References

[1] A. D. Alexandrov, *Additive set-functions in abstract spaces*, Matematicheskii Sbornik (Recueil Math.) 8 (1940), p. 307-342; 9 (1941), p. 563-621; 13 (1943), p. 169-243.

[2] J. Czipszer and L. Geher, *Extension of functions satisfying a Lipschitz condition*, Acta Math. Acad. Sci. Hungar. 6 (1955), p. 213-220.

[3] R. M. Dudley, *Lorentz-invariant Markov processes in relativistic phase space*, Arkiv før Matematik (to appear).

[4] — *Weak convergence of probabilities on non-separable metric spaces and empirical measures on Euclidean spaces*, Illinois Journal of Mathematics 10 (1966), p. 109-126.

[5] L. Kantorovitch and G. Rubinstein, *On a space of completely additive functions* (in Russian), Vestnik Leningrad Univ. 13.7 (Ser. Mat. Astr. 2) (1958), p. 52-59.

[6] J. L. Kelley, *General topology*, New York 1955.

[7] E. Marczewski and R. Sikorski, *Measures in non-separable metric spaces*, Coll. Math. 1 (1948), p. 133-139.

[8] Yu. V. Prokhorov, *Convergence of random processes and limit theorems in probability*, Theory of Probability and its Applications 1 (1956).

[9] R. Ranga Rao, *Some theorems on weak convergence of measures and applications*, Annals of Mathematical Statistics 33 (1962), p. 659-680.

[10] J. C. Shepherdson, *Inner models for set theory (II)*, Jour. Symb. Logic 17 (1952), p. 225-237.

[11] S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. 16 (1930), p. 140-150.

[12] — and J. C. Oxtoby, *On the existence of a measure invariant under a transformation*, Annals of Math. 40 (1939), p. 560.

[13] V. S. Varadarajan, *Weak convergence of measures on separable metric spaces*, Sankhya 19 (1958), p. 15-22.

[14] — *Measures on topological spaces* (in Russian), Matematicheskii Sbornik 55 (97) (1961), p. 35-100.

[15] Donald S. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. 111 (1964), p. 240-272.

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Interpolation of additive functionals

by

ROBERT KAUFMAN (Urbana, Ill.)

In this note a generalization of the theorem of Mazur and Orlicz ([1], p. 147) is presented; the proof of the latter was simplified by Sikorski [3] and Pták [2]. We state first our extension and its proof and then explain how the previous statement may be obtained as a special case.

We consider a semi-group S, composition in S being denoted by $x+y$, provided with a real functional ω subject to two conditions:

(1) $$\infty > \omega(s) \geqslant -\infty \quad \text{for} \quad s \in S,$$

(2) $$\omega(s)+\omega(t) \geqslant \omega(s+t) \quad \text{for} \quad s, t \in S.$$

In addition to ω there is given a real functional L on S, restricted as follows:

(3) $$\infty > L(s) \geqslant -\infty, \quad s \in S, L \not\equiv -\infty.$$

(4) If $\{s_1, \ldots, s_n\}$ is a finite sequence in S,
$$\omega(s_1+\ldots+s_n) \geqslant \sum_{i=1}^{n} L(s_i).$$

This condition is abbreviated: $\omega \gg L$.

THEOREM. *There exists an additive functional ξ on S such that $\omega \geqslant \xi \geqslant L$.*

Proof. We begin with the observation that if $\omega = L$ in S, then ω is already additive. Let us exclude this and choose an element $a_0 \in S$ and a number r such that $\omega(a_0) > r > L(a_0)$.

We claim now that either A or B holds, among the next two statements:

A. $\omega(ma_0+u_1+\ldots+u_n) \geqslant mr+\sum_{i=1}^{n} L(u_i)$, for any $m \geqslant 1$ and elements $u_1, \ldots, u_n$ in S.

B. $\omega(s)+m'r \geqslant \sum_{j=1}^{n'} L(t_j)$, whenever $m'a_0+s = t_1+\ldots+t_{n'}$, $m' \geqslant 1$; $s \in S$; $t_1, \ldots, t_{n'} \in S$.

