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Convergence of Baire measures

by

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Introduction. For any topological space S, let $\mathscr{C}(S)$ be the Banach space of bounded real-valued continuous functions f on S, with the supremum norm

$$\|f\|_\infty = \sup\{|f(x)|: x \in S\}.$$

A *pseudo-metric space* is a pair (S, d) where S is a set and d is a non-negative real-valued function on $S \times S$ such that for all x, y and z in S, $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leqslant d(x, y) + d(y, z)$. d then defines a topology on S in the usual way.

If (S, d) is a pseudo-metric space, then a real-valued function f on S will be called *Lipschitzian* if

$$\|f\|_L = \sup\{|f(x)-f(y)|/d(x, y): d(x, y) \neq 0\} < \infty.$$

Then $\mathrm{BL}(S, d)$ will denote the Banach space of all bounded Lipschitzian functions f on S, with the norm

$$\|f\|_{\mathrm{BL}} = \|f\|_L + \|f\|_\infty.$$

This paper is mainly concerned with weak-star convergence in the space $\mathscr{M}(S)$ of all finite, signed Baire measures on S (i.e. pointwise convergence on $\mathscr{C}(S)$), and its close but rather complicated relations with convergence for the norm $\| \quad \|_{\mathrm{BL}}^*$ in the dual space of $\mathrm{BL}(S, d)$. The results of § 3 below show that the $\| \quad \|_{\mathrm{BL}}^*$ metric metrizes a weak-star structure (topology or uniformity) whenever it is metrizable on $\mathscr{M}(S)$, or on the subset $\mathscr{M}^+(S)$ of non-negative measures, or the subset $\mathscr{P}(S)$ of probability measures on a separable metric space S. (For S completely regular, Hausdorff, but not metrizable, none of the weak-star structures in metrizable, since the set of unit point masses is homeomorphic to S (see e.g. Varadarajan [14], Teorema 13, p. 621).

The best-known metrizations of weak-star convergence have appeared in probability theory, in the work of Prokhorov [8] for complete separable metric spaces and that of P. Lévy for the real line. These metrics are suitable only on $\mathscr{M}^+(S)$. The BL* metric has the further advantage of being defined by a norm on a linear space. A norm on $\mathscr{M}(S)$ for S *compact*, very close to the BL* norm in that case, has been defined and similarly applied by Kantorovich and Rubinstein [5].

Weak-star convergence of measures seems to have been first studied extensively by Alexandrov [1]. We shall quote several of his results, as well as some from the more recent long paper by Varadarajan [14]. A related theorem has also been proved by Ranga Rao [9], as will be indicated below.

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1. Preliminaries. Let S be a topological space. There is a smallest σ-algebra $\mathscr{B}(S)$ of subsets of S for which all members of $\mathscr{C}(S)$ (or equivalently, all real continuous functions) are measurable. Elements of $\mathscr{B}(S)$ will be called *Baire sets*. $\mathscr{M}(S)$ is the set of all finite, real-valued, countably additive set functions (signed measures) on $\mathscr{B}(S)$.

Given a topological space S, a signed measure μ on S will be called *separable* if for every continuous pseudo-metric d on S, μ is concentrated in a subset of S which has a countable set dense for d. The set of all separable elements of $\mathscr{M}(S)$ will be denoted $\mathscr{M}_s(S)$.

It is consistent with all the usual axioms of set theory to assume that $\mathscr{M}_s(S) = \mathscr{M}(S)$ for every topological space S. (A cardinal number α is said to be of *measure zero* if every finite, countably additive measure on all subsets of a set of cardinal α, giving points measure zero, is identically zero. It was proved by Marczewski and Sikorski [7] that if a metric space has a non-separable finite Baire measure, then it has a subset with discrete relative topology and cardinality not of measure zero. The proposition that all cardinals have measure zero is consistent according to Ulam [11] and Shepherdson [10]. Also, the continuum hypothesis implies that the cardinal of the continuum has measure zero.)

One does have occasion to consider non-separable probability measures, defined on suitable sub-σ-algebras of $\mathscr{B}(S)$ [4].

$\mathscr{M}(S)$ is naturally a subspace of the dual space $\mathscr{C}(S)^*$. The weakest topology on $\mathscr{C}(S)^*$ (or any subset of it) making continuous each linear functional

$$T \to T(f), \quad f \in \mathscr{C}(S),$$

will be called the *weak-star* or weak* topology. $\mathscr{M}^+(S)$ denotes the set of non-negative elements of $\mathscr{M}(S)$, and we let

$$\mathscr{M}_s^+(S) = \mathscr{M}^+(S) \cap \mathscr{M}_s(S).$$

For any μ in $\mathscr{M}(S)$, there is the Jordan decomposition

$$\mu = \mu^+ - \mu^-$$

where μ^+ and μ^- are mutually singular elements of $\mathscr{M}^+(S)$, uniquely determined by these conditions. We let

$$|\mu| = \mu^+ + \mu^-.$$

$\mathscr{N}(S)$ will denote the class of all sets of the form $\{x: f(x) = 0\}$ for f in $\mathscr{C}(S)$. Clearly $\mathscr{N}(S) \subset \mathscr{B}(S)$, and $\mathscr{N}(S)$ is closed under finite unions and countable intersections (note that $\{x: f(x) = 0\} = \{x: |f|(x) = 0\}$). If S is pseudo-metrizable, $\mathscr{N}(S)$ is precisely the class of all closed sets. The following two facts are known (see e.g. Alexandrov [1] or Varadarajan [14], Teorema 18, p. 45, and Teorema 6, p. 39):

LEMMA 1. *For any μ in $\mathscr{M}^+(S)$ and A in $\mathscr{B}(S)$,*

$$\mu(A) = \sup\{\mu(B): B \subset A, B \in \mathscr{N}(S)\}.$$

LEMMA 2. *For any μ in $\mathscr{M}(S)$ and f in $\mathscr{C}(S)$, let $L(f) = \int f d\mu$. Then $\|L\|^* = |\mu|(S)$ where $\| \quad \|^*$ is the norm in $\mathscr{C}(S)^*$.*

A measure μ in $\mathscr{M}(S)$ will be called *tight* if for every $\varepsilon > 0$ there is a compact set K such that

$$|\mu|(S \sim K) < \varepsilon.$$

Then, by Lemma 1, for any $\varepsilon > 0$ and A in $\mathscr{B}(S)$ there is a compact set C such that

$$|\mu|(A \sim C) < \varepsilon.$$

The class of all tight measures will be denoted by $\mathscr{M}_t(S)$. Note that $\mathscr{M}_t(S) \subset \mathscr{M}_s(S)$. $\mathscr{M}_t^+(S)$ has the obvious meaning.

Now suppose (S, d) is a pseudo-metric space. Then $\mathrm{BL}(S) = \mathrm{BL}(S, d)$ is a Banach space. Lemmas 3-8 below present facts we shall need about this space. (Lemmas 3 and 5 also appear in Sherbert [15].) First, $\mathrm{BL}(S, d)$ is a Banach algebra:

LEMMA 3. *For any f and g in $\mathrm{BL}(S) = \mathrm{BL}(S, d)$, fg is in $\mathrm{BL}(S)$ and $\|fg\|_{BL} \leqslant \|f\|_{BL}\|g\|_{BL}$.*

Proof. Clearly $\|fg\|_\infty \leqslant \|f\|_\infty \|g\|_\infty$. For any x and y in S,

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &\leqslant |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\leqslant (\|f\|_\infty \|g\|_L + \|f\|_L \|g\|_\infty)\, d(x, y).\end{aligned}$$

The conclusion follows.

