| - \|x_1\|)^2}{\|x_1\|\,\|x_2\|} = \frac{(t + \|x_2\| - \|x_1\|)(t - \|x_2\| + \|x_1\|)}{\|x_1\|\,\|x_2\|} \leqslant \frac{2t2\,\|x_1\|}{\|x_1\|\,\|x_2\|} = \frac{4t}{\|x_2\|} \leqslant 8.$$

Therefore,

$$(33)\qquad \delta_T(t) \leqslant 3\delta^*(t) + \sqrt{8\delta^*(t/2)\,\delta^*(3t/2)} \leqslant 8\delta^*(t).$$

On the other hand, $\|x_1\| \geqslant t/2$ implies $\|x_2\| \geqslant t/2$ and since $\delta^*(t)/t$ is a decreasing function of t,

$$(34)\qquad \delta_T(t) \leqslant 3\delta^*(t) + t\frac{\delta^*(t/2)}{t/2} \leqslant 5\delta^*(t).$$

Thus, in either case $\delta_T(t) \leqslant 8\delta^*(t)$ and δ_T is weaker than δ^*. But, as δ^* and δ_T coincide along any ray out of the origin, δ_T is also stronger than δ^*, and in consequence is equivalent to it, as the Lemma asserts.

Now we pass to the calculation of $\delta_T^{\perp}$. We have

$$(34)\qquad \delta_T^{\perp}(t) = \sup_{\|x_1-x_2\|<t} \|(Tx_1 - Tx_2)^{\perp}\|$$
$$= \sup_{\|x_1-x_2\|<t} \left\|\left[x_1\left(\frac{\delta^*(\|x_1\|)}{\|x_1\|} - \frac{\delta^*(\|x_2\|)}{\|x_2\|}\right) + (x_1 - x_2)\frac{\delta^*(\|x_2\|)}{\|x_2\|}\right]^{\perp}\right\|$$
$$= \sup_{\|x_1-x_2\|<t} \left\{\|x_1^{\perp}\|\left|\frac{\delta^*(\|x_1\|)}{\|x_1\|} - \frac{\delta^*(\|x_2\|)}{\|x_2\|}\right|\right\},$$

where $x_1^{\perp}$ denotes the component of x_1 perpendicular to $x_2 - x_1$. Assuming as before $\|x_1\| \leqslant \|x_2\|$ and majorating the right hand member of (34) with help of (24),

$$\delta_T^{\perp}(t) \leqslant \sup_{\|x_2-x_1\|<t} \left\{\delta(\|x_2\|)\,\|x_1^{\perp}\|\left(\frac{1}{\|x_1\|} - \frac{1}{\|x_2\|}\right)\right\},$$

and as $\|x_2-x_1\|<t$ implies $\|x_2\|-\|x_1\|<t$, and $\|x_1^{\perp}\|\leqslant\|x_1\|$,

$$\delta_T^{\perp}(t)\leqslant\sup_{\|x_2\|-\|x_1\|<t}\left\{\delta(\|x_2\|)\frac{\|x_2\|-\|x_1\|}{\|x_2\|}\right\}.$$

Now if $\|x_2\|\geqslant t$, $t\dfrac{\delta(\|x_2\|)}{\|x_2\|}\leqslant 2\delta(t)$ by (18), and the expression in braces does not exceed $2\delta(t)$, whereas if $\|x_2\|<t$ it is smaller than $\delta(t)$ so

$$\delta_T^{\perp}(t)\leqslant 2\delta(t). \tag{35}$$

Moreover, again by (24),

$$\delta_T^{\perp}(t)\geqslant\sup_{\|x_1-x_2\|<t}\left\{\delta(\|x_1\|)\|x_1^{\perp}\|\left(\frac{1}{\|x_1\|}-\frac{1}{\|x_2\|}\right)\right\}. \tag{36}$$

Choosing for x_1 and x_2 vectors of the form $x_1=tu$, $x_2=tu+tsv$, with u and v unitary and orthogonal, and s a positive real number smaller than 1, condition $\|x_1-x_2\|<t$ is satisfied, and (36) yields

$$\delta_T^{\perp}(t)\geqslant\delta(t)t\left(\frac{1}{t}-\frac{1}{t\sqrt{1+s^2}}\right)=\left(1-\frac{1}{\sqrt{1+s^2}}\right)\delta(t). \tag{37}$$

Estimates (35) and (37) establish the equivalence of $\delta_T^{\perp}$ and δ, completing the proof of the Lemma.

We are finally in a position to state our main result:

THEOREM 2. *Any δ-cross-continuous mapping ($\delta\not\equiv 0$) over a dense subset $\mathscr{D}$ of a convex open set is δ-continuous over $\mathscr{D}^{(p)}$ for any $p>0$ if and only if*

$$\overline{\lim_{t\to 0}}\frac{1}{\delta(t)}\int_t^1\delta(t/u)\,du<+\infty. \tag{38}$$

Proof. Condition (38) is equivalent to $\delta\sim\delta^*$ by (23). It is clear then that sufficiency follows from Lemma 7 and that Lemma 8 establishes necessity.