In fact, if both A and B fail to be true for the instances given, multiplying the first by $m' \geqslant 1$ and the second by $m \geqslant 1$ we have, as $r > -\infty$,

$$m' \sum_{i=1}^{n} L(u_i) + m \sum_{j=1}^{n'} L(t_j)$$

$$> m'\omega(ma_0 + u_1 + \ldots + u_n) + m\omega(s) \geqslant \omega(m'ma_0 + m'u_1 + \ldots + m'u_n + ms),$$

$$\text{(by (2))} \qquad = \omega(mt_1 + \ldots + mt_{n'} + m'u_1 + \ldots + m'u_n).$$

This is in contradiction to (4): $\omega \geqslant L$, so that either A or B must always hold.

Let us suppose first that A holds. Define $L'(s) = L(s)$ if $s \in S$, $s \neq a_0$, and $L'(a_0) = r$. Then $\omega \geqslant L' \geqslant L$. For, in (4), if $s_i \neq a_0$ for $1 \leqslant i \leqslant n$, then $L(s_i) = L'(s_i)$, $1 \leqslant i \leqslant n$. If $s_i = a_0$ for only $1 \leqslant i \leqslant m < n$,

$$\omega(ma_0 + s_{m+1} + \ldots + s_n) \geqslant mr + \sum_{i=m+1}^{n} L(s_i) = \sum_{i=1}^{n} L'(s_i).$$

The only remaining inequality to be proved is that $\omega(ma_0) \geqslant mr$ for any $m \geqslant 1$. For any b such that $L(b) > -\infty$, and $n \geqslant 1$, $n\omega(ma_0) + \omega(b) \geqslant \omega(mna_0 + b) \geqslant nmr + L(b)$. As $n \to \infty$ we obtain in the limit $\omega(ma_0) \geqslant rm$. The conclusion we emphasize is that if A holds, L is *not* a *maximal* element in the class $\{L''\}$ of functionals L'' such that $\omega \geqslant L''$.

If B holds a similar conclusion can be obtained for ω. In what follows the convention is adopted that $0 \cdot s + t \equiv t$. The method for constructing a functional $\omega' \leqslant \omega$, $\omega'(a_0) \leqslant r$, subject to (1) and (2) is to consider at once all the restraints ω' must satisfy. These are the equations:

(E) $$na_0 + s = mt, \quad n \geqslant 0,\ m \geqslant 1,\ s, t \in S.$$

Define

$$\omega'(t) = \inf \frac{1}{m}\big(nr + \omega(s)\big),$$

the infimum over all equations (E) involving t. Clearly $\omega'(t) \leqslant \omega(t)$; for every n, $(n+1)\omega'(a_0) \leqslant nr + \omega(a_0)$. To show that ω' satisfies the hypothesis (2) suppose

(E′) $$n'a_0 + s^* = m't^*.$$

Then

$$(nm' + mn')a_0 + [ms^* + m's] = mm'(t + t^*);$$

$$\omega'(t + t^*) \leqslant \left(\frac{n}{m} + \frac{n'}{m'}\right)r + \frac{1}{mm'}\omega(ms^* + m's)$$

$$\leqslant \frac{n}{m}r + \frac{1}{m}\omega(s) + \frac{n'}{m'}r + \frac{1}{m'}\omega(s^*).$$

Thus (2) holds.

Condition (4), $\omega' \geqslant L$, is verified as follows. If $na_0 + s = m(s_1 + \ldots + s_{m'})$, and B holds,

$$\omega(s) + nr \geqslant m \sum_{j=1}^{m'} L(s_j), \qquad \frac{1}{m}\big(\omega(s) + nr\big) \geqslant \sum_{j=1}^{m'} L(s_j)$$

and finally

$$\omega'(s_1 + \ldots + s_{m'}) \geqslant \sum_{j=1}^{m'} L(s_j).$$

Conclusion: if B holds, ω is *not* a *minimal* element in the family $\{\omega''\}$ of functions ω'' such that $\omega'' \geqslant L$.

Let us now apply Hausdorff's principle to the pairs of functionals (ω', L') such that $\omega \geqslant \omega' \geqslant L' \geqslant L$, the partial order being $(\omega', L') \leqslant (\omega'', L'')$ if $\omega'' \leqslant \omega'$ and $L'' \geqslant L'$. We readily obtain a maximal element (ω_0, L_0); by what has gone before we must have $\omega_0 = L_0$ so that $\xi = \omega_0$ is additive. This completes the proof.