For any real-valued functions $f_1, \ldots, f_n$ on a set S, we let

$$f_1 \wedge \ldots \wedge f_n = \min(f_1, \ldots, f_n),$$
$$f_1 \vee \ldots \vee f_n = \max(f_1, \ldots, f_n).$$

LEMMA 4. *For any real-valued functions $f_1, \ldots, f_n$ on a pseudo-metric space (S, d), if*

$$g = f_1 \wedge \ldots \wedge f_n \quad \text{and} \quad h = f_1 \vee \ldots \vee f_n,$$

then

$$\max(\|g\|_L, \|h\|_L) \leqslant \max_{1 \leqslant i \leqslant n} \|f_i\|_L.$$

Proof. It suffices to prove the Lemma for $n = 2$ since induction then gives the general case. By symmetry, it is enough to prove that for any functions φ and ψ,

$$\|\varphi \wedge \psi\|_L \leqslant \max(\|\varphi\|_L, \|\psi\|_L) = M.$$

For any x and y in S,

$$\max(|\varphi(x) - \varphi(y)|, |\psi(x) - \psi(y)|) \leqslant Md(x, y).$$

If $(\varphi \wedge \psi)(x) = \varphi(x)$, $(\varphi \wedge \psi)(y) = \psi(y)$, then

$$\varphi(x) - \varphi(y) \leqslant \varphi(x) - \psi(y) \leqslant \psi(x) - \psi(y),$$

so $|\varphi(x) - \psi(y)| \leqslant Md(x, y)$.

In the preceding, x and y can be interchanged. Thus the proof is complete.

Suppose given a Lipschitzian function f on a subset T of a metric space (S, d). Then f can be extended to all of S without increasing $\|f\|_L$ (Czipszer and Geher [2]; one proof is a simpler form of the proof of the Hahn-Banach theorem). Now if f is in $\mathrm{BL}(T)$, and g is an extension to S with $\|g\|_L = \|f\|_L$, we let

$$h = (g \vee -\|f\|_\infty) \wedge \|f\|_\infty;$$

then by Lemma 4, $\|h\|_{\mathrm{BL}} = \|f\|_{\mathrm{BL}}$. Thus we have

LEMMA 5. *Given a metric space (S, d), $T \subset S$, and $f \in \mathrm{BL}(T, d)$, f has an extension h in $\mathrm{BL}(S, d)$ with $\|h\|_{\mathrm{BL}} = \|f\|_{\mathrm{BL}}$.*

Each μ in $\mathscr{M}(S)$ defines an element of the dual space $\mathrm{BL}(S)^*$ with the norm

$$\|\mu\|_{\mathrm{BL}}^* = \sup\{|\int f d\mu| : \|f\|_{\mathrm{BL}} = 1\}.$$

In fact, the natural map of $\mathscr{M}(S)$ into $\mathrm{BL}(S)^*$ is one-to-one:

LEMMA 6. *For any $\mu \neq 0$ in $\mathscr{M}(S)$, $\|\mu\|_{\mathrm{BL}}^* > 0$.*

Proof. Let A be a Baire set such that $\mu(A) = \mu^+(S) = \alpha$ (Hahn decomposition). If $\alpha = 0$, the result is clear. If $\alpha > 0$, we take closed sets B and C with $B \subset A$, $C \subset S \sim A$, $\mu^+(A \sim B) < \alpha/4$, and $\mu^-((S \sim A) \sim C) < \alpha/4$. Since B and C are disjoint, there exist closed sets B_n, $n = 1, 2, \ldots$, with $B_n \uparrow B$ and $d(x, y) \geqslant 1/n$ for all x in B_n and y in C. Thus there exist f_n in $\mathrm{BL}(S, d)$ with $\|f_n\|_\infty \leqslant 1$ for all n, $f_n \equiv 1$ on B_n, and $f_n \equiv 0$ on C. Thus

$$\lim_{n \to \infty} \int f_n d\mu \geqslant \mu^+(B) - \alpha/2 \geqslant \alpha/4,$$

so $\|\mu\|_{\mathrm{BL}}^* > 0$, q.e.d.

LEMMA 7. *If S is a compact metric space, $\mathrm{BL}(S)$ is dense in $\mathscr{C}(S)$ for $\| \ \|_\infty$.*

Proof. $\mathrm{BL}(S)$ is an algebra by Lemma 3, contains the constants, and separates points by the extension property (Lemma 5). Thus the Stone-Weierstrass theorem yields the result.

LEMMA 8. *For any metric space (S, d), the closure of $\mathrm{BL}(S, d)$ for $\| \ \|_\infty$ is the space $\mathscr{UC}(S)$ of all bounded uniformly continuous real-valued functions on S.*

Proof. It is clear that $\mathscr{UC}(S)$ is closed for $\| \ \|_\infty$ and includes $\mathrm{BL}(S, d)$. To show that $\mathrm{BL}(S, d)$ is dense, let $f \in \mathscr{UC}(S, d)$. For $n = 1, 2, \ldots$, let A_n be a maximal subset of S such that $d(x, y) \geqslant 1/n$ whenever $x \neq y$, $x, y \in A_n$. Let $f_n = f$ on A_n, and extend f_n to all of S without increasing $\|f_n\|_{\mathrm{BL}}$. Given $\varepsilon > 0$, take $m > 0$ such that $d(x, y) < 1/m$ implies $|f(x) - f(y)| < \varepsilon$. For any z in S and $n \geqslant m$, choose x in A_n such that $d(x, z) < 1/n$. Then

$$|f(z) - f_n(z)| \leqslant |f(z) - f(x)| + |f_n(z) - f_n(x)| \leqslant \varepsilon + \|f_n\|_L/n.$$

We next show that $\|f_n\|_L/n \to 0$ as $n \to \infty$. Suppose not; then for some $\delta > 0$, n in an infinite set N, and $x_n, y_n \in A_n$,

$$|f(x_n) - f(y_n)|/nd(x_n, y_n) \geqslant \delta.$$

If $d(x_n, y_n) \to 0$ as $n \to \infty$ through N, then $f(x_n) - f(y_n) \to 0$ by uniform continuity, while $nd(x_n, y_n) \geqslant 1$, giving a contradiction. Thus $d(x_n, y_n) \geqslant \gamma > 0$ for infinitely many n in N, and

$$|f(x_n) - f(y_n)|/nd(x_n, y_n) \leqslant 2\|f\|_\infty/n\gamma < \delta$$

for some large enough n in N, again a contradiction. The proof is complete.

Note that Lemma 7 follows from Lemma 8.

Let (S, d) be a metric space. Then the d uniformity is discrete if and only if there do not exist points $x_n \neq y_n$ in S with $d(x_n, y_n) \to 0$. In this case we shall call d *uniformly discrete*. A topological space is metrizable by a uniformly discrete metric if and only if it is discrete, but a metric defining the discrete topology need not be uniformly discrete.

LEMMA 9. *If* (S, d) *is a metric space, the following are equivalent:*

(a) $\mathscr{C}(S) = \mathscr{U}\mathscr{C}(S)$.

