References


Interpolation of additive functionals

by

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In this note a generalization of the theorem of Marur and Orlicz [11] is presented; the proof of the latter was simplified by Sikorski [13] and Pták [1]. We state first our extension and its proof and then explain how the previous statement may be obtained as a special case.

We consider a semi-group $S$, composition in $S$ being denoted by $+$, provided with a real functional $*$ subject to two conditions:

1. $\omega(s) \geq -\infty$ for $s \in S$.
2. $\omega(s) + \omega(t) \geq \omega(s + t)$ for $s, t \in S$.

In addition to $\omega$ there is given a real functional $L$ on $S$, restricted as follows:

3. $\omega(s) \geq -\infty$, $s \in S$, $L(s) = -\infty$.
4. If $(s_1, \ldots, s_n)$ is a finite sequence in $S$,
   $$\omega(s_1 + \ldots + s_n) \geq \sum_{i=1}^{n} L(s_i).$$

This condition is abbreviated: $\omega \geq L$.

There exists an additive functional $\xi$ on $S$ such that
$$\omega \geq \xi \geq L.$$

Proof. We begin with the observation that if $\omega = L$ in $S$, then $\omega$ is already additive. Let us exclude this and choose an element $s_0 \in S$ and a number $r$ such that $\omega(s_0) > r > L(s_0)$.

We claim now that either $A$ or $B$ holds, among the next two statements:

A. $\omega(ms_0 + s_1 + \ldots + s_n) \geq mr + \sum_{i=1}^{n} L(s_i)$, for any $m \geq 1$ and elements $s_1, \ldots, s_n$ in $S$.

B. $\omega(s) + m's' \geq \sum_{i=1}^{n} L(t_i)$, whenever $m's' = t_1 + \ldots + t_w$, $m' \geq 1$; $s \in S$; $t_1, \ldots, t_w \in S$. 

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In fact, if both $A$ and $B$ fail to be true for the instances given, multiplying the first by $m' \geq 1$ and the second by $m \geq 1$ we have, as $r \to -\infty$,

$$m^r \sum_{i=1}^{\infty} L(u_i) + m \sum_{i=1}^{\infty} L(v_i)$$

$$\geq m^r \omega(mu_0 + u_1 + \ldots + u_n) + \omega(s) \geq \omega(mu_0 + m' u_1 + \ldots + m' u_n + ma)$$

(by (2))

$$= \omega(mu_0 + m' u_1 + \ldots + m' u_n + ma).$$

This is in contradiction to (4): $\omega \gg L$, so that either $A$ or $B$ must always hold.

Let us suppose first that $A$ holds. Define $L'(s) = L(s)$ if $s \neq \emptyset$, and $L'(\emptyset) = r$. Then $\omega \gg L' \gg L$. For, in (4), if $s_i \neq \emptyset$ for $1 \leq i \leq n$, then $L(s_i) = L'(s_i), 1 \leq i \leq n$. If $s_i = \emptyset$ for only $1 \leq i \leq m < n$,

$$\omega(mu_0 + s_{m+1} + \ldots + s_n) \geq mr + \sum_{i=m+1}^{\infty} L(s_i) = \sum_{i=m+1}^{\infty} L(s_i).$$

The only remaining inequality to be proved is that $\omega(ma) \geq mr$ for any $m \geq 1$. For any $b$ such that $L(b) \geq -\infty$, and $n \geq 1, n \omega(ma) + + \omega(b) \geq \omega(ma_0 + b) \geq nmr + L(b)$. As $n \to \infty$ we obtain in the limit $\omega(ma) \geq mr$. The conclusion we emphasize is that if $A$ holds, $L'$ is a maximal element in the class $\{L'\}$ of functionals $L'$ such that $\omega \gg L'$.

If $B$ holds a similar conclusion can be obtained for $\omega$. In what follows the convention is adopted that $0 \cdot i = 1 = t$. The method for constructing a functional $\omega' \leq \omega$, $\omega'(\emptyset) \leq t$, subject to (1) and (2) is to consider at once all the restraints $\omega'$ must satisfy. These are the equations:

(E) Define

$$a_n + s = m, \quad n \geq 0, m \geq 1, t \in S \Rightarrow 0.$$  

Define

$$\omega'(t) = \inf \frac{1}{m} (mr + \omega(s)),$$

the infimum over all equations (E) involving $t$. Clearly $\omega'(t) \leq \omega(t)$; for every $n$, $(n+1)\omega'(a_n) \leq n\omega'(a_n)$. To show that $\omega'$ satisfies the hypothesis (2) suppose

$$n^r a_n + s' = m', t' = m, m'n^r$$

Then

$$\omega'(t + t') \leq \frac{n}{m} + \frac{n'}{m'} \langle r + \frac{1}{m} \omega(ms + m's) \rangle$$

$$\leq \frac{n}{m} r + \frac{1}{m} \omega(s) + \frac{n'}{m'} r + \frac{1}{m'} \omega(s').$$

Thus (2) holds.

