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# Concerning the convergence of iterates to fixed points

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This paper is devoted to a generalization of a theorem of Ostrowski ([2], p. 119) concerning fixed points.

One of the earliest and most useful fixed point theorems is the Picard fixed point theorem (*alias* contraction principle) which states that a contraction mapping $f$ of a complete metric space into itself has a unique fixed point. The theorem goes on to say that if $x$ is any point of the metric space, then the sequence of iterates of $x$ under $f$, $\{f^n(x)\}$, converges to this fixed point. The theorem proved here is concerned with the following question: given that a mapping $f$ has a fixed point $x^*$, when is it true that iterates (under $f$) of nearby points converge to $x^*$? Such a question is clearly of interest in numerical analysis since many numerical problems can be reduced to the problem of locating fixed points. In Ostrowski's work $f$ is taken to be a function of class $C^1$ throughout an open set in $R^n$. The proofs involve rather elaborate calculations with matrices and make heavy use of the Jordan canonical form. In this paper $R^n$ is replaced by an arbitrary Banach space and the regularity assumptions on $f$ are somewhat weakened. The price for so strengthening and generalizing Ostrowski's result proved to be small; in fact a more transparent proof was made possible, the key to it being the well-known spectral radius formula.

THEOREM. *Let $f$ be a mapping whose domain and range are subsets of a Banach space $X$. Suppose that*

1) *$x^* \in X$ is a fixed point of $f$;*

2) *$f$ is differentiable at $x^*$;*

3) *the spectral radius of the derivative of $f$ at $x^*$ is less than one.*

*Then there exists a neighborhood $N$ of $x^*$ such that*

$$\lim_{n\to\infty} f^n(x) = x^*$$

*for each $x \in N$.*

Proof. Without loss of generality we may suppose that $x^* = 0$. Since $f$ is differentiable at $x^* = 0$, we can write

$$f(x) = T(x) + h(x),$$

where $T$ is a bounded linear transformation of $X$ into itself and

$$\lim_{x\to 0} \|h(x)\|/\|x\| = 0.$$

The transformation $T$ is (in the terminology of Dieudonné [1]) what we have called the *derivative* of $f$ at $x^*$, and so, by assumption $\|T\|_{sp} < 1$.

CASE 1. Suppose that $\|T\| < 1$. We pick a number $r$ such that $\|T\| < r < 1$. Since

$$\lim_{x\to 0} \|h(x)\|/\|x\| = 0,$$

there exists an $\varepsilon > 0$ such that

$$\|h(x)\| < (1-r)\|x\|$$

whenever $\|x\| < \varepsilon$. Let $N$ be the sphere $\{x \in X: \|x\| < \varepsilon\}$. We shall now prove that $N$ has the required properties.

By a simple argument involving the triangle inequality one can show that for each $x \in N$,

$$\|f(x)\| \leqslant R\|x\|,$$

where $R = \|T\| + 1 - r$. Since $0 < R < 1$, it follows that $N$ is $f$-invariant. Consequently, if $x \in N$, then its entire sequence of iterates $\{f^n(x)\}$ is also contained in $N$, and we get by induction on the inequality displayed above

$$\|f^n(x)\| \leqslant R^n \|x\|.$$

Since $R^n \to 0$, it follows that $f^n(x)$ converges to $0 = x^*$.

CASE 2. General case. By the spectral radius formula

$$1 > \|T\|_{sp} = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Thus, there exists an integer $m$ such that $\|T^m\| < 1$. We can now apply Case 1 to the function $f^m$. (By the chain rule, $f^m$ is differentiable at $x^* = 0$, and $T^m$ is its derivative at the point.) Hence there exists a sphere $M$ about $x^* = 0$ such that

a) $M$ is $f^m$-invariant,

and

b) for each $x \in M$,

$$\lim_{n\to\infty} (f^m)^n(x) = \lim_{n\to\infty} f^{mn}(x) = 0.$$



Since $f$ is differentiable at $x^* = 0$, it is continuous there and we can therefore find a sphere $N$ about $x^* = 0$ such that $N, f(N), f^2(N), \ldots, f^{m-1}(N)$ are all contained in $M$. Since $\|f^m(x)\| \leqslant \|x\|$ for each $x \in M$, we conclude that for each $x \in N$ the sequence of iterates $\{f^n(x)\}$ is entirely contained in $M$. Also, since $f$ is continuous at $x^* = 0$,

$$\lim_{n\to\infty} f^{mn+1}(x) = \lim_{n\to\infty} f(f^{mn}(x)) = f(0) = 0$$

for each $x \in N$. More generally,

$$\lim_{n\to\infty} f^{mn+k}(x) = 0,$$

if $x \in N$ and $k$ is one of the integers $0, 1, 2, \ldots, m-1$. The fact that each of these $m$ different subsequences converges to $x^* = 0$ implies that the entire sequence $\{f^n(x)\}$ (obtained simply by interlacing these subsequences) converges to $x^* = 0$ also.

### References

[1] J. Dieudonné, *Foundations of modern analysis*, New York 1960.

[2] A. M. Ostrowski, *Solutions of equations and systems of equations*, New York 1960.

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