(b) *If* $\{x_n\}$ *is a sequence of distinct points such that* $d(x_{2n}, x_{2n+1}) \to 0$ *as* $n \to \infty$, *then* $\{x_n\}$ *has a convergent subsequence.*

(c) *There is a compact set* $K \subset S$ *such that for every* $\varepsilon > 0$, *the set of points at distance* ε *or more from* K *is uniformly discrete.*

Proof. Suppose (b) is false for a sequence $\{x_n\}$, which then forms a discrete, closed set. Then we can let $f(x_{2n}) = 1$ and $f(x_{2n+1}) = -1$ for all n, and extend f to a bounded continuous function on S which is not uniformly continuous. Thus (a) implies (b).

Next, assume (b). Let K be the set of all accumulation points (non-isolated points) of S. Then K is compact, and clearly (c) holds for this K.

Finally, it is easy to see that (c) implies (a) (cf. Lemma 1 of [4]), so the proof is complete.

A metric space S satisfying (a), (b) or (c) of Lemma 9 will be called a *uniform continuity space* (u. c. space).

2. Convergence of measures. UW* will denote the weak-star uniformity on $\mathscr{M}(S)$ or any subset $\mathscr{A}$ of it. A base of UW* consists of all sets

$$\{\langle\mu, \nu\rangle \in \mathscr{A}\times\mathscr{A}: \left|\int f_j d(\mu-\nu)\right| < \varepsilon,\ j = 1, \ldots, n\}$$

where $\varepsilon > 0$ and $f_1, \ldots, f_n$ is any finite set of elements of $\mathscr{C}(S)$. TW* will denote the weak-star topology. UBL* and TBL* will denote the uniformity and topology, respectively, defined by $\|\ \|_{BL}^*$, and likewise UV* and TV* for the total variation norm $\|\ \|_\infty^*$ ($\|\mu\|_\infty^* = |\mu|(S)$). Sequences $\{\mu_n: n = 1, 2, \ldots\}$ will be written simply $\{\mu_n\}$.

The following basic result is due to Alexandrov [1], 1943, Theorem 1, p. 202, and Theorem 3, p. 209. It has also been proved by Varadarajan [14], Teorema 19, p. 68, and we give it approximately as in the latter reference:

THEOREM 1. *Let* S *be any topological space, and let* $\{\mu_n\}$ *be a* UW*-*Cauchy sequence of elements of* $\mathscr{M}(S)$. *Then*

(a) $\{\mu_n\}$ *converges for* TW* *to an element of* $\mathscr{M}(S)$. ($\mathscr{M}(S)$ *is* UW*-*sequentially complete*)

(b) *Suppose* $C_k \in \mathscr{N}(S)$, $k = 1, 2, \ldots$, C_k *is included in the interior of* C_{k+1} *for each* k, *and the union of the* C_k *is* S. *Then for any* $\varepsilon > 0$ *there is a* k *such that for all* n,

$$|\mu_n|(S \sim C_k) < \varepsilon.$$

If S is a non-compact completely regular Hausdorff space, then S is a proper dense subset of its Stone-Čech compactification $\check{S}$, and each member of $\mathscr{C}(S)$ extends to an element of $\mathscr{C}(\check{S})$, so there exist weak-star Cauchy *nets* or filters of elements of $\mathscr{M}(S)$ which do not converge to elements of $\mathscr{M}(S)$.

Let (S, d) be a pseudo-metric space. If $\delta \geqslant 0$, a subset B of S is called *δ-totally bounded* if there is a finite set $F \subset B$ such that for each y in B, $d(x, y) \leqslant \delta$ for some x in F. B is *totally bounded* if and only if it is δ-totally bounded for every $\delta > 0$. (A set is *compact* if and only if it is complete and totally bounded.) The following is a consequence of Theorem 1:

THEOREM 2. *If* (S, d) *is a pseudo-metric space and* $\{\mu_n\}$ *is a* TW*-*convergent sequence of elements of* $\mathscr{M}_s(S)$, *then for every* $\varepsilon > 0$ *there is a totally bounded set* B *such that* $|\mu_n|(S \sim B) < \varepsilon$ *for all* n.

Proof. We may assume S is separable since the union of the supports of the μ_n is separable. It suffices to show that, given a positive integer n, there is a $1/n$-totally bounded set B_n with

$$|\mu_n|(S \sim B_n) < \varepsilon/2^n \quad \text{for all } n$$

(since then we can let $B = \bigcap B_n$). Let $\{x_i\}$ be dense in S. For each positive integer k let

$$C_k = \{x: d(x, x_i) \leqslant (k-1)/kn \text{ for some } i \leqslant k\}.$$

Then the C_k satisfy the conditions of Theorem 1 and are $1/n$-totally bounded for all k, so the proof is complete.

Since total boundedness is not a topological property (in fact, every separable metric space is homeomorphic to a totally bounded one), Theorem 2 is of most interest in complete spaces where "totally bounded" can be replaced by "compact". Thus we have a

COROLLARY (Ulam and Oxtoby [12]). *If* S *is a complete metric space,* $\mathscr{M}_s(S) = \mathscr{M}_t(S)$.

We shall call a topological space S *inner regular* if $\mathscr{M}_s(S) = \mathscr{M}_t(S)$. We then have (see Varadarajan [14], b, p. 97):

THEOREM 3. *A separable metric space* S *is inner regular if and only if* S *is Carathéodory measurable in its completion* $\bar{S}$ *for every* μ *in* $\mathscr{M}^+(\bar{S})$.

It follows from Theorem 3 that S is inner regular if it is a Borel or analytic set in its completion. Using the axiom of choice, one can obtain an S which is not inner regular by taking a subset of the interval (0,1) which is not Lebesgue measurable.

A set B in a topological space T is called *relatively compact* if its closure $\bar{B}$ is compact. B is *sequentially relatively compact* if every sequence in B has a subsequence convergent to a point of T. The following result is due to Varadarajan [14], and generalizes a result of Prokhorov [8]:

THEOREM 4. *Let (S, d) be an inner regular separable metric space and $B \subset \mathcal{M}(S)$. Then the following are equivalent:*

(I) *B is TW^*-relatively compact.*

(II) *B is TW^*-sequentially relatively compact.*

(III) *$\sup\{|\mu|(S): \mu \in B\}$ is finite, and for every $\varepsilon > 0$ there is a compact set K such that $|\mu|(S \sim K) < \varepsilon$ for all μ in B.*

Proof. (I) is equivalent to (II) in any separable metric space ([14], Teorema 27, p. 76). (I) is equivalent to (III) under our hypotheses ([14], Teorema 2, p. 96).

Thus, in Theorem 2, "totally bounded" can be replaced by "compact" for a large class of incomplete spaces S.

We next have a joint sequential continuity result:

THEOREM 5. *Suppose S is an inner regular separable metric space, $\mu_n \in \mathcal{M}(S)$, $\mu_n \to \mu$ for TW^*, $f_n \in \mathcal{C}(S)$, $\|f_n\|_\infty \leqslant M < \infty$ for all n, and $f_n \to f$ uniformly on compact sets. Then*

$$\int f_n d\mu_n \to \int f d\mu.$$

Proof. There is an N such that

$$|\mu_n|(S) \leqslant N < \infty \quad \text{for all } n$$

(Banach-Steinhaus theorem, or Theorem 4). Also, given $\varepsilon > 0$ there is a compact set $K \subset S$ such that

$$|\mu_n|(S \sim K) < \varepsilon/2M$$

for all n. Then

$$\left|\int f_n d\mu_n - \int f d\mu\right| \leqslant \left|\int f d(\mu_n - \mu)\right| + \left|\int_K (f_n - f)\, d\mu_n\right| + \left|\int_{S\sim K} (f_n - f)\, d\mu_n\right|.$$

The first two terms approach zero as $n \to \infty$, and the last is at most ε. Letting $\varepsilon \downarrow 0$, the proof is finished.