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Condition (4), $\omega' \gg L$, is verified as follows. If $na_n + s = m(s_1 + \ldots + s_m)$ and $B$ holds,

$$\omega(s) + nr \geq m \sum_{i=1}^{\infty} L(s_i), \quad \frac{1}{m} \frac{\omega(s) + nr}{\omega(s)} \geq \sum_{i=1}^{\infty} L(s_i),$$

and finally

$$\omega'(s_1 + \ldots + s_m) \geq \sum_{i=1}^{\infty} L(s_i).$$

Conclusion: if $B$ holds, $\omega$ is not a minimal element in the family $\{\omega'\}$ of functions $\omega'$ such that $\omega' \gg L$.

Let us now apply Hausdorff's principle to the pairs of functionals $(\omega', L')$ such that $\omega' \gg \omega \gg L$, the partial order being $(\omega', L') \leq (\omega'', L'')$ if $\omega' \leq \omega''$ and $L' \gg L''$. We readily obtain a maximal element $(\omega_0, L_0)$; by what has gone before we must have $\omega_0 = L_0$ so that $\xi = \omega_0$ is additive. This completes the proof.

The theorem of Marzur and Orlicz may be derived from the present one as follows. We give an abstract set $Z$, a real function $c$ on $Z$, and a mapping $f$ of $Z$ into $S$ such that

(5)

$$\omega[f(z_1) + f(z_2) + \ldots + f(z_n)] \geq n \omega(c(z_1)),$$

for any finite sequence $(z_1, \ldots, z_n)$ in $Z$. Define

(6)

$$L(s) = \sup \{\omega(c): c \in Z, f(c) = c\},$$

if $c \in Z$, and $L(s) = -\infty$ if $c \notin Z$. Then conditions (1) and (5) yield conditions (3) and (4) for the functionals $\omega$ and $L$ on $S$. Thus there is an additive functional $\xi$ on $S$ such that $\omega \gg \xi \gg L$, whence $\omega[f(c)] = \xi[f(c)] \gg \xi[f(c)] > \xi[c] \gg c(c),$ for $c$ in $Z$.

If, in addition, $S$ is a real vector space and $\omega$ satisfies the condition

(7)

$$\omega(0) = \omega(0), \quad s \in S, \quad a \geq 0,$$

it is required that $\xi$ be homogeneous. If $s$ is fixed and $F(a) = \xi(a), a$ real, then $F$ is an additive transformation of the real numbers which is bounded in a neighborhood of $a = 0$. As is well known $F$ must then be homogeneous, so that $\xi(a) = F(a) = aF(1) = a\xi(1)$.

To deduce the well-known Hahn-Banach Theorem, we suppose that $Z$ is a linear subspace of $S$, that $f$ is the identity mapping of $Z$ into $S$, and that $c$ is a linear functional on $Z$. We find then a linear functional $\xi$ on $S$ such that $\omega \gg \xi$ and $\xi \gg c$ on $Z$. But then $-\xi \gg -c$ and $\xi = c$.

The Hahn-Banach Theorem itself may be generalized to semi-groups; we hope to announce this elsewhere.
Cross-continuity vs. continuity

by

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A mapping $T$ of the Hilbert space $\mathcal{H}$ (real or complex but not necessarily separable) into itself is said to be $\delta$-continuous or $\delta$-cross-continuous if there are constants $C$ and $C'$ such that for any couple of distinct points $x_1$ and $x_2$ in its domain

\begin{align}
(1) \quad & \|Tx_1 - Tx_2\| \leq C\delta(\|x_1 - x_2\|), \\
(2) \quad & \|\|Tx_1 - Tx_2\|\| \leq C'\delta(\|x_1 - x_2\|)
\end{align}

respectively, where $\delta(t)$ is a non-decreasing, non-negative, sub-additive function defined on the open positive half line ("triangular function"), and where

\[
(Tx_1 - Tx_2)^\perp = Tx_1 - Tx_2 - \frac{(Tx_1 - x_1)(x_2 - x_1)}{\|x_1 - x_2\|^2}.
\]

For $\delta(t) = t^r$, $0 < r < 1$, $\delta$-continuity coincides with the usual Hölder condition of exponent $r$, also called Lipschitz condition if $r = 1$, and $\delta$-cross-continuity yields the notion of cross-Hölder condition of exponent $r$ (or cross-Lipschitz condition if $r = 1$) introduced by one of the authors in a recent study of non-linear operators in Hilbert space [3], where the idea was immediately put to use without any further inquiry into its meaning. Clearly, since $(Tx_1 - Tx_2)^\perp$ is the component of $Tx_1 - Tx_2$ orthogonal to $x_1 - x_2$, a Hölder condition implies a cross-Hölder condition of the same exponent. At the beginning the apparent absence of counterexamples led to the conjecture that perhaps the converse of this was also true and the new notion altogether superfluous. Counter-examples such as $z\log(1/|z|)$ — which is cross Lipschitzian but not Lipschitzian — arrived to later showed the conjecture false but were insufficient to establish for which $\delta$'s any $\delta$-cross-Hölder mapping is $\delta$-Hölder, or more generally, for which $\delta$'s $\delta$-cross-continuity and $\delta$-continuity are equivalent, if ever. This broader question is our main concern in this article, to which we give the following somewhat unexpected answer (boundary behaviour being ignored): There is equivalence if and only if

\[
\lim_{t\to 0} \frac{1}{\delta(t)} \int_1^t \delta(t/u) du < \infty.
\]