The theorem of Mazur and Orlicz may be derived from the present one as follows. We are given an abstract set Z, a real function c on Z, and a mapping f of Z into S such that

(5) $$\omega\big(f(z_1) + f(z_2) + \ldots + f(z_n)\big) \geqslant \sum_{i=1}^{n} c(z_i),$$

for any finite sequence $\{z_1, \ldots, z_n\}$ in Z. Define

(6) $$L(s) = \sup\{c(z) \colon z \in Z, f(z) = s\},$$

if $s \in f(Z)$, and $L(s) = -\infty$ if $s \notin f(Z)$. Then conditions (1) and (5) yield conditions (3) and (4) for the functionals ω and L on S. Thus there is an additive functional ξ on S such that $\omega \geqslant \xi \geqslant L$, whence $\omega\big(f(z)\big) \geqslant \xi\big(f(z)\big) \geqslant L\big(f(z)\big) \geqslant c(z)$, for z in Z.

If, in addition, S is a real vector space and ω satisfies the condition

(7) $$\omega(as) = a\omega(s), \quad s \in S,\ a \geqslant 0,$$

it is required that ξ be homogeneous. If s is fixed and $F(a) = \xi(as)$, a real, then F is an additive transformation of the real numbers which is bounded in a neighborhood of $a = 0$. As is well known F must then be homogeneous, so that $\xi(as) = F(a) = aF(1) = a\xi(s)$.

To deduce the well-known Hahn-Banach Theorem, we suppose that Z is a linear subspace of S, that f is the identity mapping of Z into S, and that c is a linear functional on Z. We find then a linear functional ξ on S such that $\omega \geqslant \xi$ and $\xi \geqslant c$ on Z. But then $-\xi \geqslant -c$ and $\xi = c$.

The Hahn-Banach Theorem itself may be generalized to semi-groups; we hope to announce this elsewhere.

References

[1] S. Mazur and W. Orlicz, *Sur les espaces métriques linéaires (II)*, Studia Math. 13 (1953), p. 137-179.

[2] V. Pták, *On a theorem of Mazur and Orlicz*, ibidem 15 (1956), p. 365-366.

[3] R. Sikorski, *On a theorem of Mazur and Orlicz*, ibidem 13 (1953), p. 180-182.

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Cross-continuity vs. continuity

by

F. GRÜNBAUM and E. H. ZARANTONELLO (Córdoba)

A mapping T of the Hilbert space $\mathscr{H}$ (real or complex but not necessarily separable) into itself is said to be *δ-continuous* or *δ-cross-continuous* if there are constants C and $C^{\perp}$ such that for any couple of distinct points x_1 and x_2 in its domain

$$\|Tx_1 - Tx_2\| \leqslant C\delta(\|x_1 - x_2\|), \tag{1}$$

$$\|(Tx_1 - Tx_2)^{\perp}\| \leqslant C^{\perp}\delta(\|x_1 - x_2\|) \tag{2}$$

respectively, where $\delta(t)$ is a non-decreasing, non-negative, sub-additive function defined on the open positive half line ("triangular function"), and where

$$(Tx_1 - Tx_2)^{\perp} = Tx_1 - Tx_2 - \frac{(Tx_1 - Tx_2, x_1 - x_2)}{\|x_1 - x_2\|^2}(x_1 - x_2).$$

For $\delta(t) = t^{\nu}$, $0 \leqslant \nu \leqslant 1$, δ-continuity coincides with the usual Hölder condition of exponent ν, also called *Lipschitz condition* if $\nu = 1$, and δ-cross-continuity yields the notion of cross-Hölder condition of exponent ν (or cross-Lipschitz condition if $\nu = 1$) introduced by one of the authors in a recent study of non-linear operators in Hilbert space [3], where the idea was immediately put to use without any further inquiry into its meaning. Clearly, since $(Tx_1 - Tx_2)^{\perp}$ is the component of $Tx_1 - Tx_2$ orthogonal to $x_1 - x_2$, a Hölder condition implies a cross-Hölder condition of the same exponent. At the beginning the apparent absence of counter-examples led to the conjecture that perhaps the converse of this was also true and the new notion altogether superfluous. Counter-examples such as $x\log(1/\|x\|)$ — which is cross Lipschitzian but not Lipschitzian — arrived to later showed the conjecture false but were insufficient to establish for which ν's any ν-cross-Hölder mapping is ν-Hölder, or more generally still, for which δ's δ-cross-continuity and δ-continuity are equivalent, if ever. This broader question is our main concern in this article, to which we give the following somewhat unexpected answer (boundary behaviour being ignored): There is equivalence if and only if

$$\varlimsup_{t\to 0} \frac{1}{\delta(t)} \int_t^1 \delta(t/u)\,du < \infty.$$