We now begin our investigation of the relations between weak-star and $\|\ \|^*_{\mathrm{BL}}$ convergence of measures.

THEOREM 6. *Let (S, d) be a pseudo-metric space, $\mu_n \in \mathcal{M}_s(S)$, and $\mu_n \to \mu$ weak-star. Then $\|\mu_n - \mu\|^*_{\mathrm{BL}} \to 0$.*

Proof. First, we can identify points at zero distance, and assume (S, d) is a metric space. Let $\bar{S}$ be the completion of S. Then the μ_n and μ naturally define elements $\bar{\mu}_n$ and $\bar{\mu}$ of $\mathcal{M}(S)$, with $\bar{\mu}_n \to \bar{\mu}$ weak-star. The spaces $\mathrm{BL}(S, d)$ and $\mathrm{BL}(\bar{S}, d)$ are naturally isometric. Thus we can assume S is complete. Since $\mu_n \in \mathcal{M}_s(S)$ for all n, we can assume S is separable.

We want to show that

$$\sup\left(\left|\int f d(\mu_n - \mu)\right|: \|f\|_{\mathrm{BL}} \leqslant 1\right) \to 0.$$

Suppose not. Then, passing to a subsequence, there exist an $\varepsilon > 0$ and f_n with $\|f_n\|_{\mathrm{BL}} \leqslant 1$ such that

$$\left|\int f_n d\mu_n - \int f_n d\mu\right| \geqslant \varepsilon, \qquad n = 1, 2, \ldots$$

Taking another subsequence, we can assume $f_n(x)$ converges to some $f(x)$ at each point x of a countable dense set in S. Then since $\|f_n\|_{\mathrm{BL}} \leqslant 1$, we have $f_n \to f$ uniformly on compact sets and $\|f_n\|_\infty \leqslant 1$ for all n. Thus by Theorem 5,

$$\int f_n d\mu_n \to \int f d\mu, \quad \text{and} \quad \int f_n d\mu \to \int f d\mu.$$

Thus we have a contradiction, completing the proof.

Sequences can be replaced by general nets in Theorem 6 if and only if S is finite; see Theorem 17 (k) below.

Here is a result proved for non-negative measures on (separable) metric spaces by Ranga Rao [9]:

THEOREM 7. *Let S be any topological space, and let μ_n in $\mathcal{M}_s(S)$ converge weak-star to μ. Then $\mu_n \to \mu$ uniformly on any equicontinuous and uniformly bounded class $\mathcal{F}$ of functions on S.*

Proof. Let $d(x, y) = \sup\{|f(x) - f(y)|: f \in \mathcal{F}\}$. Then d is a continuous pseudo-metric on S, and for every $f \in \mathcal{F}$, $f \in \mathrm{BL}(S, d)$ and $\sup\{\|f\|_{\mathrm{BL}}: f \in \mathcal{F}\} < \infty$. Thus Theorem 6 applies.

The converse of Theorem 6 is true if and only if S is uniformly discrete (Theorem 11 below), but it holds for non-negative measures:

THEOREM 8. *Let (S, d) be any metric space, $\mu_n, \mu \in \mathcal{M}^+(S)$, and $\|\mu_n - \mu\|^*_{\mathrm{BL}} \to 0$ as $n \to \infty$. Then $\mu_n \to \mu$ weak-star.*

Proof. $\mu_n(S) \to \mu(S)$ since the constant 1 belongs to $\mathrm{BL}(S, d)$. Thus by a well-known characterization of weak-star convergence ([1], 1943, p. 180) it suffices to show that for any open set U in S, $\underline{\lim}\, \mu_n(U) \geqslant \mu(U)$. Let F_m be the closed set of points x such that $d(x, y) \geqslant 1/m$ for all $y \notin U$. Then $\{F_m\}$ is an increasing sequence of closed sets whose union is U. By Lemmas 4 and 5 there is an f_m in $\mathrm{BL}(S, d)$ such that $f_m \equiv 1$ on F_m, $f_m \equiv 0$ outside U, and $0 \leqslant f_m \leqslant 1$ everywhere. Given $\varepsilon > 0$, choose m so that $\mu(F_m) > \mu(U) - \varepsilon/2$, implying

$$\int f_m d\mu > \mu(U) - \varepsilon/2.$$

We can then choose n_0 so that for $n \geqslant n_0$,

$$\left|\int f_m d(\mu_n - \mu)\right| < \varepsilon/2,$$

$$\mu_n(U) \geqslant \int f_m d\mu_n \geqslant \mu(U) - \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we have the result.

THEOREM 9. *Suppose (S, d) is a complete metric space, $\mu_n \in \mathcal{M}_s^+(S)$, and $\{\mu_n\}$ is a Cauchy sequence for UBL*. Then μ_n converges weak-star to some μ in $\mathcal{M}_s^+(S)$, so $\mu_n \to \mu$ for TBL*.*

Proof. We may assume S has a countable dense subset $\{x_i\}$. Given $\varepsilon > 0$, let

$$f_j(x) = \begin{cases} 0 & \text{if} \quad d(x, x_i) \leqslant \varepsilon/2 \text{ for some } i \leqslant j, \\ 1 & \text{if} \quad d(x, x_i) \geqslant \varepsilon \text{ for all } i \leqslant j. \end{cases}$$

Then for each j, $\|f_j\|_\infty = 1$ and $\|f_j\|_L \leqslant 2/\varepsilon$. By Lemmas 4 and 5, the f_j can be defined on all of S so as to form a decreasing sequence of functions with

$$\|f_j\|_{\mathrm{BL}} \leqslant 1 + 2/\varepsilon \quad \text{for all} \quad j.$$

Given $\delta > 0$, we choose m so that

$$\|\mu_n - \mu_m\|_{\mathrm{BL}}^* \leqslant \delta\varepsilon/3(\varepsilon+2)$$

for $n \geqslant m$, and i such that $\int f_i d\mu_m < \delta/3$. Then $\int f_i d\mu_n \leqslant 2\delta/3$ for $n \geqslant m$. For each $r = 1, 2, \ldots, m-1$, we choose $i(r)$ so that $\int f_{i(r)} d\mu_r < \delta$. Let $j = \max(i(1), \ldots, i(m-1), i)$. Then $\int f_j d\mu_r < \delta$ for all r. Thus there is an ε-totally bounded set B_ε such that $\mu_r(S \sim B_\varepsilon) < \delta$ for all r. Thus, as in the proof of Theorem 2, for any $\gamma > 0$ there is a totally bounded set B such that $\mu_r(S \sim B) < \gamma$ for all r. Of course B can be closed, so by Theorem 4, the μ_r form a TW*-sequentially relatively compact set and have a subsequence $\mu_{r(n)}$ converging for TW* to some μ in $\mathcal{M}^+(S)$. Then by Theorem 5, $\|\mu_{r(n)} - \mu\|_{\mathrm{BL}}^* \to 0$, so $\|\mu_r - \mu\|_{\mathrm{BL}}^* \to 0$. This determines μ in $\mathcal{M}^+(S)$ uniquely by Lemma 6. Thus all weak-star convergent subsequences of $\{\mu_r\}$ converge to μ, so since $\{\mu_r\}$ is TW* sequentially relatively compact, $\mu_r \to \mu$ weak-star, q.e.d.