The functions t^ν, $0\leqslant\nu\leqslant 1$ — sort of "eigenfunctions" of the operator $\delta\to\delta^*$ — play a special role in this theory. The following lemma partially reflects this fact:

LEMMA 8. *If $\delta(t)$ is a triangular function,*

$$t^{1-\mu}\precsim\delta(t)\precsim t^{1-M} \tag{39}$$

for any couple of real numbers μ and M such that

$$0\leqslant\mu<\underline{\lim_{t\to 0}}\frac{\delta(t)}{\delta^*(t)}\leqslant\overline{\lim_{t\to 0}}\frac{\delta(t)}{\delta^*(t)}<M\leqslant 1. \tag{40}$$

Proof. If (40) is fulfilled there is a positive η such that $\mu\delta^*(t)\leqslant\delta(t)\leqslant M\delta^*(t)$ for $0<t\leqslant\eta$. Recalling that $\delta^*(t)$ satisfies the differential equation $d\delta^*(t)/dt=\big(\delta^*(t)-\delta(t)\big)/t$ one obtains for $0<t\leqslant\eta$,

$$(1-M)\frac{\delta^*(t)}{t}\leqslant\frac{d\delta^*(t)}{dt}\leqslant(1-\mu)\frac{\delta^*(t)}{t},$$

whence

$$\frac{d}{dt}\big(\delta^*(t)/t^{1-\mu}\big)\leqslant 0\leqslant\frac{d}{dt}\big(\delta^*(t)/t^{1-M}\big).$$

Therefore

$$\frac{\delta^*(\eta)}{\eta^{1-\mu}}t^{1-\mu}\leqslant\delta^*(t)\leqslant\frac{\delta^*(\eta)}{\eta^{1-M}}t^{1-M}$$

and

$$\frac{\mu\delta^*(\eta)}{\eta^{1-\mu}}t^{1-\mu}\leqslant\delta(t)\leqslant\frac{M\delta^*(\eta)}{\eta^{1-M}}t^{1-M},\qquad 0\leqslant t\leqslant\eta,$$

proving the Lemma's assertion.

A consequence useful to verify if a particular δ satisfies (38) or not can be derived at once from Lemma 8. Noticing that $\delta\sim\delta^*$ amounts to

$$\underline{\lim_{t\to 0}}\,\delta(t)/\delta^*(t)\neq 0$$

one sees that in order that $\delta\sim\delta^*$ it is necessary that $\delta>t^\nu$ for some ν in the interval $0\leqslant\nu<1$, and hence that $\delta<t^\nu$ for a ν, $0<\nu<1$, is sufficient for $\delta\sim\delta^*$. It is now easy to produce examples at either side of condition (38). A simple computation shows that if $\delta(t)=t^\nu$, $0\leqslant\nu<1$, then

$$\delta^*(t)=\frac{t^\nu-\nu t}{1-\nu},$$

and hence that (38) is met by these functions. On the other hand, the above remark indicates that (38) is not satisfied by function of the type

$$t\left(\log\frac{1}{t}\right)^{s_1}\left(\log\log\frac{1}{t}\right)^{s_2}\dots\left(\log\log\dots\log\frac{1}{t}\right)^{s_n}$$

for any choice of the integer n and the non-negative reals $s_1,s_2,\dots,s_n$. Particularizing this to the functions t^ν, $0\leqslant\nu\leqslant 1$, we may state:

COROLLARY 1. *If $\nu<1$ any cross-Hölder condition of exponent ν over $\mathscr{D}$ implies a Hölder condition of the same exponent over $\mathscr{D}^{(\varrho)}$, $\varrho>0$.*

COROLLARY 2. *The mapping*

$$Tx=x\log\frac{1}{\|x\|}$$

is cross-Lipschitzian but not Lipschitzian over any open ball about the origin.

For linear mappings our result may be phrased so as to say:

COROLLARY 3. *For any densely defined linear mapping* $T:\mathscr{H}\to\mathscr{H}$,

$$K\inf_{\lambda}\|T-\lambda I\|\leqslant\|T\|^{\perp}\leqslant\inf_{\lambda}\|T-\lambda I\|, \tag{41}$$

where

$$\|T\|^{\perp}=\sup_{\|x\|\leqslant 1}\{\|Tx\|^2-|(Tx,x)|^2\}^{1/2},$$

and K *is a positive constant independent of* T.

The best value of K is not known to the authors, they can only say that it is not smaller than $5^{-1/2}$. It may be shown that the value is one if the dimension of the space is 2 or if the mapping is normal. Is it always so?

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Properties of the orthonormal Franklin system, II

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1. Introduction. This is to continue the investigations undertaken in the paper [1]. Most of the results were announced without proofs in [2].

In Sections 3 and 4 sharp estimates from above and from below for the single Franklin functions and for the Dirichlet kernel of the Franklin system are obtained. Actually, we work out an explicite formula for the Dirichlet kernel.

Theorem 4 shows that the Fourier-Franklin series of an integrable function converges at each weak Lebesgue point. Using Theorem 3 and Lemma 8 one could deduce this result from the general criterion for singular integrals of Krein and Levin [10]. However, with the help of generalized Natanson Lemma, proved by Taberski in [15], the straightforward proof of the Theorem 4 becomes very simple and therefore it is presented here.

The next part of this paper deals with the best approximation and with the approximation by the partial sums of the Fourier-Franklin expansions in the $L_p\langle 0,1\rangle$ spaces. Most of the corresponding results for the space $C\langle 0,1\rangle$ were discussed in [1]. Theorem 9 shows that there is a non-trivial difference in the order of approximation of smooth functions by the partial sums of the Fourier-Franklin and Haar-Fourier expansions.

Theorem 12 extends the results obtained in [3] for the case $p=\infty$ to the Lipschitz classes in $L_p\langle 0,1\rangle$. It shows that there is a constructive linear isomorphism between any two L_p Lipschitz classes with the exponents α, $0<\alpha<1$. Again, the limit case $\alpha=1$, like for $p=\infty$ [3], is singular. We do not know whether the isomorphism exists for $1<p<\infty$ and $\alpha=1$. If $p=\infty$ and $\alpha=1$ then it exists but the known proof is not constructive [12].

Theorem 6 is a generalization of the main inequality proved in [1]. It plays an important role in the proofs of the absolute convergence theorems of Section 7.