Varadarajan has proved that if S is a metric space, then $(\mathcal{M}_s^+(S), \mathrm{TW}^*)$ is metrizable, by a complete metric if S is complete ([14], IV, p. 49, Teorema 13, p. 62, and Teorema 18, p. 68). Theorem 9 and Theorem 18 below yield new proofs of these facts.

The next theorem is very close to a result of Kantorovich and Rubinstein [5], who start from a differently defined norm on $\mathcal{M}(S)$ but arrive at essentially the same conclusion:

THEOREM 10. *Let S be a compact metric space. Then a sequence $\{\mu_n\}$ of elements of $\mathcal{M}(S) = \mathcal{C}(S)^*$ converges weak-star if and only if*

(a) $\sup_n |\mu_n|(S) < \infty$,

(b) *$\{\mu_n\}$ is a Cauchy sequence for UBL*.*

Proof. "If" follows from density of $\mathrm{BL}(S)$ in $\mathcal{C}(S)$ (Lemma 7), and "only if" from Theorem 6 and the Banach-Steinhaus theorem.

We shall see below (Theorem 17) that condition (a) of Theorem 10 cannot be removed unless S is finite.

Non-negativity of the measures in Theorems 8 and 9 cannot be weakened to boundedness of $|\mu_n|(S)$: let S be the real line and let μ_n have mass 1 at n and -1 at $(n^2+1)/n$. I have an argument to show that if S is complete, $\{\mu_n\}$ is a UBL*-Cauchy sequence in $\mathcal{M}_s(S)$ and $|\mu_n|(S)$ is bounded, then $\{\mu_n\}$ converges for TBL* to an element of $\mathcal{M}(S)$, but this result seems irrelevant to the main purpose of this paper, and the argument seems too long to be worth giving.

3. Topologies and uniformities. Throughout this section we assume that (S, d) is a *separable* metric space. We investigate in detail the possible inclusions between TW*, TBL* and TV* and their uniformities, and metrizability of the weak-star structures, on $\mathcal{M}(S)$, $\mathcal{M}^+(S)$ and $\mathcal{P}(S)$. We show that whenever a weak-star structure is metrizable, $\|\ \|_{\mathrm{BL}}^*$ metrizes it. We first note two useful meta-results:

LEMMA 10. *Suppose T_1 and T_2 are filters of sets containing 0 in $\mathcal{M}(S)$. For any $\mathcal{A} \subset \mathcal{M}(S)$, let U_i be the filter of subsets of $\mathcal{A} \times \mathcal{A}$ with a base consisting of all sets*

$$\{\langle \mu, \nu\rangle : \mu - \nu \in B\}, \quad B \in T_i,$$

for $i = 1, 2$. Then the following are equivalent:

(a) $T_1 \subset T_2$,

(b) $U_1 \subset U_2$ *on* $\mathcal{M}(S)$,

(c) $U_1 \subset U_2$ *on* $\mathcal{M}^+(S)$.

Proof. Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Given (c), we have that for every $A \in T_1$ there is a $B \in T_2$ such that

$$\{\langle \mu, \nu\rangle \in \mathcal{M}^+(S) \times \mathcal{M}^+(S) : \mu - \nu \in A\} \supset \{\langle \mu, \nu\rangle \in \mathcal{M}^+(S) \times \mathcal{M}^+(S) : \mu - \nu \in B\}.$$

But by the Jordan decomposition, this implies (a), q.e.d.

We shall see below that "$U_1 \subset U_2$ on $\mathcal{P}(S)$" is *not* equivalent to the conditions of Lemma 10. However, we have

LEMMA 11. *Suppose T_1 and T_2 are topologies on $\mathcal{M}(S)$ making it a topological linear space for which the linear functional $\mu \to \mu(S)$ is continuous. Then $T_1 \subset T_2$ on $\mathcal{M}^+(S)$ if and only if $T_1 \subset T_2$ on $\mathcal{P}(S)$.*

Proof. "Only if" is obvious. To prove "if", suppose $T_1 \subset T_2$ on $\mathcal{P}(S)$. We must show that if $\{\mu_a\}$ is a net in $\mathcal{M}^+(S)$ and $\mu_a \to \mu$ for T_2, then $\mu_a \to \mu$ for T_1. Now $\mu_a(S) \to \mu(S)$. Let $\nu_a = \mu_a/\mu_a(S)$ if $\mu_a(S) > 0$, otherwise $\nu_a \equiv 0$. If $\mu(S) > 0$, then $\mu_a(S) > 0$ for a sufficiently large, and $\nu_a \to \mu/\mu(S)$ for T_2, hence for T_1, thus $\mu_a \to \mu$ for T_1. If $\mu(S) = 0$, i.e., $\mu \equiv 0$, let $\sigma \in \mathcal{M}^+(S)$ satisfy $\sigma(S) > 0$. Then $\sigma + \mu_a \to \sigma + \mu$ for T_2, hence for T_1, thus $\mu_a \to \mu$ for T_1, q.e.d.

It is trivial that for any S, UW* $\subset$ UV*, UBL* $\subset$ UV*, TW* $\subset$ TV*, and TBL* $\subset$ TV* on $\mathcal{M}(S)$, hence on any subset. We shall discover when each of these inclusions is proper (all are if S is not discrete, even on $\mathcal{P}(S)$).

THEOREM 11. *The following are equivalent:*

(a) *S is uniformly discrete,*
(b) UBL* = UV* *on* $\mathcal{M}(S)$,
(c) UBL* = UV* *on* $\mathcal{M}^+(S)$,
(d) TBL* = TV* *on* $\mathcal{M}(S)$,
(e) TW* $\subset$ TBL* *on* $\mathcal{M}(S)$,
(f) UW* $\subset$ UBL* *on* $\mathcal{M}(S)$,
(g) UW* $\subset$ UBL* *on* $\mathcal{M}^+(S)$,
(h) UBL* = UV* *on* $\mathcal{P}(S)$.

Proof. Suppose S is uniformly discrete, i.e. for some $\varepsilon > 0$, $d(x, y) > \varepsilon$ whenever $x \neq y$. Given μ in $\mathcal{M}(S)$, let $f = 1$ on the support of μ^+, $f = -1$ elsewhere. Then $\|f\|_{\mathrm{BL}} < (\varepsilon+2)/\varepsilon$, and $\int f d\mu = |\mu|(S)$, so

$$|\mu|(S) \geqslant \|\mu\|_{\mathrm{BL}}^* \geqslant \varepsilon|\mu|(S)/(\varepsilon+2).$$

Thus UV* = UBL*, and (a) $\Rightarrow$ (b). We have (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) and (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g) by Lemma 10. (d) $\Rightarrow$ (e) and (b) $\Rightarrow$ (h) are obvious.

Suppose S is not uniformly discrete, and take x_n and y_n with

$$0 < \varepsilon_n = d(x_n, y_n) \to 0.$$

Let p_n and q_n be unit masses at x_n and y_n respectively, and let

$$\mu_n = \varepsilon_n^{-1/2}(p_n - q_n).$$

Then $\|\mu_n\|_{\mathrm{BL}}^* \to 0$, but $|\mu_n|(S) \to \infty$ so $\mu_n \nrightarrow 0$ for TW*, and (e) $\Rightarrow$ (a). Also $\|p_n - q_n\| \to 0$ and $|p_n - q_n|(S) \equiv 2$, so *(h)* $\Rightarrow$ *(a)*, and the proof is complete.

THEOREM 12. *For any (separable metric) space S, TBL* = TW* on $\mathcal{M}^+(S)$, hence on $\mathcal{P}(S)$ and TW* is metrizable on both spaces.*

Proof. S is a Lindelöf space ([6], Theorem 15, p. 49). Thus TW* on $\mathcal{M}^+(S)$ is metrizable (Varadarajan [14], IV, p. 49, Teorema 13, p. 62); an independent proof of metrizability of $\mathcal{M}^+(S)$ will be given below, in § 4). The identity map of $\mathcal{M}^+(S)$ is sequentially continuous from TW* to TBL* by Theorem 6, hence continuous, i.e. TBL* $\subset$ TW* on $\mathcal{M}^+(S)$. Theorem 9 asserts that the identity is sequentially continuous in the opposite direction, hence continuous, q.e.d.

We now need a lemma. For any set S, we let $l_\infty(S)$ be the set of all bounded real-valued functions on S, with the supremum norm

$$\|f\|_\infty = \|f\|_{\infty,S} = \sup\{|f(x)| : x \in S\}.$$

LEMMA 12. *For any set S, f in $l_\infty(S)$, and finite-dimensional subspace H of $l_\infty(S)$,*

$$d(f, H) = \inf_{h \in H}\|f-h\|_\infty = \sup\{\inf_{h \in H}\|f-h\|_{\infty,B} : B \text{ finite}\}.$$

Proof. Suppose H has dimension n. We may assume $f \notin H$. Let J be the subspace spanned by f and H. Then there is a subset N of S, containing $n+1$ points, such that the natural projection (restriction) of $l_\infty(S)$ onto $l_\infty(N)$ is one-to-one on J. Thus for some $\delta > 0$, we have

$$\|j\|_{\infty,N} \geqslant \delta\|j\|_\infty$$

for all j in J. Now the set K of all h in H such that

$$\|h\|_\infty \leqslant \|f\|_\infty + 2d(f, H)/\delta$$

is compact. Let $\varepsilon > 0$. For each h in K there is a finite set B such that

$$\|f-h\|_{\infty,B} > \|f-h\|_\infty - \varepsilon.$$

For each finite B, the set of all h for which the inequality holds is open for $\| \ \|_\infty$. Thus we have an open cover of K. We take a finite subcover, and let C be the union of the corresponding finite sets. Let $D = N \cup C$. Then

$$\|f-h\|_{\infty,C} > \|f-h\|_\infty - \varepsilon$$

for all h in K, so that

$$\inf_{h \in K}\|f-h\|_{\infty,D} \geqslant d(f, H) - \varepsilon.$$

For h in $H \sim K$, we have

$$\|f-h\|_{\infty,D} \geqslant \|f-h\|_{\infty,N} \geqslant \delta\|f-h\|_\infty \geqslant 2d(f, H).$$

Thus

$$\inf_{h \in H}\|f-h\|_{\infty,D} \geqslant d(f, H) - \varepsilon.$$

Letting $\varepsilon \downarrow 0$, the proof is complete.

THEOREM 13. *The following are equivalent:*

(a) *S is compact,*
(b) *UW* = UBL* on* $\mathcal{P}(S)$,
(c) *UW* is metrizable on* $\mathcal{P}(S)$.

Proof. If S is compact, then $\mathcal{P}(S)$ is TW*-relatively compact by Theorem 4. It is TW*-closed, hence TW*-compact. By Theorem 12, TW* = TBL* on $\mathcal{P}(S)$. Thus UBL* and UW* on $\mathcal{P}(S)$ are each the unique uniformity yielding the compact Hausdorff topology TW* on $\mathcal{P}(S)$ (see Kelley [6], Theorems 29, 30, pp. 197-198; I thank C. M. Deo for this simple proof). Thus (a) $\Rightarrow$ (b).

Clearly (b) $\Rightarrow$ (c). Finally, suppose (c) holds. Then UW* on $\mathscr{P}(S)$ has a countable base, i.e. there exist countably many functions $f_1, f_2, \ldots$, in $\mathscr{C}(S)$, such that for any f in $\mathscr{C}(S)$ and $\varepsilon > 0$ there exist n and $\delta > 0$ such that for any p and q in $\mathscr{P}(S)$,

$$(*) \qquad \left|\int f_i d(p-q)\right| < \delta,\ i = 1, \ldots, n, \text{ implies } \left|\int f d(p-q)\right| < \varepsilon.$$

We may assume $f_1 \equiv 1$. Suppose that S is not compact, so that it contains an infinite set A with no accumulation points. Then there is an f in $\mathscr{C}(S)$ with $\|f\|_\infty = 1$ and $\|f-g\|_{\infty,A} \geqslant \frac{3}{4}$ for every g in the subspace spanned by the f_i. Choose n and δ so that $(*)$ holds with $\varepsilon = 1$. Then by Lemma 12, there is a finite set F such that

$$\|f-g\|_{\infty,F} \geqslant \tfrac{1}{2}$$

for every g in the linear span of $f_1, \ldots, f_n$. Then we take μ in $\mathscr{M}(F)$, by the Hahn-Banach theorem, such that

$$\int f_i d\mu = 0,\ i = 1, \ldots, n, \qquad \int f d\mu = 1, \qquad |\mu|(F) \leqslant 2.$$

Since $f_1 \equiv 1$, we have $\mu = \lambda(p-q)$ where $\lambda \leqslant 1$ and $p, q \in \mathscr{P}(S)$. Now

$$\int f_i d(p-q) = 0,\ i = 1, \ldots, n, \qquad \int f d(p-q) = 1/\lambda \geqslant 1,$$

a contradiction. Thus S is compact, q.e.d.

THEOREM 14. *$UBL^* \subset UW^*$ on $\mathscr{P}(S)$ if and only if S is totally bounded.*

Proof. We have $\mathrm{TBL}^* = \mathrm{TW}^*$ on $\mathscr{P}(S)$ in general. If S is totally bounded, let $\bar{S}$ be its completion, which is compact. The natural map of $\mathscr{M}(S)$ into $\mathscr{M}(\bar{S})$ is weak-star uniformly continuous, and $\mathrm{BL}(S, d)$ is naturally isometric to $\mathrm{BL}(\bar{S}, d)$. Thus $\mathrm{UBL}^* \subset \mathrm{UW}^*$ on $\mathscr{P}(S)$ by Theorem 13.

If S is not totally bounded, there is a $\delta > 0$ and an infinite set $A \subset S$ such that $d(x, y) > \delta$ for any distinct x and y in A. Suppose that for some $f_1, \ldots, f_k$ in $\mathscr{C}(S)$ and $\varepsilon > 0$,

$\left|\int f_j d(p-q)\right| < \varepsilon,\ j = 1, \ldots, k$, implies $\|p-q\|^*_{\mathrm{BL}} < \delta/(\delta+2)$ for any p and q in $\mathscr{P}(S)$.

We can assume $f_1 \equiv 1$. Let H be the linear space spanned by the f_j. There is an f in $l_\infty(A)$ such that $\|f\|_\infty = 1$ and $d(f, H) \geqslant \frac{3}{4}$. Then by Lemma 12, there is a finite set $B \subset A$ such that

$$\|f-h\|_{\infty,B} \geqslant \tfrac{1}{2}$$

for all h in H. Thus by the Hahn-Banach theorem there is a μ in $\mathscr{M}(B)$ such that $\int f_j d\mu = 0,\ j = 1, \ldots, k$, $\int f d\mu = 1$, $|\mu|(B) \leqslant 2$. f can be extended to all of S with $\|f\|_{\mathrm{BL}} \leqslant (\delta+2)/\delta$, so that $\|\mu\|^*_{\mathrm{BL}} \geqslant \delta/(\delta+2)$. Also $\mu = \lambda(p-q)$ for some p and q in $\mathscr{P}(S)$ and $0 < \lambda \leqslant 1$, so that $\|p-q\|^*_{\mathrm{BL}} \geqslant \delta/(\delta+2)$. This is a contradiction, so the proof is complete.

THEOREM 15. *$UW^* \subset UBL^*$ on $\mathscr{P}(S)$ if and only if S is a u.c. space.*

Proof. If S is a u.c. space, then the identity on $\mathscr{P}(S)$ is uniformly continuous from UBL* to UW* since $\mathrm{BL}(S, d)$ is uniformly dense in $\mathscr{C}(S)$ (Lemma 8).

Conversely, if S is not a u.c. space, then by Lemma 9 we take distinct x_n in S with $d(x_{2n}, x_{2n+1}) \to 0$ and $\{x_n\}$ having no convergent subsequence. Let $f(x_{2n}) = 1$, $f(x_{2n+1}) = -1$, and extend f to a continuous function on S. Let p_n be the unit mass at x_{2n}, and q_n at x_{2n+1}. Then

$$\|p_n - q_n\|^*_{\mathrm{BL}} \to 0, \qquad \int f d(p_n - q_n) \equiv 2.$$

Thus $\mathrm{UW}^* \not\subset \mathrm{UBL}^*$, and the proof is complete.

In [3], in the proof of Theorem 5.1, I considered weak-star "uniform" continuity of a function whose values are probability measures. In view of the differences shown by Theorems 14 and 15, it now appears that the uniform continuity assertions should refer to UBL*, not to UW*. With this interpretation, one obtains a correct proof of the theorem.

THEOREM 16. *The following are equivalent*:

(a) *S is discrete,*
(b) *$TV^* = TW^*$ on $\mathscr{M}^+(S)$,*
(c) *$TV^* = TW^*$ on $\mathscr{P}(S)$,*
(d) *$TV^* = TBL^*$ on $\mathscr{M}^+(S)$,*
(e) *$TV^* = TBL^*$ on $\mathscr{P}(S)$.*

Proof. Suppose S is discrete. To prove (b), we note that TV* and TW* depend only on the topology of S, not on the metrization. Thus we may assume S is uniformly discrete, and apply Theorems 11(d) and 12 to obtain (b).

(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) by Lemma 11 and Theorem 12.

Suppose $\mathrm{TV}^* = \mathrm{TBL}^*$ on $\mathscr{P}(S)$. If S is not discrete, let x_n be a sequence of distinct points converging to a point x. Then the unit masses at the x_n converge to the unit mass at x for TBL* but not for TV*, a contradiction, so (a) holds, q.e.d.

THEOREM 17. *The following are all equivalent*:

(a) *S is finite,*
(b) *UW^* on $\mathscr{M}(S)$ is metrizable,*
(c) *UW^* on $\mathscr{M}^+(S)$ is metrizable,*
(d) *TW^* on $\mathscr{M}(S)$ is metrizable,*
(e) *$TW^* = TV^*$ on $\mathscr{M}(S)$,*
(f) *$UW^* = UV^*$ on $\mathscr{M}(S)$,*
(g) *$UW^* = UV^*$ on $\mathscr{M}^+(S)$,*
(h) *$UW^* = UBL^*$ on $\mathscr{M}(S)$,*
(i) *$UW^* = UBL^*$ on $\mathscr{M}^+(S)$,*
(j) *$TW^* = TBL^*$ on $\mathscr{M}(S)$,*
(k) *$TBL^* \subset TW^*$ on $\mathscr{M}(S)$,*
(l) *$UBL^* \subset UW^*$ on $\mathscr{M}(S)$,*
(m) *$UBL^* \subset UW^*$ on $\mathscr{M}^+(S)$,*
(n) *$UW^* = UV^*$ on $\mathscr{P}(S)$.*

Proof. It is easy to see that (a) implies all the other conditions, specifically (h) which in turn implies (b). (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) as in Lemma 10. (d) implies (e) by a result of Varadarajan [13]. (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g) by Lemma 10.

(e) implies that S is discrete (by Theorem 16) and (g) implies that S is compact (by Theorem 13), thus either implies that S is finite, and (a) through (h) are equivalent.

(h) $\Leftrightarrow$ (i) $\Leftrightarrow$ (j) $\Rightarrow$ (k) $\Leftrightarrow$ (l) $\Leftrightarrow$ (m) by Lemma 10. (k) implies that there exist $f_1, \ldots, f_n$ in $\mathscr{C}(S)$ and $\varepsilon > 0$ such that $\left|\int f_i d\mu\right| < \varepsilon$, $i = 1, \ldots, n$, implies $\|\mu\|_{BL}^* < 1$ for μ in $\mathscr{M}(S)$. If S is infinite, let F be a set of $n+1$ points of S. By Lemma 5, we obtain an f in $BL(S, d)$ which is not a linear combination of $f_1, \ldots, f_n$, even on F. Thus there is a ν in $\mathscr{M}(F)$ such that

$$\int f_i d\nu = 0, \; i = 1, \ldots, n, \qquad \int f d\nu \neq 0.$$

Letting $\mu = M\nu$ for M large enough, we have a contradiction. Thus (k) $\Rightarrow$ (a). Since (g) $\Rightarrow$ (n), it remains only to prove (n) $\Rightarrow$ (a). This follows from Theorems 13 and 11. The proof is finished.

4. Metrizability of $\mathscr{M}_s^+(S)$. In this section, (S, d) will be an arbitrary (not necessarily separable) metric space. For any subset A of S and $\varepsilon > 0$, we let

$$A^\varepsilon = \{x \in S : d(x, y) < \varepsilon \text{ for some } y \text{ in } A\}.$$

We shall show that $TBL^* = TW^*$ on $\mathscr{M}_s^+(S)$. The proof (unlike that of Theorem 12 above) does not use the fact that TW^* on $\mathscr{M}_s^+(S)$ is metrizable. Thus a new proof of the latter fact is obtained, shorter than the original proof of Varadarajan [14], p. 61-64.

LEMMA 13. *Suppose* $\mu \in \mathscr{M}_t^+(S)$. *Then* TW^* *on* $\mathscr{M}^+(S)$ *has a countable neighborhood-base at* μ.

Proof. Let K_n be an increasing sequence of compact sets such that

$$\lim_{n\to\infty} \mu(K_n) = \mu(S).$$

For $n = 1, 2, \ldots$, let F_n be a countable set of functions dense in $\mathscr{C}(K_n)$, and extended continuously to all of S without increasing their supremum norms (Tietze extension). (Incidentally, the known fact that $\mathscr{C}(K)$ is separable for any compact metric space K follows directly from Lemma 7 and the fact that a set of functions bounded for $\| \;\; \|_{BL}$ is uniformly relatively compact (Ascoli), hence separable.) For $m, n = 1, 2, \ldots$, let h_{mn} be a continuous function on S such that $h_{mn}(x) = 0$ for all x in K_n, $h_{mn}(x) = 1$ if $x \in S \sim K_n^{1/m}$, and $0 \leqslant h_{mn}(x) \leqslant 1$ for all x. Let F be the union of all the sets F_n and the set of all functions h_{mn} and the constant function 1.

Let $f \in \mathscr{C}(S)$, $0 < \varepsilon < 1$, and

$$\max(\mu(S), \|f\|_\infty, 1) = M.$$

We take n such that $\mu(S \sim K_n) < \varepsilon/27M$, and g in F so that

$$|f(x) - g(x)| < \varepsilon/4M \quad \text{for all} \quad x \text{ in } K_n.$$

Then for some positive integer m, $|f(x) - g(x)| < \varepsilon/3M$ for all x in $K_n^{1/m}$. Suppose $\nu \in \mathscr{M}^+(S)$, and

$$|(\mu - \nu)(S)| < M, \quad \left|\int g d(\mu - \nu)\right| < \varepsilon/3, \quad \left|\int h_{mn} d(\mu - \nu)\right| < \varepsilon/27M.$$

Then $\nu(S \sim K_n^{1/m}) \leqslant \int h_{mn} d\nu \leqslant 2\varepsilon/27M$,

$$\left|\int f d(\mu - \nu)\right| \leqslant \left|\int g d(\mu - \nu)\right| + \left|\int (f - g) d(\mu - \nu)\right| < \varepsilon/3 + \int |f - g| d(\mu + \nu)$$
$$\leqslant \varepsilon/3 + \int_{K_n^{1/m}} \varepsilon/3M d(\mu + \nu) + \int_{S \sim K_n^{1/m}} 3M d(\mu + \nu) < \varepsilon.$$

Now for each finite subset G of F and positive integer k, let

$$N(G, k) = \left\{\nu \in \mathscr{M}^+(S) : \left|\int g d(\mu - \nu)\right| < 1/k \text{ for all } g \text{ in } G\right\}.$$

Then the set of all $N(G, k)$ is a countable neighborhood-base at μ for TW^* on $\mathscr{M}^+(S)$, q.e.d.

THEOREM 18. *For any metric space* S, $TW^* = TBL^*$ *on* $\mathscr{M}_s^+(S)$.

Proof. Suppose μ_α is a net in $\mathscr{M}_s^+(S)$ which converges to μ for TW^*. Then if $\bar{S}$ is the completion of S, μ_α converges to μ for TW^* on $\mathscr{M}_s^+(\bar{S})$. Suppose μ_α does not converge to μ for TBL^* on $\bar{S}$. Then, by Lemma 13, we can replace the net μ_α by a sequence u_n which converges for TW^* but not for TBL^*. By Theorem 6, this is impossible. Thus $\mu_\alpha \to \mu$ for TBL^* on $\bar{S}$ and hence on S.

Continuity in the converse direction holds by Theorem 8 since TBL^* is a metric topology. The proof is complete.

For any topological space X, $\mathscr{M}_\sigma(X)$ is the set of all measures μ in $\mathscr{M}(X)$ such that if f_α is any net in $\mathscr{C}(X)$ decreasing pointwise to 0,

$$\int f_\alpha d\mu \to 0.$$

Varadarajan proved that for any metric space S, $\mathscr{M}_\sigma(S) = \mathscr{M}_s(S)$ ([14], Corollary, p. 50), and that $\mathscr{M}_\sigma(S)$ is metrizable. For a general topological space X, we have the inclusion $\mathscr{M}_\sigma(X) \subset \mathscr{M}_s(X)$ since $\mathscr{M}_s(X)$ is defined in terms of continuous pseudo-metrics on X. Whether the converse inclusion holds seems unclear.

References

[1] A. D. Alexandrov, *Additive set-functions in abstract spaces*, Matematicheskii Sbornik (Recueil Math.) 8 (1940), p. 307-342; 9 (1941), p. 563-621; 13 (1943), p. 169-243.

[2] J. Czipszer and L. Geher, *Extension of functions satisfying a Lipschitz condition*, Acta Math. Acad. Sci. Hungar. 6 (1955), p. 213-220.

[3] R. M. Dudley, *Lorentz-invariant Markov processes in relativistic phase space*, Arkiv før Matematik (to appear).

[4] — *Weak convergence of probabilities on non-separable metric spaces and empirical measures on Euclidean spaces*, Illinois Journal of Mathematics 10 (1966), p. 109-126.

[5] L. Kantorovitch and G. Rubinstein, *On a space of completely additive functions* (in Russian), Vestnik Leningrad Univ. 13.7 (Ser. Mat. Astr. 2) (1958), p. 52-59.

[6] J. L. Kelley, *General topology*, New York 1955.

[7] E. Marczewski and R. Sikorski, *Measures in non-separable metric spaces*, Coll. Math. 1 (1948), p. 133-139.

[8] Yu. V. Prokhorov, *Convergence of random processes and limit theorems in probability*, Theory of Probability and its Applications 1 (1956).

[9] R. Ranga Rao, *Some theorems on weak convergence of measures and applications*, Annals of Mathematical Statistics 33 (1962), p. 659-680.

[10] J. C. Shepherdson, *Inner models for set theory (II)*, Jour. Symb. Logic 17 (1952), p. 225-237.

[11] S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. 16 (1930), p. 140-150.

[12] — and J. C. Oxtoby, *On the existence of a measure invariant under a transformation*, Annals of Math. 40 (1939), p. 560.

[13] V. S. Varadarajan, *Weak convergence of measures on separable metric spaces*, Sankhya 19 (1958), p. 15-22.

[14] — *Measures on topological spaces* (in Russian), Matematicheskii Sbornik 55 (97) (1961), p. 35-100.

[15] Donald S. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. 111 (1964), p. 240-272.

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Interpolation of additive functionals

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In this note a generalization of the theorem of Mazur and Orlicz ([1], p. 147) is presented; the proof of the latter was simplified by Sikorski [3] and Pták [2]. We state first our extension and its proof and then explain how the previous statement may be obtained as a special case.

We consider a semi-group S, composition in S being denoted by $x+y$, provided with a real functional ω subject to two conditions:

$$(1) \qquad \infty > \omega(s) \geqslant -\infty \quad \text{for} \quad s \in S,$$

$$(2) \qquad \omega(s)+\omega(t) \geqslant \omega(s+t) \quad \text{for} \quad s, t \in S.$$

In addition to ω there is given a real functional L on S, restricted as follows:

$$(3) \qquad \infty > L(s) \geqslant -\infty, \quad s \in S,\ L \not\equiv -\infty.$$

(4) If $\{s_1, \ldots, s_n\}$ is a finite sequence in S,

$$\omega(s_1+\ldots+s_n) \geqslant \sum_{i=1}^{n} L(s_i).$$

This condition is abbreviated: $\omega \gg L$.

THEOREM. *There exists an additive functional ξ on S such that $\omega \geqslant \xi \geqslant L$.*

Proof. We begin with the observation that if $\omega = L$ in S, then ω is already additive. Let us exclude this and choose an element $a_0 \in S$ and a number r such that $\omega(a_0) > r > L(a_0)$.

We claim now that either A or B holds, among the next two statements:

A. $\omega(ma_0+u_1+\ldots+u_n) \geqslant mr+\sum_{i=1}^{n} L(u_i)$, for any $m \geqslant 1$ and elements $u_1, \ldots, u_n$ in S.

B. $\omega(s)+m'r \geqslant \sum_{j=1}^{n'} L(t_j)$, whenever $m'a_0+s = t_1+\ldots+t_{n'}$, $m' \geqslant 1$; $s \in S$; $t_1, \ldots, t_{n'} \in S$.